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ON CLASS OF INTEGRAL EQUATIONS WITH PARTIAL INTEGRALS AND ITS APPLICATIONS

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Abstract. We prove the existence and uniqueness of the solution to one class of systems of integral equations with partial integrals. Equations with partial integrals are equations containing an unknown function in the integrands of integrals of different dimension. The feature of the considered class of integral equations is that the equations involve integrals with both variables and constant upper integration limits. We first prove the unique solvability theorem for integral equations in the three-dimensional space. A similar statement is proved for equations with arbitrary many independent variables. Some applications of the obtained result are provided. For a hyperbolic system with dominant derivatives of the second order with three independent variables, we prove the existence and uniqueness of the solution of the main characteristic problem. The main characteristic problem for the system of equations with higher derivatives of the second order can be considered as an analogue of the Goursat problem for a hyperbolic system with no multiple characteristics. The solution of this problem is constructed explicitly in terms of the Riemann matrix. The Riemann matrix is defined as the solution of a system of Volterra integral equations. The problem with boundary conditions on five sides of the characteristic parallelepiped for this system of equations with higher derivatives of the second order is formulated. By reducing the problem to a system of equations with partial integrals and basing on our results, we prove the existence and uniqueness of the solution to this problem.

Keywords: integral equation with partial integrals, problem with conditions on the characteristics.

Mathematics Subject Classification: 45A05, 45F05, 35L51

1. Introduction

In the present work we consider a class of integral equations with partial integrals, that is, the equations involving integrals of various multiplicities for an unknown function of many variables [1]–[3]. We obtain conditions ensuring unique solvability of some systems of integral equations and the author found information on this issue in the literature. These conditions are applied to solving one problem for a system of partial differential equations with boundary conditions on the characteristics.

The study of boundary value problems for hyperbolic systems is of a significant theoretical interest. Various aspects of the theory of the mentioned systems were studied by many authors [4]–[11]. In the present paper we give some developing of the results in work [12], in which there was proposed a version of Riemann method for a system of differential equation with multiple characteristics and in terms of the Riemann matrix solutions for Cauchy and Goursat problem were constructed. We also develop the results of works [13]–[14], in which the Riemann method was applied for studying problems for one system of equations with two independent variables and multiple characteristics.

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2. Unique solvability of one class of system of integral equations

It was shown in work [15] that an integral Volterra equation with partial integrals with continuous kernels and with a free term

$$
\begin{aligned}
w(x,y,z) &- \int_{x_0}^{x} K_1(x,y,z,\alpha)w(\alpha,y,z)\,d\alpha \\
&- \int_{y_0}^{y} K_2(x,y,z,\beta)w(x,\beta,z)\,d\beta \\
&- \int_{z_0}^{z} K_3(x,y,z,\gamma)w(x,y,\gamma)\,d\gamma \\
&- \int_{x_0}^{x}\int_{y_0}^{y} K_4(x,y,z,\alpha,\beta)w(\alpha,\beta,z)\,d\beta\,d\alpha \\
&- \int_{x_0}^{x}\int_{z_0}^{z} K_5(x,y,z,\alpha,\gamma)w(\alpha,y,\gamma)\,d\gamma\,d\alpha \\
&- \int_{y_0}^{y}\int_{z_0}^{z} K_6(x,y,z,\beta,\gamma)w(x,\beta,\gamma)\,d\gamma\,d\beta \\
&- \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} K_7(x,y,z,\alpha,\beta,\gamma)w(\alpha,\beta,\gamma)\,d\gamma\,d\beta\,d\alpha = F(x,y,z),
\end{aligned} \tag{1}
$$

as well as its n-dimensional analogue have a unique solution in the class of continuous functions. More precisely, the following theorem holds.

Theorem 1. *The equation*

$$
\begin{aligned}
w(x_1,\dots,x_n) - \sum_{k=1}^{n}\sum_{Q_{k,n}} \int_{x_{q_1}^0}^{x_{q_1}} \cdots \int_{x_{q_k}^0}^{x_{q_k}} & K_{q_1\dots q_k}(x_1,\dots,x_n,\alpha_{q_1},\dots,\alpha_{q_k}) \\
& \cdot w(x_1,\dots,x_{q_1-1},\alpha_{q_1},x_{q_1+1},\dots,x_{q_k-1},\alpha_{q_k},x_{q_k+1},\dots,x_n) \\
& \cdot d\alpha_{q_k}\dots d\alpha_{q_1} = F(x_1,\dots,x_n),
\end{aligned} \tag{2}
$$
$$Q_{k,n} = \{(q_1,\dots,q_k) \mid 1 \leqslant q_1 < \dots < q_k \leqslant n\},$$

where $x_i^0 \leqslant x_i \leqslant x_i^1$, $i = \overline{1,n}$, $q_1,\dots,q_k$ are natural numbers, $K_{q_1\dots q_k}$, $(q_1,\dots,q_k) \in Q_{k,n}$, $k = \overline{1,n}$, F are continuous functions in corresponding closed parallelepipeds possesses a unique continuous solution $w(x_1,\dots,x_n)$ in the parallelepiped

$$\Omega = [x_1^0,x_1^1] \times \dots \times [x_n^0,x_n^1].$$

Here we propose a generalization of this result for a wider class of integral equations.

1. Three-dimensional equation. We consider the equation

$$
\begin{aligned}
w(x,y,z) &- \int_{x_0}^{x} K_1(x,y,z,\alpha)w(\alpha,y,z)\,d\alpha - \int_{y_0}^{y} K_2(x,y,z,\beta)w(x,\beta,z)\,d\beta \\
&- \int_{z_0}^{z} K_3(x,y,z,\gamma)w(x,y,\gamma)\,d\gamma - \int_{x_0}^{x}\int_{y_0}^{y_1} K_{12}(x,y,z,\alpha,\beta)w(\alpha,\beta,z)\,d\beta\,d\alpha \\
&- \int_{x_0}^{x}\int_{z_0}^{z_1} K_{13}(x,y,z,\alpha,\gamma)w(\alpha,\gamma,z)\,d\gamma\,d\alpha \\
&- \int_{y_0}^{y}\int_{x_0}^{x_1} K_{21}(x,y,z,\alpha,\beta)w(\alpha,\beta,z)\,d\alpha\,d\beta \\
&- \int_{y_0}^{y}\int_{z_0}^{z_1} K_{23}(x,y,z,\beta,\gamma)w(\beta,\gamma,z)\,d\gamma\,d\beta \\
&- \int_{z_0}^{z}\int_{x_0}^{x_1} K_{31}(x,y,z,\alpha,\gamma)w(\alpha,y,\gamma)\,d\alpha\,d\gamma \\
&- \int_{z_0}^{z}\int_{y_0}^{y_1} K_{32}(x,y,z,\beta,\gamma)w(x,\beta,\gamma)\,d\beta\,d\gamma \\
&- \int_{x_0}^{x}\int_{y_0}^{y_1}\int_{z_0}^{z_1} K_{123}(x,y,z,\alpha,\beta,\gamma)w(\alpha,\beta,\gamma)\,d\gamma\,d\beta\,d\alpha \\
&- \int_{y_0}^{y}\int_{x_0}^{x_1}\int_{z_0}^{z_1} K_{213}(x,y,z,\alpha,\beta,\gamma)w(\alpha,\beta,\gamma)\,d\gamma\,d\alpha\,d\beta \\
&- \int_{z_0}^{z}\int_{x_0}^{x_1}\int_{y_0}^{y_1} K_{312}(x,y,z,\alpha,\beta,\gamma)w(\alpha,\beta,\gamma)\,d\beta\,d\alpha\,d\gamma = F(x,y,z),
\end{aligned}
\tag{3}
$$

where $(x,y,z) \in D = [x_0,x_1] \times [y_0,y_1] \times [z_0,z_1]$, $x_0 < x_1$, $y_0 < y_1$, $z_0 < z_1$, $w = \mathrm{colon}(w^1,\dots,w^m)$, $F = \mathrm{colon}(f^1,\dots,f^m)$, K_ω are matrix-valued function of dimension $m \times m$ and ω is the set of indices. The coefficients and the right hand side in equation (3) are supposed to be continuous in corresponding closed parallelepipeds. We write equation (3) shortly as

$$w - Bw = F. \tag{4}$$

Without loss of generality we can assume that $x_0 = y_0 = z_0 = 0$.

For the matrices $A = (a_{ij})$, where the functions a_{ij} are defined on a bounded closed set D, we shall employ the norm [16]

$$\|A\| = \max_i \sum_j \max_D |a_{ij}|.$$

Suppose that the estimates hold:

$$
\begin{aligned}
&x + y + z < s \quad \text{in} \quad D, \\
&1 + \|K_1\| < M, \quad 1 + \|K_2\| < M, \quad 1 + \|K_3\| < M,
\end{aligned}
$$

$$1+\left\|\int\limits_0^{y_1} K_{12}d\beta\right\| < M, \qquad 1+\left\|\int\limits_0^{z_1} K_{13}d\gamma\right\| < M,$$
$$1+\left\|\int\limits_0^{x_1} K_{21}d\alpha\right\| < M, \qquad 1+\left\|\int\limits_0^{z_1} K_{23}d\gamma\right\| < M,$$
$$1+\left\|\int\limits_0^{x_1} K_{31}d\alpha\right\| < M, \qquad 1+\left\|\int\limits_0^{y_1} K_{32}d\beta\right\| < M,$$
$$1+\left\|\int\limits_0^{y_1}\int\limits_0^{z_1} K_{123}d\gamma d\beta\right\| < M, \qquad 1+\left\|\int\limits_0^{x_1}\int\limits_0^{z_1} K_{213}d\gamma d\alpha\right\| < M,$$
$$1+\left\|\int\limits_0^{x_1}\int\limits_0^{y_1} K_{312}d\beta d\alpha\right\| < M.$$

Let us prove that the operator B is continuous on the set of continuous vector functions defined on D. Let w_1 and w_2 be continuous vector functions defined on the set. It is obvious that

$$\|Bw_1 - Bw_2\| < 12mM(x+y+z)\|w_1 - w_2\| < 12mMs\|w_1 - w_2\|.$$

It is clear that for each $\varepsilon > 0$ there exists $\delta = \varepsilon/(12mMs)$ such that the inequality $\|w_1 - w_2\| < \delta$ implies $\|Bw_1 - Bw_2\| < \varepsilon$. This proves the continuity of the operator B.

Let us show that some power of the operator B is a contracting mapping. We have:

$$\|B^2 w_1 - B^2 w_2\| <(12mM)^2\frac{s^2}{2!}\|w_1 - w_2\|,$$
$$\dots$$
$$\|B^k w_1 - B^k w_2\| <(12mM)^k\frac{s^k}{k!}\|w_1 - w_2\|.$$

It is clear that

$$\frac{(12mMs)^k}{k!} < 1$$

for some k. This is why B^k is a contracting mapping for some k.

If B is a continuous mapping of a complete metric space into itself such that some power of this mapping is contracting, the equation

$$w - Bw = 0$$

is uniquely solvable [17]; in our case this solution is zero. But then linear equation (4) possesses a unique solution in the class of continuous vector functions [18].

2. General case. The above arguing are generalized for a n-dimensional space $(x_1, \dots, x_n)$. We consider the equation

$$\begin{aligned} w(x_1,\dots,x_n) - \sum_{k=1}^{n}\int\limits_{x_k^0}^{x_k}\sum_{l=0}^{n-1}\sum_{Q_{l,n}^k}\int\limits_{x_{q_1}^0}^{x_{q_1}^1}\cdots\int\limits_{x_{q_l}^0}^{x_{q_l}^1} & K_{kq_1\dots q_l}(x_1,\dots,x_n,\alpha_{q_1},\dots,\alpha_{q_l},\alpha_{q_k}) \\ & \cdot w(x_1,\dots,x_n)|_{x_k=\alpha_k}|_{x_{q_1}=\alpha_{q_1}}\cdots|_{x_{q_l}=\alpha_{q_l}} \\ & \cdot d\alpha_{q_l}\dots d\alpha_{q_1}d\alpha_k = F(x_1,\dots,x_n), \end{aligned} \qquad (5)$$

$$Q_{l,n}^k = \{(q_1,\dots,q_l) \mid 1 \leqslant q_1 < \dots < q_l \leqslant n,$$
$$q_i \in \{1,\dots,k-1,k+1,\dots,n\},\ 1 \leqslant i \leqslant l\}.$$

Here $(x_1,\dots,x_n)\in\Omega$ (we recall that the set Ω was defined in the formulation of Theorem 1), $x_1^0<x_1^1,\dots,x_n^0<x_n^1$, $w=\mathrm{colon}(w^1,\dots,w^m)$, $F=\mathrm{colon}(f^1,\dots,f^m)$, K_ω are matrix functions of dimension $m\times m$. The coefficients and right hand side in equation (5) are continuous in corresponding closed domain. In an operator for, equation (5) reads as

$$w-B_nw=F. \tag{6}$$

Without loss of generality we can assume that $x_i^0=0$, $i=\overline{1,n}$.

The norm of a matrix $A=(a_{ij})$, where the functions a_{ij} are defined on Ω, is determined by the relation

$$\|A\|=\max_i\sum_j\max_\Omega|a_{ij}|.$$

Assume that the estimates hold:

$$\begin{aligned}
&\sum_{i=1}^n x_i<s, \qquad (x_1,\dots,x_n)\in\Omega,\\
&1+\|K_j\|<M, \qquad 1\leqslant j\leqslant n,\\
&1+\left\|\int_0^{x_{q_1}^1}\cdots\int_0^{x_{q_l}^1}K_{kq_1\dots q_l}d\alpha_{q_l}\dots d\alpha_{q_1}\right\|<M \qquad \text{for all } k \text{ and } (q_1,\dots,q_l).
\end{aligned}$$

Let us prove that the operator B_n is continuous. Let w_1 and w_2 be continuous vector functions defined on Ω. Then

$$\|B_nw_1-B_nw_2\|<n2^{n-1}mM(x_1+\dots+x_n)\|w_1-w_2\|<n2^{n-1}mMs\|w_1-w_2\|. \tag{7}$$

It is obvious that (7) implies the continuity of the operator B_n.

We have

$$\begin{aligned}
\|B_n^2w_1-B_n^2w_2\|<&(n2^{n-1}mM)^2\frac{s^2}{2!}\|w_1-w_2\|,\\
&\dots\\
\|B_n^kw_1-B_n^kw_2\|<&(n2^{n-1}mM)^k\frac{s^k}{k!}\|w_1-w_2\|,
\end{aligned}$$

It is clear that

$$\frac{(n2^{n-1}mMs)^k}{k!}<1$$

for some k and B_n^k is a contracting mapping for some k. Hence, linear equation (6) possesses the only solution in the class of continuous vector functions and the following theorem holds true.

Theorem 2. *If in equation (6) all coefficients K_ω and the right hand side F are continuous in corresponding closed parallelepipeds, then in the parallelepiped $\Omega=[x_1^0,x_1^1]\times\dots\times[x_n^0,x_n^1]$ there exist a unique continuous solution $w(x_1,\dots,x_n)$ to this equation.*

Systems of Volterra type (2), where w, F are vectors, $K_{q_1\dots q_k}$ are matrices, are a particular case of system (6).

We mention separately that once we deal with an equation or a system of equations of Volterra type, its solution has the same smoothness as the kernel(s) and the right hand(s) in the equation(s) [19].

3. Unique solvability of boundary value problems for systems of hyperbolic equations

We provide an application of Theorem 2.

1. Construction of solution for main characteristic problem for a system in the three-dimensional space by the Riemann method. We consider a system of differential equations:

$$\begin{cases} u_{xx} = a_1(x,y,z)v_x + b_1(x,y,z)w_x + c_1(x,y,z)u \\ \qquad + d_1(x,y,z)v + e_1(x,y,z)w + f_1(x,y,z), \\ v_{yy} = a_2(x,y,z)u_y + b_2(x,y,z)w_y + c_2(x,y,z)u \\ \qquad + d_2(x,y,z)v + e_2(x,y,z)w + f_2(x,y,z), \\ w_{zz} = a_3(x,y,z)u_z + b_3(x,y,z)v_z + c_3(x,y,z)u \\ \qquad + d_3(x,y,z)v + e_3(x,y,z)w + f_3(x,y,z). \end{cases} \tag{8}$$

We assume that in the closure of the considered domain D in the space (x,y,z), we have $a_i,\, b_i \in C^2$, $c_i,\, d_i,\, e_i,\, f_i \in C^1$, $i = \overline{1,3}$. A solution of (8) in the class $u,\, v,\, w \in C^1(D)$, $u_{xx},\, v_{yy},\, w_{zz} \in C(D)$ is called regular in D.

System (8) arises as a reduction of a system with higher derivatives

$$\begin{cases} u^*_{xx} = a^*_1(x,y,z)u^*_x + b^*_1(x,y,z)v^*_x + c^*_1(x,y,z)w^*_x + d^*_1(x,y,z)u^* \\ \qquad + e^*_1(x,y,z)v^* + f^*_1(x,y,z)w^* + g^*_1(x,y,z), \\ v^*_{yy} = a^*_2(x,y,z)u^*_y + b^*_2(x,y,z)v^*_y + c^*_2(x,y,z)w^*_y + d^*_2(x,y,z)u^* \\ \qquad + e^*_2(x,y,z)v^* + f^*_2(x,y,z)w^* + g^*_2(x,y,z), \\ w^*_{zz} = a^*_3(x,y,z)u^*_z + b^*_3(x,y,z)v^*_z + c^*_3(x,y,z)w^*_z + d^*_3(x,y,z)u^* \\ \qquad + e^*_3(x,y,z)v^* + f^*_3(x,y,z)w^* + g^*_3(x,y,z) \end{cases}$$

by the substitutions

$$u^* = \exp\left(\frac{1}{2}\int\limits_{x_0}^{x} a^*_1(\alpha,y,z)\,d\alpha\right)u, \qquad v^* = \exp\left(\frac{1}{2}\int\limits_{y_0}^{y} b^*_2(x,\beta,z)\,d\beta\right)v,$$

$$w^* = \exp\left(\frac{1}{2}\int\limits_{z_0}^{z} c^*_3(x,y,\gamma)\,d\gamma\right)w.$$

Let us formulate a main characteristic problem playing the same role in the theory of system (8) as the Goursat problem [5] does in the theory of the hyperbolic system

$$\begin{cases} u_x = a_1(x,y)u + b_1(x,y)v, \\ v_y = a_2(x,y)u + b_2(x,y)v. \end{cases}$$

Main characteristic problem. Let

$$G = \{x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1\}.$$

We denote by X, Y, Z the sides of G as $x = x_0$, $y = y_0$, $z = z_0$, respectively. We need to find a regular in domain G solution to system (8) satisfying the conditions

$$\begin{aligned} & u(x_0,y,z) = \varphi_1(y,z), \qquad v(x,y_0,z) = \varphi_2(x,z), \\ & w(x,y,z_0) = \varphi_3(x,y), \\ & (u_x - a_1v - b_1w)(x_0,y,z) = \psi_1(y,z), \\ & (v_y - a_2u - b_2w)(x,y_0,z) = \psi_2(x,z), \\ & (w_z - a_3u - b_3v)(x,y,z_0) = \psi_3(x,y), \end{aligned} \tag{9}$$

$\varphi_1,\ \psi_1 \in C^1(\overline{X}),\ \varphi_2,\ \psi_2 \in C^1(\overline{Y}),\ \varphi_3,\ \psi_3 \in C^1(\overline{Z})$.

The main characteristic problem is uniquely solvable. Indeed, we rewrite (8) as

$$\begin{cases} u_x = u_1 + a_1 v + b_1 w, \\ u_{1x} = c_1 u + (d_1 - a_{1x})v + (e_1 - b_{1x})w + f_1, \\ v_y = a_2 u + v_1 + b_2 w, \\ v_{1y} = (c_2 - a_{2y})u + d_2 v + (e_2 - b_{2y})w + f_2, \\ w_z = a_3 u + b_3 v + w_1, \\ w_{1z} = (c_3 - a_{3z})u + (d_3 - b_{3z})v + e_3 w + f_3. \end{cases} \tag{10}$$

We denote

$$\begin{aligned} &d_{10} = d_1 - a_{1x}, && e_{10} = e_1 - b_{1x}, && c_{20} = c_2 - a_{2y}, \\ &e_{20} = e_2 - b_{2y}, && c_{30} = c_3 - a_{3z}, && d_{30} = d_3 - b_{3z}. \end{aligned}$$

System (8) with conditions (9) is reduced to the system of integral equations

$$\begin{cases} u(x,y,z) = \varphi_1(y,z) + \int\limits_{x_0}^{x} (u_1 + a_1 v + b_1 w)(\alpha, y, z)d\alpha, \\ u_1(x,y,z) = \psi_1(y,z) + \int\limits_{x_0}^{x} (c_1 u + d_{10} v + e_{10} w + f_1)(\alpha, y, z)d\alpha, \\ v(x,y,z) = \varphi_2(x,z) + \int\limits_{y_0}^{y} (v_1 + a_2 u + b_2 w)(x, \beta, z)d\beta, \\ v_1(x,y,z) = \psi_2(x,z) + \int\limits_{y_0}^{y} (c_{20} u + d_2 v + e_{20} w + f_2)(x, \beta, z)d\beta, \\ w(x,y,z) = \varphi_3(x,y) + \int\limits_{z_0}^{z} (w_1 + a_3 u + b_3 v)(x, y, \gamma)d\gamma, \\ w_1(x,y,z) = \psi_3(x,y) + \int\limits_{z_0}^{z} (c_{30} u + d_{30} v + e_3 w + f_3)(x, y, \gamma)d\gamma. \end{cases} \tag{11}$$

It is obvious that solution to (11) exists and unique in the class of continuous functions.

It is clear that (11) is equivalent to main characteristic problem (8), (9). Hence, the following theorem holds.

Theorem 3. *If in the closure of the domain G we have $a_i,\ b_i \in C^2,\ c_i,\ d_i,\ e_i,\ f_i \in C^1$, $i = \overline{1,3}$, then main characteristic problem (8), (9) is uniquely solvable.*

We are going to construct the solution to the main characteristic problem in terms of the Riemann problem. We rewrite (10) in a matrix form:

$$\begin{aligned} &L(\mathbf{U}) = \mathbf{F}, \qquad L(\mathbf{U}) \equiv \mathbf{A}_1 \mathbf{U}_x + \mathbf{A}_2 \mathbf{U}_y + \mathbf{A}_3 \mathbf{U}_z - \mathbf{B}\mathbf{U}, \\ &\mathbf{U} = \mathrm{colon}(u, u_1, v, v_1, w, w_1), \end{aligned} \tag{12}$$

$$\mathbf{A}_1 = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$\mathbf{A}_3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0&1&a_1&0&b_1&0\\ c_1&0&d_{10}&0&e_{10}&0\\ a_2&0&0&1&b_2&0\\ c_{20}&0&d_2&0&e_{20}&0\\ a_3&0&b_3&0&0&1\\ c_{30}&0&d_{30}&0&e_3&0 \end{pmatrix},$$

$$\mathbf{F} = \mathrm{colon}(0, f_1, 0, f_2, 0, f_3).$$

We introduce the Riemann matrix $\mathbf{R} = \mathrm{colon}(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4, \mathbf{R}_5, \mathbf{R}_6)$, where

$$\mathbf{R}_i(x, y, z, \xi, \eta, \zeta) = (r_{i1}, r_{i2}, r_{i3}, r_{i4}, r_{i5}, r_{i6}), \qquad i = \overline{1, 6},$$

are solutions to the systems

$$\begin{cases} r_{i1}(x,y,z) = \delta_{i1} - \int\limits_{\xi}^{x} (c_1 r_{i2} + a_2 r_{i3} + c_{20} r_{i4} + a_3 r_{i5} + c_{30} r_{i6})(\alpha, y, z)\,d\alpha, \\ r_{i2}(x,y,z) = \delta_{i2} - \int\limits_{\xi}^{x} r_{i1}(\alpha, y, z)\,d\alpha, \\ r_{i3}(x,y,z) = \delta_{i3} - \int\limits_{\eta}^{y} (a_1 r_{i1} + d_{10} r_{i2} + d_2 r_{i4} + b_3 r_{i5} + d_{30} r_{i6})(x, \beta, z)\,d\beta, \\ r_{i4}(x,y,z) = \delta_{i4} - \int\limits_{\eta}^{y} r_{i3}(x, \beta, z)\,d\beta, \\ r_{i5}(x,y,z) = \delta_{i5} - \int\limits_{\zeta}^{z} (b_1 r_{i1} + e_{10} r_{i2} + b_2 r_{i3} + e_{20} r_{i4} + e_3 r_{i6})(x, y, \gamma)\,d\gamma, \\ r_{i6}(x,y,z) = \delta_{i6} - \int\limits_{\zeta}^{z} r_{i5}(x, y, \gamma)\,d\gamma, \end{cases} \tag{13}$$

δ_{ij} is the Kronecker delta. For each i, systems (13) are uniquely solvable in the class of continuous functions. With respect to the first triple of independent variables (x, y, z), the matrix $\mathbf{R}$ solves the adjoint to (12) system

$$L^*(\mathbf{V}) = 0, \qquad L^*(\mathbf{V}) \equiv -(\mathbf{V}\mathbf{A}_1)_x - (\mathbf{V}\mathbf{A}_2)_y - (\mathbf{V}\mathbf{A}_3)_z - \mathbf{V}\mathbf{B}.$$

The identity holds:

$$\mathbf{R}L(\mathbf{U}) = (\mathbf{R}\mathbf{A}_1\mathbf{U})_x + (\mathbf{R}\mathbf{A}_2\mathbf{U})_y + (\mathbf{R}\mathbf{A}_3\mathbf{U})_z, \tag{14}$$

which can be checked straightforwardly.

Let us find $\mathbf{U}(\xi, \eta, \zeta)$ as $(\xi, \eta, \zeta) \in G$. The first line in (14) gives

$$r_{12} f_1 + r_{14} f_2 + r_{16} f_3 = (r_{11} u + r_{12} u_1)_x + (r_{13} v + r_{14} v_1)_y + (r_{15} w + r_{16} w_1)_z, \tag{15}$$

where $r_{1j} = r_{1j}(x, y, z, \xi, \eta, \zeta)$, other functions depend on (x, y, z). We integrate (15) over the domain $G_1 = \{x_0 < x < \xi,\ y_0 < y < \eta,\ z_0 < z < \zeta\}$:

$$\begin{aligned}
&\int_{y_0}^{\eta}\int_{z_0}^{\zeta}(r_{11}u + r_{12}u_1)(\xi, \beta, \gamma, \xi, \eta, \zeta)d\gamma\, d\beta - -\int_{y_0}^{\eta}\int_{z_0}^{\zeta}(r_{11}u + r_{12}u_1)(x_0, \beta, \gamma, \xi, \eta, \zeta)d\gamma\, d\beta \\
&+\int_{x_0}^{\xi}\int_{z_0}^{\zeta}(r_{13}v + r_{14}v_1)(\alpha, \eta, \gamma, \xi, \eta, \zeta)d\gamma\, d\alpha - \int_{x_0}^{\xi}\int_{z_0}^{\zeta}(r_{13}v + r_{14}v_1)(\alpha, y_0, \gamma, \xi, \eta, \zeta)d\gamma\, d\alpha \\
&+\int_{x_0}^{\xi}\int_{y_0}^{\eta}(r_{15}w + r_{16}w_1)(\alpha, \beta, \zeta, \xi, \eta, \zeta)d\beta\, d\alpha - \int_{x_0}^{\xi}\int_{y_0}^{\eta}(r_{15}w + r_{16}w_1)(\alpha, \beta, z_0, \xi, \eta, \zeta)d\beta\, d\alpha \\
&= \iiint\limits_{G_1}(r_{12}f_1 + r_{14}f_2 + r_{16}f_3)(\alpha, \beta, \gamma, \xi, \eta, \zeta)d\alpha\, d\beta\, d\gamma.
\end{aligned} \tag{16}$$

By (13)

$$\begin{aligned}
r_{11}(\xi, \beta, \gamma, \xi, \eta, \zeta) \equiv& 1, \\
r_{12}(\xi, \beta, \gamma, \xi, \eta, \zeta) =& r_{13}(\alpha, \eta, \gamma, \xi, \eta, \zeta) = r_{14}(\alpha, \eta, \gamma, \xi, \eta, \zeta) \\
=& r_{15}(\alpha, \beta, \zeta, \xi, \eta, \zeta) = r_{16}(\alpha, \beta, \zeta, \xi, \eta, \zeta) \equiv 0.
\end{aligned}$$

Thus, (16) casts into the form

$$\int_{y_0}^{\eta}\int_{z_0}^{\zeta} u(\xi, \beta, \gamma)d\gamma\, d\beta = g_1(\xi, \eta, \zeta)$$

with the function $g_1(\xi, \eta, \zeta)$ expressed via data in (9) and the entries of the matrix $\mathbf{R}$. This implies:

$$u(\xi, \eta, \zeta) = \frac{\partial^2 g_1(\xi, \eta, \zeta)}{\partial\eta\partial\zeta}.$$

In the same way, the third and fifth lines in (14) give rise to formulae obtained from (16) via replacing r_{1j} by r_{3j} and r_{5j}, respectively. Employing (13), we obtain

$$v(\xi, \eta, \zeta) = \frac{\partial^2 g_2(\xi, \eta, \zeta)}{\partial\xi\partial\zeta}, \qquad w(\xi, \eta, \zeta) = \frac{\partial^2 g_3(\xi, \eta, \zeta)}{\partial\xi\partial\eta},$$

where $g_2(\xi, \eta, \zeta)$ and $g_3(\xi, \eta, \zeta)$ are expressed via conditions (9) and the entries of the matrix $\mathbf{R}$.

2. Existence and uniqueness of solution to problem with conditins on five sides of the characteristic rectangle. Various problem for a system with multiple characteristics with two independent variables were considered in [14]. In the parallelepiped $G = \{x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1\}$ we consider one problem with boundary conditions on five sides X, Y, Z, $X_1 = \{(x, y, z)|x = x_1, y_0 < y < y_1, z_0 < z < z_1\}$, $Y_1 = \{(x, y, z)|y = y_1, x_0 < x < x_1, z_0 < z < z_1\}$. We note that in distinction to the below considered problem, to obtain the solvability conditions for the problem in [14], Theorem 2 is not needed.

Problem 1. Find a regular in G solution to (8) satisfying the conditions

$$\begin{aligned}
u(x_0, y, z) &= \varphi_1(y, z), \qquad (u_x - a_1v - b_1w)(x_1, y, z) = \chi_1(y, z), \\
v(x, y_0, z) &= \varphi_2(x, z), \qquad (v_y - a_2u - b_2w)(x, y_1, z) = \chi_2(x, z), \\
w(x, y, z_0) &= \varphi_3(x, y), \qquad (w_z - a_3u - b_3v)(x, y, z_0) = \psi_3(x, y),
\end{aligned} \tag{17}$$

$\varphi_1(y,z) \in C^1(\overline{X})$, $\chi_1(y,z) \in C^1(\overline{X}_1)$, $\varphi_2(x,z) \in C^1(\overline{Y})$, $\chi_2(x,z) \in C^1(\overline{Y}_1)$, $\varphi_3(x,y)$, $\psi_3(x,y) \in C^1(\overline{Z})$.

We shall need formulae for u, u_1, v, v_1, w, w_1, obtained while solving the main characteristic problem by the Riemann method; hereafter the differentiation of the entries in the Riemann matrix in ξ, η, ζ means the differentiation with respect to the second triple of the independent variables:

$$
\begin{aligned}
u(x,y,z) &= r_{11}(x_0,y,z,x,y,z)u(x_0,y,z) + r_{12}(x_0,y,z,x,y,z)u_1(x_0,y,z) \\
&+ \int\limits_{y_0}^{y} (r_{11\eta}(x_0,\beta,z,x,y,z)u(x_0,\beta,z) + r_{12\eta}(x_0,\beta,z,x,y,z)u_1(x_0,\beta,z))d\beta \\
&+ \int\limits_{z_0}^{z} (r_{11\zeta}(x_0,y,\gamma,x,y,z)u(x_0,y,\gamma) + r_{12\zeta}(x_0,y,\gamma,x,y,z)u_1(x_0,y,\gamma))d\gamma \\
&+ \int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{11\eta\zeta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma) + r_{12\eta\zeta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta \\
&+ \int\limits_{x_0}^{x} (r_{13\eta}(\alpha,y_0,z,x,y,z)v(\alpha,y_0,z) + r_{14\eta}(\alpha,y_0,z,x,y,z)v_1(\alpha,y_0,z))d\alpha \\
&+ \int\limits_{x_0}^{x}\int\limits_{z_0}^{z} (r_{13\eta\zeta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma) + r_{14\eta\zeta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,z))d\gamma d\alpha \\
&+ \int\limits_{x_0}^{x} (r_{15\zeta}(\alpha,y,z_0,x,y,z)w(\alpha,y,z_0) + r_{16\zeta}(\alpha,y,z_0,x,y,z)w_1(\alpha,y,z_0))d\alpha \\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} (r_{15\eta\zeta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0) + r_{16\eta\zeta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha \\
&+ \int\limits_{x_0}^{x} r_{12}(\alpha,y,z,x,y,z)f_1(\alpha,y,z)d\alpha \\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} (r_{12\eta}(\alpha,\beta,z,x,y,z)f_1(\alpha,\beta,z) + r_{14\eta}(\alpha,\beta,z,x,y,z)f_2(\alpha,\beta,z) \\
&\qquad + r_{16\eta}(\alpha,\beta,z,x,y,z)f_3(\alpha,\beta,z))d\beta d\alpha \\
&+ \int\limits_{x_0}^{x}\int\limits_{z_0}^{z} (r_{12\zeta}(\alpha,y,\gamma,x,y,z)f_1(\alpha,y,\gamma) + r_{14\zeta}(\alpha,y,\gamma,x,y,z)f_2(\alpha,y,\gamma) \\
&\qquad + r_{16\zeta}(\alpha,y,\gamma,x,y,z)f_3(\alpha,y,\gamma))d\gamma d\alpha \\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{12\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{14\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma) \\
&\qquad + r_{16\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha,
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
u_1(x,y,z) =& r_{21}(x_0,y,z,x,y,z)u(x_0,y,z) + r_{22}(x_0,y,z,x,y,z)u_1(x_0,y,z) \\
&+ \int_{y_0}^{y} (r_{21\eta}(x_0,\beta,z,x,y,z)u(x_0,\beta,z) + r_{22\eta}(x_0,\beta,z,x,y,z)u_1(x_0,\beta,z))d\beta \\
&+ \int_{z_0}^{z} (r_{21\zeta}(x_0,y,\gamma,x,y,z)u(x_0,y,\gamma) \\
&\quad + r_{22\zeta}(x_0,y,\gamma,x,y,z)u_1(x_0,y,\gamma))d\gamma \\
&+ \int_{y_0}^{y}\int_{z_0}^{z} (r_{21\eta\zeta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma) \\
&\quad + r_{22\eta\zeta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta \\
&+ \int_{x_0}^{x} (r_{23\eta}(\alpha,y_0,z,x,y,z)v(\alpha,y_0,z) + r_{24\eta}(\alpha,y_0,z,x,y,z)v_1(\alpha,y_0,z))d\alpha \\
&+ \int_{x_0}^{x}\int_{z_0}^{z} (r_{23\eta\zeta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma) \\
&\quad + r_{24\eta\zeta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,z))d\gamma d\alpha \\
&+ \int_{x_0}^{x} (r_{25\zeta}(\alpha,y,z_0,x,y,z)w(\alpha,y,z_0) + r_{26\zeta}(\alpha,y,z_0,x,y,z)w_1(\alpha,y,z_0))d\alpha \qquad (19) \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} (r_{25\eta\zeta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0) \\
&\quad + r_{26\eta\zeta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha \\
&+ \int_{x_0}^{x} r_{22}(\alpha,y,z,x,y,z)f_1(\alpha,y,z)d\alpha \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} (r_{22\eta}(\alpha,\beta,z,x,y,z)f_1(\alpha,\beta,z) + r_{24\eta}(\alpha,\beta,z,x,y,z)f_2(\alpha,\beta,z) \\
&\quad + r_{26\eta}(\alpha,\beta,z,x,y,z)f_3(\alpha,\beta,z))d\beta d\alpha + \\
&+ \int_{x_0}^{x}\int_{z_0}^{z} (r_{22\zeta}(\alpha,y,\gamma,x,y,z)f_1(\alpha,y,\gamma) + r_{24\zeta}(\alpha,y,\gamma,x,y,z)f_2(\alpha,y,\gamma) \\
&\quad + r_{26\zeta}(\alpha,y,\gamma,x,y,z)f_3(\alpha,y,\gamma))d\gamma d\alpha \\
&+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} (r_{22\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{24\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma) \\
&\quad + r_{26\eta\zeta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha,
\end{aligned}
$$

$$\begin{aligned}
v(x,y,z) =& r_{33}(x,y_0,z,x,y,z)v(x,y_0,z) + r_{34}(x,y_0,z,x,y,z)v_1(x,y_0,z)\\
&+ \int\limits_{x_0}^{x} (r_{33\xi}(\alpha,y_0,z,x,y,z)v(\alpha,y_0,z) + r_{34\xi}(\alpha,y_0,z,x,y,z)v_1(\alpha,y_0,z))d\alpha\\
&+ \int\limits_{z_0}^{z} (r_{33\zeta}(x,y_0,\gamma,x,y,z)v(x,y_0,\gamma) + r_{34\zeta}(x,y_0,\gamma,x,y,z)v_1(x,y_0,\gamma))d\gamma\\
&+ \int\limits_{x_0}^{x}\int\limits_{z_0}^{z} (r_{33\xi\zeta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma)\\
&\qquad + r_{34\xi\zeta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,\gamma))d\gamma d\alpha\\
&+ \int\limits_{y_0}^{y} (r_{31\xi}(x_0,\beta,z,x,y,z)u(x_0,\beta,z)\\
&\qquad + r_{32\xi}(x_0,\beta,z,x,y,z)u_1(x_0,\beta,z))d\beta\\
&+ \int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{31\xi\zeta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma)\\
&\qquad + r_{32\xi\zeta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta\\
&+ \int\limits_{y_0}^{y} (r_{35\zeta}(x,\beta,z_0,x,y,z)w(x,\beta,z_0)\\
&\qquad + r_{36\zeta}(x,\beta,z_0,x,y,z)w_1(x,\beta,z_0))d\beta\\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} (r_{35\xi\zeta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0)\\
&\qquad + r_{36\xi\zeta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha\\
&+ \int\limits_{y_0}^{y} r_{34}(x,\beta,z,x,y,z)f_2(x,\beta,z)d\beta\\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} (r_{32\xi}(\alpha,\beta,z,x,y,z)f_1(\alpha,\beta,z) + r_{34\xi}(\alpha,\beta,z,x,y,z)f_2(\alpha,\beta,z)\\
&+ r_{36\xi}(\alpha,\beta,z,x,y,z)f_3(\alpha,\beta,z))d\beta d\alpha\\
&+ \int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{32\zeta}(x,\beta,\gamma,x,y,z)f_1(x,\beta,\gamma) + r_{34\zeta}(x,\beta,\gamma,x,y,z)f_2(x,\beta,\gamma)\\
&\qquad + r_{36\zeta}(x,\beta,\gamma,x,y,z)f_3(x,\beta,\gamma))d\gamma d\beta\\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{32\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{34\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma)\\
&\qquad + r_{36\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha,
\end{aligned} \tag{20}$$

$$
\begin{aligned}
v_1(x,y,z) =& r_{43}(x,y_0,z,x,y,z)v(x,y_0,z) + r_{44}(x,y_0,z,x,y,z)v_1(x,y_0,z) \\
&+ \int_{x_0}^{x} (r_{43\xi}(\alpha,y_0,z,x,y,z)v(\alpha,y_0,z) + r_{44\xi}(\alpha,y_0,z,x,y,z)v_1(\alpha,y_0,z))d\alpha \\
&+ \int_{z_0}^{z} (r_{43\zeta}(x,y_0,\gamma,x,y,z)v(x,y_0,\gamma) + r_{44\zeta}(x,y_0,\gamma,x,y,z)v_1(x,y_0,\gamma))d\gamma \\
&+ \int_{x_0}^{x}\int_{z_0}^{z} (r_{43\xi\zeta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma) \\
&\qquad + r_{44\xi\zeta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,\gamma))d\gamma d\alpha \\
&+ \int_{y_0}^{y} (r_{41\xi}(x_0,\beta,z,x,y,z)u(x_0,\beta,z) \\
&\qquad + r_{42\xi}(x_0,\beta,z,x,y,z)u_1(x_0,\beta,z))d\beta \\
&+ \int_{y_0}^{y}\int_{z_0}^{z} (r_{41\xi\zeta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma) \\
&\qquad + r_{42\xi\zeta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta \\
&+ \int_{y_0}^{y} (r_{45\zeta}(x,\beta,z_0,x,y,z)w(x,\beta,z_0) + r_{46\zeta}(x,\beta,z_0,x,y,z)w_1(x,\beta,z_0))d\beta \qquad (21) \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} (r_{45\xi\zeta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0) \\
&\qquad + r_{46\xi\zeta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha \\
&+ \int_{y_0}^{y} r_{44}(x,\beta,z,x,y,z)f_2(x,\beta,z)d\beta \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} (r_{42\xi}(\alpha,\beta,z,x,y,z)f_1(\alpha,\beta,z) + r_{44\xi}(\alpha,\beta,z,x,y,z)f_2(\alpha,\beta,z) \\
&\qquad + r_{46\xi}(\alpha,\beta,z,x,y,z)f_3(\alpha,\beta,z))d\beta d\alpha \\
&+ \int_{y_0}^{y}\int_{z_0}^{z} (r_{42\zeta}(x,\beta,\gamma,x,y,z)f_1(x,\beta,\gamma) + r_{44\zeta}(x,\beta,\gamma,x,y,z)f_2(x,\beta,\gamma) \\
&\qquad + r_{46\zeta}(x,\beta,\gamma,x,y,z)f_3(x,,\beta,\gamma))d\gamma d\beta \\
&+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} (r_{42\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{44\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma) \\
&\qquad + r_{46\xi\zeta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha,
\end{aligned}
$$

$$
\begin{aligned}
w(x,y,z) =& r_{55}(x,y,z_0,x,y,z)w(x,y,z_0) + r_{56}(x,y,z_0,x,y,z)w_1(x,y,z_0) \\
&+ \int\limits_{x_0}^{x} (r_{55\xi}(\alpha,y,z_0,x,y,z)w(\alpha,y,z_0) + r_{56\xi}(\alpha,y,z_0,x,y,z)w_1(\alpha,y,z_0))d\alpha \\
&+ \int\limits_{y_0}^{y} (r_{55\eta}(x,\beta,z_0,x,y,z)w(x,\beta,z_0) + r_{56\eta}(x,\beta,z_0,x,y,z)w_1(x,\beta,z_0))d\beta \\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} (r_{55\xi\eta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0) \\
&\qquad + r_{56\xi\eta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha \\
&+ \int\limits_{z_0}^{z} (r_{51\xi}(x_0,y,\gamma,x,y,z)u(x_0,y,\gamma) + r_{52\xi}(x_0,y,\gamma,x,y,z)u_1(x_0,y,\gamma))d\gamma \\
&+ \int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{51\xi\eta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma) \\
&\qquad + r_{52\xi\eta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta \\
&+ \int\limits_{z_0}^{z} (r_{53\eta}(x,y_0,\gamma,x,y,z)v(x,y_0,\gamma) + r_{54\eta}(x,y_0,\gamma,x,y,z)v_1(x,y_0,\gamma))d\gamma \\
&+ \int\limits_{x_0}^{x}\int\limits_{z_0}^{z} (r_{53\xi\eta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma) \\
&\qquad + r_{54\xi\eta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,\gamma))d\gamma d\alpha \\
&+ \int\limits_{z_0}^{z} r_{56}(x,y,\gamma,x,y,z)f_3(x,y,\gamma)d\gamma \\
&+ \int\limits_{x_0}^{x}\int\limits_{z_0}^{z} (r_{52\xi}(\alpha,y,\gamma,x,y,z)f_1(\alpha,y,\gamma) + r_{54\xi}(\alpha,y,\gamma,x,y,z)f_2(\alpha,y,\gamma) \\
&\qquad + r_{56\xi}(\alpha,y,\gamma,x,y,z)f_3(\alpha,y,\gamma))d\gamma d\alpha \\
&+ \int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{52\eta}(x,\beta,\gamma,x,y,z)f_1(x,\beta,\gamma) + r_{54\eta}(x,\beta,\gamma,x,y,z)f_2(x,\beta,\gamma) \\
&\qquad + r_{56\eta}(x,\beta,\gamma,x,y,z)f_3(x,,\beta,\gamma))d\gamma d\beta \\
&+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y}\int\limits_{z_0}^{z} (r_{52\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{54\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma) \\
&\qquad + r_{56\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha;
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
w_1(x,y,z) =& r_{65}(x,y,z_0,x,y,z)w(x,y,z_0) + r_{66}(x,y,z_0,x,y,z)w_1(x,y,z_0) \\
&+ \int_{x_0}^{x} (r_{65\xi}(\alpha,y,z_0,x,y,z)w(\alpha,y,z_0) + r_{66\xi}(\alpha,y,z_0,x,y,z)w_1(\alpha,y,z_0))d\alpha \\
&+ \int_{y_0}^{y} (r_{65\eta}(x,\beta,z_0,x,y,z)w(x,\beta,z_0) \\
&\quad + r_{66\eta}(x,\beta,z_0,x,y,z)w_1(x,\beta,z_0))d\beta \\
&+ \int_{x_0}^{x}\int_{y_0}^{y} (r_{65\xi\eta}(\alpha,\beta,z_0,x,y,z)w(\alpha,\beta,z_0) \\
&\quad\quad + r_{66\xi\eta}(\alpha,\beta,z_0,x,y,z)w_1(\alpha,\beta,z_0))d\beta d\alpha \\
&+ \int_{z_0}^{z} (r_{61\xi}(x_0,y,\gamma,x,y,z)u(x_0,y,\gamma) + r_{62\xi}(x_0,y,\gamma,x,y,z)u_1(x_0,y,\gamma))d\gamma \\
&+ \int_{y_0}^{y}\int_{z_0}^{z} (r_{61\xi\eta}(x_0,\beta,\gamma,x,y,z)u(x_0,\beta,\gamma) \\
&\quad\quad + r_{62\xi\eta}(x_0,\beta,\gamma,x,y,z)u_1(x_0,\beta,\gamma))d\gamma d\beta \\
&+ \int_{z_0}^{z} (r_{63\eta}(x,y_0,\gamma,x,y,z)v(x,y_0,\gamma) + r_{64\eta}(x,y_0,\gamma,x,y,z)v_1(x,y_0,\gamma))d\gamma \qquad (23) \\
&+ \int_{x_0}^{x}\int_{z_0}^{z} (r_{63\xi\eta}(\alpha,y_0,\gamma,x,y,z)v(\alpha,y_0,\gamma) \\
&\quad\quad + r_{64\xi\eta}(\alpha,y_0,\gamma,x,y,z)v_1(\alpha,y_0,\gamma))d\gamma d\alpha \\
&+ \int_{z_0}^{z} r_{66}(x,y,\gamma,x,y,z)f_3(x,y,\gamma)d\gamma \\
&+ \int_{x_0}^{x}\int_{z_0}^{z} (r_{62\xi}(\alpha,y,\gamma,x,y,z)f_1(\alpha,y,\gamma) + r_{64\xi}(\alpha,y,\gamma,x,y,z)f_2(\alpha,y,\gamma) \\
&\quad\quad + r_{66\xi}(\alpha,y,\gamma,x,y,z)f_3(\alpha,y,\gamma))d\gamma d\alpha \\
&+ \int_{y_0}^{y}\int_{z_0}^{z} (r_{62\eta}(x,\beta,\gamma,x,y,z)f_1(x,\beta,\gamma) + r_{64\eta}(x,\beta,\gamma,x,y,z)f_2(x,\beta,\gamma) \\
&\quad\quad + r_{66\eta}(x,\beta,\gamma,x,y,z)f_3(x,,\beta,\gamma))d\gamma d\beta \\
&+ \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} (r_{62\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_1(\alpha,\beta,\gamma) + r_{64\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_2(\alpha,\beta,\gamma) \\
&\quad\quad\quad + r_{66\xi\eta}(\alpha,\beta,\gamma,x,y,z)f_3(\alpha,\beta,\gamma))d\gamma d\beta d\alpha.
\end{aligned}
$$

While writing these formulae, we have taken into account equations (13), $i=\overline{1,6}$, for the entries of the Riemann matrix.

In order to reduce Problem 1 to the main characteristic problem, we need to determine missing data

$$u_1(x_0,y,z)=(u_x-a_1v-b_1w)(x_0,y,z), \qquad v_1(x,y_0,z)=(v_y-a_2u-b_2w)(x,y_0,z).$$

In order to do this, we let $x=x_1$ in (19), while in (21) we let $y=y_1$. We obtain a system of two integral equations

$$\begin{aligned}
&r_{22}(x_0,y,z,x_1,y,z)u_1(x_0,y,z)+\int\limits_{y_0}^{y} r_{22\eta}(x_0,\beta,z,x_1,y,z)u_1(x_0,\beta,z)d\beta\\
&+\int\limits_{z_0}^{z} r_{22\zeta}(x_0,y,\gamma,x_1,y,z)u_1(x_0,y,\gamma)d\gamma+\int\limits_{y_0}^{y}\int\limits_{z_0}^{z} r_{22\eta\zeta}(x_0,\beta,\gamma,x_1,y,z)u_1(x_0,\beta,\gamma)d\gamma d\beta\\
&+\int\limits_{x_0}^{x_1} r_{24\eta}(\alpha,y_0,z,x_1,y,z)v_1(\alpha,y_0,z)d\alpha+\int\limits_{x_0}^{x_1}\int\limits_{z_0}^{z} r_{24\eta\zeta}(\alpha,y_0,\gamma,x_1,y,z)v_1(\alpha,y_0,\gamma)d\gamma d\alpha\\
&=F_{11}(y,z)+\chi_1(y,z),
\end{aligned} \tag{24}$$

$$\begin{aligned}
&r_{44}(x,y_0,z,x,y_1,z)v_1(x,y_0,z)+\int\limits_{x_0}^{x} r_{44\xi}(\alpha,y_0,z,x,y_1,z)v_1(\alpha,y_0,z)d\alpha\\
&+\int\limits_{z_0}^{z} r_{44\zeta}(x,y_0,\gamma,x,y_1,z)v_1(x,y_0,\gamma)d\gamma+\int\limits_{x_0}^{x}\int\limits_{z_0}^{z} r_{44\xi\zeta}(\alpha,y_0,\gamma,x,y_1,z)v_1(\alpha,y_0,\gamma)d\gamma d\alpha\\
&+\int\limits_{y_0}^{y_1} r_{42\xi}(x_0,\beta,z,x,y_1,z)u_1(x_0,\beta,z)d\beta+\int\limits_{y_0}^{y_1}\int\limits_{z_0}^{z} r_{42\xi\zeta}(x_0,\beta,\gamma,x,y_1,z)u_1(x_0,\beta,\gamma)d\gamma d\beta\\
&=F_{12}(x,z)+\chi_2(x,z),
\end{aligned} \tag{25}$$

the functions F_{11}, F_{12} are known.

Let

$$a_1(x,y,z)=d_{10}(x,y,z)=d_2(x,y,z)\equiv 0, \qquad c_1\geqslant 0. \tag{26}$$

Then $r_{24\eta}(x,y,z,\xi,\eta,z)\equiv 0$, $r_{22}(x_0,y,z,x_1,y,z)\neq 0$. By equation (24), we determine uniquely $u_1(x_0,y,z)$, see Theorem 2. Substituting the found $u_1(x_0,y,z)$ into equation (25), we obtain a second kind Volterra equation for $v_1(x,y_0,z)$; as $d_2(x,y,z)\equiv 0$, $r_{44}(x,y_0,z,x,y_1,z)\neq 0$. This equation is uniquely solvable. Thus, Problem 1 is reduced to the main characteristic problem.

In the same way we resolve system (24)–(25) under the assumptions

$$c_1(x,y,z)=a_2(x,y,z)=c_{20}(x,y,z)\equiv 0, \quad d_2\geqslant 0. \tag{27}$$

At that, first by equation (25) we find $v_1(x,y_0,z)$, and then we find $u_1(x_0,y,z)$ by (24).

Theorem 4. *If a_1, b_1, a_2, b_2, a_3, $b_3\in C^2(\overline{G})$, c_1, c_{20}, c_{30}, d_{10}, d_2, d_{30}, e_{10}, e_{20}, e_3, f_1, f_2, $f_3\in C^1(\overline{G})$, and one of conditions (26), (27) holds, then Problem 1 is uniquely solvable.*

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