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CLASSIFICATION OF A SUBCLASS OF QUASILINEAR TWO-DIMENSIONAL LATTICES BY MEANS OF CHARACTERISTIC ALGEBRAS

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Abstract. We consider a classification problem of integrable cases of the Toda type two-dimensional lattices $u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y})$. The function $f = f(x_1, x_2, \cdots x_5)$ is assumed to be analytic in a domain $D \subset \mathbb{C}^5$. The sought function $u_n = u_n(x, y)$ depends on real x, y and integer n. Equations with three independent variables are complicated objects for study and especially for classification. It is commonly accepted that for a given equation, the existence of a large class of integrable reductions indicates integrability. Our classification algorithm is based on this observation. We say that a constraint $u_0 = \varphi(x, y)$ defines a degenerate cutting off condition for the lattice if it divides this lattice into two independent semi-infinite lattices defined on the intervals $-\infty < n < 0$ and $0 < n < +\infty$, respectively. We call a lattice integrable if there exist cutting off boundary conditions allowing us to reduce the lattice to an infinite number of hyperbolic type systems integrable in the sense of Darboux. Namely, we require that lattice is reduced to a finite system of such kind by imposing degenerate cutting off conditions at two different points $n = N_1$, $n = N_2$ for arbitrary pair of integers N_1, N_2. Recall that a system of hyperbolic equations is called Darboux integrable if it admits a complete set of integrals in both characteristic directions. An effective criterion of the Darboux integrability of the system is connected with properties of an associated algebraic structures. More precisely, the characteristic Lie-Rinehart algebras assigned to both characteristic directions have to be of a finite dimension. Since the obtained hyperbolic system is of a very specific form, the characteristic algebras are effectively studied. Here we focus on a subclass of quasilinear lattices of the form

$$u_{n,xy} = p(u_{n-1}, u_n, u_{n+1})u_{n,x} + r(u_{n-1}, u_n, u_{n+1})u_{n,y} + q(u_{n-1}, u_n, u_{n+1}).$$

Keywords: two-dimensional lattice, integrable reduction, characteristic Lie algebra, degenerate cutting off condition, Darboux integrable system, x-integral.

Mathematics Subject Classification: 37K10, 37K30, 37D99

1. Introduction

The problem of classifying integrable two-dimensional chains of the form

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \qquad -\infty < n < \infty, \tag{1.1}$$

is topical and currently remains open. The function $f = f(x_1, x_2, \cdots x_5)$ is assumed to be analytic in a domain $D \subset \mathbb{C}^5$, and the sought function $u_n = u_n(x, y)$ depends on real x, y and integer n.

In this paper we focus on the following subclass of quasilinear lattices (1.1):

$$u_{n,xy} = p(u_{n+1}, u_n, u_{n-1})u_{n,x} + r(u_{n+1}, u_n, u_{n-1})u_{n,y} + q(u_{n+1}, u_n, u_{n-1}), \qquad -\infty < n < \infty. \tag{1.2}$$

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Here functions $p(x_1, x_2, x_3)$, $r(x_1, x_2, x_3)$, $q(x_1, x_2, x_3)$ are assumed to be analytic in a domain $D \subset \mathbb{C}^3$.

Equations with three independent variables are complicated objects for study and especially for classification. Currently, there are different approaches to studying integrable multidimensional equations [1]–[10]. The presence of a wide class of integrable reductions indicates the integrability of the equation. This fact is often used in the study of multidimensional equations, see [1, 2, 3], where the existence of integrable reductions of a hydrodynamic type is taken to determine the integrability. Here we use a similar idea by treating integrability as the presence of an infinite sequence of Darboux integrable hyperbolic systems.

In describing Darboux integrable systems of hyperbolic equations of a special type, the concept of the characteristic Lie algebra [11, 12, 13] was used a lot. The transition to a more general characteristic Lie-Rinehart algebra opens up new possibilities [14]–[18].

The characteristic Lie algebra for two-dimensional lattices was introduced in [19]. Namely, the structure of this algebra was described for two-dimensional Toda lattice. It was observed in paper [16] that any integrable lattice of the form (1.1) admits a so-called degenerate cutting off boundary conditions. When such kind boundary conditions are imposed at two different points $n = N_1$ and $n = N_2$, the lattice reduces to a Darboux integrable system of the hyperbolic type equations. In our works [16], [17], [18], we suggested and developed a classification algorithm based on this observation. Let us briefly discuss the essence of the method.

We say that a constraint

$$u_0 = \varphi(x, y)$$

defines a degenerate cutting off condition for lattice (1.1) if it divides (1.1) into two independent semi-infinite lattices defined on the intervals $-\infty < n < 0$ and $0 < n < +\infty$, respectively.

Definition 1.1. *Lattice* (1.1) *is called integrable if there exist functions* φ_1 *and* φ_2 *such that for any pair of integers* N_1, N_2, *where* $N_1 < N_2 - 1$, *the hyperbolic system*

$$\begin{aligned} &u_{N_1} = \varphi_1(x, y), \\ &u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \qquad N_1 < n < N_2, \\ &u_{N_2} = \varphi_2(x, y) \end{aligned}$$

obtained from lattice (1.1) *by imposing degenerate boundary conditions is integrable in the sense of Darboux.*

Recall that a system of hyperbolic equations is called Darboux integrable if it admits a complete set of integrals in both characteristic directions, see [14], [15]. An effective criterion of the Darboux integrability of the system is connected with properties of an associated algebraic structures. More precisely, the characteristic Lie-Rinehart algebras [20, 21] assigned to both characteristic directions have to be of a finite dimension. Since the obtained hyperbolic system is of a very specific form, this allows us to study effectively the characteristic algebras. The method was shown to be effective in our articles [17], [18]. A large class of the integrable lattices of form (1.1) was represented in [22], where they were studied in the framework of the symmetry approach. It is remarkable that all equations of this class turned out to be integrable in the sense of Definition 1.1. Another argument in favor of our definition is that the resulting hyperbolic systems admit explicit solutions, which are extended to solutions of the original nonlinear chain. So, the chains integrable in our sense have a very wide class of explicit solutions.

In this article we continue the study initiated in [17], [18], where the integrable in the sense of Definition 1.1 cases of the two-dimensional quasilinear lattices of the form

$$u_{n,xy} = \alpha_n u_{n,x} u_{n,y} + p_n u_{n,x} + r_n u_{n,y} + q_n, \tag{1.3}$$

were described under the non-degeneracy condition $\frac{\partial \alpha_n}{\partial u_{n\pm1}} \neq 0$. Here the coefficients depend on three successive variables

$$\alpha_n = \alpha(u_{n+1}, u_n, u_{n-1}), \qquad p_n = p(u_{n+1}, u_n, u_{n-1}),$$
$$r_n = r(u_{n+1}, u_n, u_{n-1}), \qquad q_n = q(u_{n+1}, u_n, u_{n-1}).$$

We mention review [23], where a complete classification of lattices of the form

$$u_{n,xx} = f(u_{n-1}, u_n, u_{n+1}, u_{n,x}), \qquad \frac{\partial f}{\partial u_{n+1}} \frac{\partial f}{\partial u_{n+1}} \neq 0,$$

was presented. In our paper [18], we found two new equations of form (1.3), which were integrable in the sense of Definition 1.1. We note that these equations were two-dimensional generalizations of the equations from the list in paper [23].

Now we focus on a particular case (1.2) of lattice (1.3), as α_n vanishes identically. We suppose that the following conditions are satisfied: at least one of the following derivatives is non-zero:

$$\frac{\partial r_n}{\partial u_{n+1}} \neq 0 \qquad \text{or} \qquad \frac{\partial r_n}{\partial u_{n-1}} \neq 0, \tag{1.4}$$
$$\frac{\partial p_n}{\partial u_{n+1}} \neq 0 \qquad \text{or} \qquad \frac{\partial p_n}{\partial u_{n-1}} \neq 0. \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.1. *If chain* (1.2), (1.4) *is integrable in the sense of Definition 1.1, then by point transformations it is reduced to one of the following forms:*

$$u_{n,xy} = \left(e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}\right) u_{n,y}, \tag{1.6}$$
$$u_{n,xy} = \left(-u_{n+1} + 2u_n - u_{n-1}\right) u_{n,y}. \tag{1.7}$$

Lattices (1.6), (1.7) were known before [22]. Condition (1.5) implies that lattices obtained under classification procedure coincide with these lattices up to the change $x \leftrightarrow y$.

In the next section we describe briefly a theoretical base of the main research method; a detailed explanation was presented in [17, 18].

2. Preliminaries

According Definition 1.1, there exist cutting off conditions at two points that reduce (1.2) to the finite hyperbolic type system:

$$\begin{aligned} &u_{-1} = \varphi_1, \\ &u_{n,xy} = p_n u_{n,x} + r_n u_{n,y} + q_n, \quad 0 \leqslant n \leqslant N, \\ &u_{N+1} = \varphi_2. \end{aligned} \tag{2.1}$$

Here $p_n = p(u_{n-1}, u_n, u_{n+1})$, $r_n = r(u_{n-1}, u_n, u_{n+1})$, $q_n = q(u_{n-1}, u_n, u_{n+1})$.

We recall that a hyperbolic system of partial differential equations (2.1) is integrable in the sense of Darboux if it admits a complete set of functionally independent x- and y-integrals (see [14]). A function I depending on finitely many dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots$ is called y-integral if it solves the equation $D_y I = 0$ (see [14]), where D_y is the operator of total derivative with respect to variable y and $\mathbf{u}$ is a vector with coordinates $u_0, u_1, \ldots, u_N$. Since system (2.1) is autonomous, we consider autonomous y-integrals depending at least on one of the dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots$.

We suppose that system (2.1) is Darboux integrable and denote by $I(\mathbf{u}, \mathbf{u}_x, \ldots)$ its nontrivial y-integral. By definition, I solves the equation $D_y I = 0$. Let us calculate an action of the operator D_y on functions of the form $I(\mathbf{u}, \mathbf{u}_x, \ldots)$. It is determined by the rule $D_y I = YI$, where

$$Y = \sum_{i=0}^{N} \left(u_{i,y} \frac{\partial}{\partial u_i} + f_i \frac{\partial}{\partial u_{i,x}} + f_{i,x} \frac{\partial}{\partial u_{i,xx}} + \cdots \right).$$

Here $f_i = p_i u_{i,x} + r_i u_{i,y} + q_i$ is the right hand side of lattice (1.2). Therefore, the function I satisfies the equation $YI = 0$. Coefficients of the equation $YI = 0$ depend on the variables $u_{i,y}$, whereas a solution I is independent of $u_{i,y}$. Hence, I satisfies the system of linear equations:

$$YI = 0, \quad X_j I = 0, \quad j = 0, \ldots, N, \tag{2.2}$$

where $X_i = \frac{\partial}{\partial u_{i,y}}$. It follows from (2.2) that the commutator $Y_i = [X_i, Y] = X_i Y - Y X_i$ of operators Y and X_i, $i = 0, 1, \ldots, N$ also annihilates I. In the case of lattice (1.2) operator Y can be represented as:

$$Y = \sum_{i=0}^{N} u_{i,y} Y_i + R, \tag{2.3}$$

where Y_i and R are defined as

$$\begin{aligned} Y_i =& \frac{\partial}{\partial u_i} + X_i(f_i) \frac{\partial}{\partial u_{i,x}} + X_i(D_x f_i) \frac{\partial}{\partial u_{i,xx}} + \cdots \\ =& \frac{\partial}{\partial u_i} + r_i \frac{\partial}{\partial u_{i,x}} + \big(D_x(r_i) + r_i^2\big) \frac{\partial}{\partial u_{i,xx}} + \cdots \\ R =& \sum_{i=0}^{N} (u_{i,x} p_i + q_i) \frac{\partial}{\partial u_{i,x}} + \big(D_x(u_{i,x} p_i + q_i) + (u_{i,x} p_i + q_i) r_i\big) \frac{\partial}{\partial u_{i,xx}} + \cdots \end{aligned} \tag{2.4}$$

Let $\mathbf{F}$ be a ring of locally analytical functions of the dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots$. We consider the Lie-Rinehart algebra $\mathcal{L}(y, N)$ over the ring $\mathbf{F}$ generated by differential operators $Y, Y_0, Y_1, \ldots, Y_N$. We call this algebra the characteristic Lie algebra of system (2.1) along the y-direction. We shall show that we can multiply the elements in the algebra by functions depending on finitely many dynamical variables; this fact distinguishes our algebra from an ordinary Lie algebra. The characteristic Lie algebra of system (2.1) along the x-direction is defined in the same way.

Now we shall work with the operators in the algebra $\mathcal{L}(y, N)$. Algebra $\mathcal{L}(y, N)$ is of a finite dimension if there exist a finite basis $Z_1, Z_2, \ldots, Z_k$ consisting of linearly independent operators such that each element $Z \in \mathcal{L}(y, N)$ is represented as a linear combination $Z = a_1 Z_1 + a_2 Z_2 + \ldots + a_k Z_k$, where the coefficients a_1, a_2, $\ldots$, a_k are analytic functions depending on the dynamical variables defined in an open domain. Then the identity $a_1 Z_1 + a_2 Z_2 + \ldots + a_k Z_k = 0$ implies that $a_1 = a_2 = \ldots = a_k = 0$. System (2.1) is integrable in the sense of Darboux if and only if the characteristic Lie algebras in both directions are of a finite dimension [14].

In our study, we shall apply the operator D_x to smooth functions depending on the dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots$. On this class of functions, we obtain the following commutation relations for the operators Y_i, R:

$$\begin{aligned} [D_x, Y_i] &= -r_i Y_i, \\ [D_x, R] &= -\sum_{i=0}^{N} (u_{i,x} p_i + q_i) Y_i. \end{aligned} \tag{2.5}$$

The following statement holds [13, 19, 14]:

Lemma 2.1. *If a vector field*

$$Z = \sum_i z_{1,i} \frac{\partial}{\partial u_{i,x}} + z_{2,i} \frac{\partial}{\partial u_{i,xx}} + \cdots$$

solves the equation $[D_x, Z] = 0$*, then* $Z = 0$.

We shall also use the standard notation $\mathrm{ad}_X(Z) := [X, Z]$.

The key method, on which the classification algorithm is based, is a test sequence method. We call a sequence of operators $W_0, W_1, W_2, \dots$ in the algebra $\mathcal{L}(y, N)$ a test sequence if

$$[D_x, W_m] = \sum_{j=0}^{m} w_{j,m} W_j$$

holds true for all m. The test sequence allows us to derive integrability conditions for hyperbolic type system (2.1), see [24, 14, 15].

The first step of our study is to define the functions p_n, r_n. Let us note that when we search the function r_n we study the subalgebra Lie generated by the operators Y_i, see (2.4). It follows from (2.3), (2.4), (2.5) that this subalgebra coincides with the Lie algebra of a hyperbolic type system corresponding the lattice

$$u_{n,xy} = r_n(u_{n+1}, u_n, u_{n-1}) u_{n,y}. \tag{2.6}$$

The following statement holds true for this lattice.

Lemma 2.2. *If lattice* (2.6) *is integrable in the sense of Definition 1.1, then it is reduced by point transformations to one of the following forms:*

$$u_{n,xy} = \left(e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}\right) u_{n,y}, \tag{2.7}$$

$$u_{n,xy} = \left(-u_{n+1} + 2u_n - u_{n-1}\right) u_{n,y}. \tag{2.8}$$

Proof of Lemma (2.2) is given in Section 3.

Remark 2.1. *If the function* r_n *depends only on the variable* u_n*, that is* $r_n = r_n(u_n)$*, then*

$$[Y_k, Y_j] = 0$$

for all k*,* j *and system* (2.6) *splits into the system of independent equations* $u_{n,xy} = r_n(u_n) u_{n,y}$*. This system has integrals in the direction we consider. We mention that a wide class of scalar equations of the form* $u_{x,y} = f(u, u_x, u_x)$ *was studied in* [14] *within the characteristic Lie algebras approach. But the case* $r_n = r_n(u_n)$ *or* $p_n = p_n(u_n)$ *holds for lattice* (1.2) *and is to be studied, see Section 4.*

3. Integrability conditions

3.1. The first test sequence. Let us define a sequence of operators in the characteristic algebra $\mathcal{L}(y, N)$ by the reccurent formula:

$$Y_0, \quad Y_1, \quad W_1 = [Y_0, Y_1], \quad W_2 = [Y_0, W_1], \quad \dots \quad W_{k+1} = [Y_0, W_k], \quad \dots \tag{3.1}$$

The following commutation relations are valid for the first elements of the sequence (3.1), see formula (2.5):

$$[D_x, Y_0] = -r_0 Y_0, \quad [D_x, Y_1] = -r_1 Y_1. \tag{3.2}$$

By using the Jacobi identity we get the formulae

$$[D_x, W_1] = -(r_1 + r_0)W_1 - Y_0(r_1)Y_1 + Y_1(r_0)Y_0, \tag{3.3}$$

$$[D_x, W_2] = -(r_1 + 2r_0)W_2 - Y_0(2r_1 + r_0)W_1 - Y_0^2(r_1)Y_1 + Y_0Y_1(r_0)Y_0, \tag{3.4}$$

$$\begin{aligned}[D_x, W_3] = &- (r_1 + 3r_0)W_3 - Y_0(3r_1 + 3r_0)W_2 - Y_0^2(3r_1 + r_0)W_1\\ &- Y_0^3(r_1)Y_1 + Y_0^2Y_1(r_0)Y_0,\end{aligned} \tag{3.5}$$

$$\begin{aligned}[D_x, W_4] = &- (r_1 + 4r_0)W_4 - Y_0(4r_1 + 6r_0)W_3 - Y_0^2(6r_1 + 4r_0)W_2\\ &- Y_0^3(4r_1 + r_0)W_1 - Y_0^4(r_1)Y_1 + Y_0^3Y_1(r_1)Y_0.\end{aligned} \tag{3.6}$$

It can be proved by induction that (3.1) is a test sequence. Moreover, for $k \geqslant 4$

$$[D_x, W_k] = a_kW_k + b_kW_{k-1} + s_kW_{k-2} + t_kW_{k-3} + \cdots, \tag{3.7}$$

where

$$\begin{aligned} a_k &= -(r_1 + kr_0), \quad b_k = \frac{k-k^2}{2}Y_0(r_0) - Y_0(r_1)k,\\ s_k &= -Y_0^2(3r_1 + r_0) + \frac{1}{2}(k-3)Y_0(q_3 + q_{k-1}),\\ t_k &= -Y_0^3(4r_1 + r_0) + \frac{1}{2}(k-4)Y_0(s_4 + s_{k-1}).\end{aligned} \tag{3.8}$$

By assumption, in the algebra $\mathcal{L}(y, N)$ there are finitely many linearly independent elements of sequence (3.1). Therefore, there exists M such that

$$W_M = \lambda W_{M-1} + \cdots, \tag{3.9}$$

the operators Y_0, Y_1, W_1, $\ldots$, W_{M-1} are linearly independent, the dots stand for a linear combination of the operators Y_0, Y_1, W_1, $\ldots$, W_{M-2}.

Let us consider the first three elements.

Lemma 3.1. *If condition* (1.4) *holds, then the operators* Y_0, Y_1, W_1 *are linear independent. Otherwise, if* $r_0 = r_0(u_0)$ *depends only on the variable* u_0*, then* $W_1 = 0$.

Proof. Let r_0 depend on at least one of the variables u_{-1}, u_1. We are going to prove that Y_0, Y_1, W_1 are linear independent in this case. We argue by contradiction assuming that

$$\lambda_1W_1 + \mu_1Y_1 + \mu_0Y_0 = 0.$$

The operators Y_0, Y_1 are of the form

$$Y_0 = \frac{\partial}{\partial u_0} + \cdots, \qquad Y_1 = \frac{\partial}{\partial u_1} + \cdots,$$

while W_1 contains terms of the form $\frac{\partial}{\partial u_0}$ and $\frac{\partial}{\partial u_1}$. Hence, the coefficients μ_1, μ_0 are equal to zero. If $\lambda_1 \neq 0$, then $W_1 = 0$. We apply the operator ad_{D_x} to both sides of the last identity, then by virtue of (3.2) we obtain the equation

$$Y_0(r_1)Y_1 - Y_1(r_0)Y_0 = 0.$$

It implies that $Y_0(r_1) = r_{1,u_0} = 0$ and $Y_1(r_0) = r_{0,u_1} = 0$. This is equivalent to $r_{0,u_{-1}} = 0$, $r_{0,u_1} = 0$ and we arrive at a contradiction to condition (1.4).

By direct calculation of the operator

$$W_1 = [Y_0, Y_1] = Y_0Y_1 - Y_1Y_0$$

and using formula (2.4), we prove the second part of the lemma. The proof is complete. □

In what follows we assume that $M \geqslant 2$ and condition (1.4) holds.

Lemma 3.2. *If relation (3.9) holds true for $M \geqslant 2$, then the function r_0 has one of the following forms:*
i) if $\lambda = 0$, then

$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) - \frac{2}{M-1}\alpha(u_0) + \delta(u_1); \tag{3.10}$$

ii) if $\lambda \neq 0$, then

$$r_0(u_1, u_0, u_{-1}) = \beta(u_{-1})e^{-\frac{2}{M(M-1)}\lambda u_0} + \psi(u_0, u_1), \tag{3.11}$$

where functions β and ψ satisfy the equation

$$\lambda\psi(u_0, u_1) + \frac{1}{2}M(M-1)\psi_{u_0}(u_0, u_1) + Me^{-\frac{2}{M(M-1)}\lambda u_1}\beta'(u_0) = 0. \tag{3.12}$$

Proof. We apply the operator ad_{D_x} to both sides of identity (3.9). Combining the coefficients before W_{M-1}, we get the equation:

$$D_x(\lambda) = \lambda(a_M - a_{M-1}) + b_M. \tag{3.13}$$

We substitute formulae (3.8) into (3.13):

$$D_x(\lambda) = -r_0\lambda - \frac{M(M-1)}{2}r_{0,u_0} - Mr_{1,u_0}. \tag{3.14}$$

From identity (3.14) it follows that λ is a constant and

$$r_0\lambda + \frac{M(M-1)}{2}r_{0,u_0} + Mr_{1,u_0} = 0. \tag{3.15}$$

Let us apply the operator $\frac{\partial}{\partial u_2}$ to (3.15):

$$Mr_{1,u_0u_2} = 0.$$

This is equivalent to $r_{0,u_{-1}u_1} = 0$ and, hence,

$$r_0(u_1, u_0, u_{-1}) = \varphi(u_{-1}, u_0) + \psi(u_0, u_1). \tag{3.16}$$

We substitute function (3.15) into (3.14)

$$\lambda\varphi_{u_{-1}} + \frac{M(M-1)}{2}\varphi_{u_0u_{-1}} = 0. \tag{3.17}$$

We consider two different cases:
i) $\lambda = 0$;
ii) $\lambda \neq 0$.
If i) holds, then $\varphi_{u_0u_{-1}} = 0$, so that $\varphi(u_{-1}, u_0) = \alpha(u_{-1}) + \beta(u_0)$ and

$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) + \beta(u_0) + \psi(u_0, u_1).$$

We re-denote $\beta + \psi \to \psi$ and we get

$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) + \psi(u_0, u_1). \tag{3.18}$$

We substitute (3.18) and $\lambda = 0$ into (3.15):

$$\frac{M(M-1)}{2}\psi_{u_0}(u_0, u_1) + M\alpha'(u_0) = 0. \tag{3.19}$$

Applying the operator $\frac{\partial}{\partial u_1}$ to identity (3.19), we obtain $\psi_{u_0u_1} = 0$ and, hence,

$$\psi(u_0, u_1) = \gamma(u_0) + \delta(u_1).$$

We substitute ψ into (3.19) and we find

$$\gamma(u_0) = -\frac{2}{M-1}\alpha(u_0) + c_1$$

and, then

$$r_0(u_1,u_0,u_{-1}) = \alpha(u_{-1}) - \frac{2}{M-1}\alpha(u_0) + \delta(u_1).$$

Let us consider case ii). Solution of equation (3.17) is the function

$$\varphi(u_{-1},u_0) = \alpha(u_0) + e^{-\frac{2}{M(M-1)}\lambda u_0}\beta(u_{-1}).$$

Then function (3.16) becomes

$$r_0(u_1,u_0,u_{-1}) = \alpha(u_0) + e^{-\frac{2}{M(M-1)}\lambda u_0}\beta(u_{-1}) + \psi(u_0,u_1).$$

We redenote $\alpha + \psi \to \psi$ and we get

$$r_0(u_1,u_0,u_{-1}) = e^{-\frac{2}{M(M-1)}\lambda u_0}\beta(u_{-1}) + \psi(u_0,u_1).$$

Substituting r_0 into (3.15), we obtain (3.12). The proof is complete. □

3.2. Second test sequence. We construct the test sequence containing operators Y_0, Y_1, Y_2 and their multiple commutators:

$$Z_0 = Y_0, \quad Z_1 = Y_1, \quad Z_2 = Y_2, \quad Z_3 = [Y_1, Y_0], \quad Z_4 = [Y_2, Y_1],$$
$$Z_5 = [Y_2, Z_3], \quad Z_6 = [Y_1, Z_3], \quad Z_7 = [Y_1, Z_4], \quad Z_8 = [Y_1, Z_5].$$

The elements Z_m, $m > 8$ are defined by the recurrent formula $Z_m = [Y_1, Z_{m-3}]$.

The following commutation relations hold:

$$[D_x, Y_0] = -r_0Y_0, \quad [D_x, Y_1] = -r_1Y_1, \quad [D_x, Y_2] = -r_2Y_2, \tag{3.20}$$
$$[D_x, Z_3] = -(r_1 + r_0)Z_3 + Y_0(r_1)Y_1 - Y_1(r_0)Y_0, \tag{3.21}$$
$$[D_x, Z_4] = -(r_2 + r_1)Z_4 + Y_1(r_2)Y_2 - Y_2(r_1)Y_1, \tag{3.22}$$
$$\begin{aligned}[D_x, Z_5] = &-(r_0 + r_1 + r_2)Z_5 - Y_2(r_1 + r_0)Z_3 + Y_0(r_1)Z_4\\ &+ Y_2Y_0(r_1)Y_1 - Y_2Y_1(r_0)Y_0,\end{aligned} \tag{3.23}$$
$$[D_x, Z_6] = -(r_0 + 2r_1)Z_6 - Y_1(2r_0 + r_1)Z_3 + Y_1Y_0(r_1)Y_1 - Y_1^2(r_0)Y_0,$$
$$[D_x, Z_7] = -(2r_1 + r_2)Z_7 - Y_1(r_1 + 2r_2)Z_4 + Y_1^2(r_2)Y_2 - Y_1Y_2(r_1)Y_1,$$
$$\begin{aligned}[D_x, Z_8] = &-(r_0 + 2r_1 + r_2)Z_8 + Y_0(r_1)Z_7 - Y_2(r_0 + r_1)Z_6 - Y_1(r_0 + r_1 + r_2)Z_5\\ &+ Y_1Y_0(r_1)Z_4 - Y_1Y_2(r_1)Z_3 + Y_1Y_2Y_0(r_1)Y_1 - Y_1Y_2Y_1(r_0)Y_0.\end{aligned} \tag{3.24}$$

We recall that we assume condition (1.4), otherwise, starting with Z_3, all elements of the sequence vanish.

Lemma 3.3. *The operators Z_0, Z_1, ..., Z_5 are linearly independent.*

Proof. It is easy to show that the operators Z_0, Z_1, ..., Z_4 are linearly independent; this is similar to the proof of Lemma 3.1. We prove Lemma 3.3 by arguing by contradiction. We suppose that

$$Z_5 = \sum_{j=0}^{4} \lambda_j Z_j. \tag{3.25}$$

We apply the operator ad_{D_x} to both sides of identity (3.25), and we use formulae (3.21)–(3.23) to simplify the obtained identity:

$$\begin{aligned} -(r_0+r_1+r_2)\sum_{j=0}^{4}\lambda_j Z_j + Y_0(r_1)Z_4 - Y_2(r_1)Z_3 + Y_2Y_0(r_1)Y_1 - Y_2Y_1(r_0)Y_0 \\ =\sum_{j=0}^{4} D_x(\lambda_j)Z_j + \lambda_4\big(-(r_2+r_1)Z_4 + Y_1(r_2)Y_2 - Y_2(r_1)Y_1\big) \\ + \lambda_3\big(-(r_1+r_0)Z_3 + Y_0(r_1)Y_1 - Y_1(r_0)Y_0\big) \\ - \lambda_2 r_2 Y - 2 - \lambda_1 r_1 Y_1 - \lambda_0 r_0 Y_0. \end{aligned} \tag{3.26}$$

Combining the coefficients at Z_4 in (3.26), we get the equation:

$$D_x(\lambda_4) = -r_0\lambda_4 + r_{1,u_0}.$$

This identity implies that λ_4 is a constant and

$$-r_0\lambda_4 + r_{1,u_0} = 0. \tag{3.27}$$

We shall study this equation in two different cases i) and ii):

i) If r_0 is defined by formula (3.10), then (3.27) casts into the form

$$-\left(\alpha(u_{-1}) - \frac{2}{M-1}\alpha(u_0) + \delta(u_1)\right)\lambda_4 + \alpha'(u_0) = 0. \tag{3.28}$$

If $\lambda_4 \neq 0$, then (3.28) implies that functions α, δ are constants since the variables u_1, u_0, u_{-1} are independent. Then we get that r_0 is a constant that contradicts condition (1.4). If $\lambda_4 = 0$, then it follows from (3.28) that $\alpha'(u_0) = 0$ and

$$r_0(u_1,u_0,u_{-1}) = \delta(u_1). \tag{3.29}$$

ii) If r_0 is defined by (3.11), then identity (3.27) becomes

$$-\left(\beta(u_{-1})e^{-\frac{2}{M(M-1)\lambda u_0}} + \psi(u_0,u_1)\right)\lambda_4 + \beta'(u_0)e^{-\frac{2}{M(M-1)}\lambda u_1} = 0. \tag{3.30}$$

We apply the operator $\frac{\partial}{\partial u_{-1}}$ to (3.30):

$$\beta'(u_{-1})e^{-\frac{2}{M(M-1)}\lambda u_0}\lambda_4 = 0.$$

If $\lambda_4 = 0$, then it follows from (3.30) that $\beta'(u_0) = 0$ and, hence, $\beta = c_4$, where c_4 is a constant. If $\lambda_4 \neq 0$, then it follows from (3.30) that

$$\beta(u_{-1})e^{-\frac{2}{M(M-1)\lambda u_0}} + \psi(u_0,u_1) = 0.$$

The expression in the left hand side of the last identity coincides exactly with $r_0(u_1,u_0,u_{-1})$. Therefore, the last identity contradicts condition (1.4).

Thus, we obtain that $\lambda_4 = 0$ and r_0 is defined by the formula

$$r_0(u_1,u_0,u_{-1}) = c_4 e^{-\frac{2}{M(M-1)}\lambda u_0} + \psi(u_0,u_1). \tag{3.31}$$

We collect the coefficients at Z_3 in (3.26), take into consideration that $\lambda_4 = 0$, and we obtain the equation

$$D_x(\lambda_3) = -r_2\lambda_3 - r_{1,u_2}.$$

Hence, λ_3 is a constant, and

$$r_2\lambda_3 + r_{1,u_2} = 0.$$

Applying the shift operator, we get the equation

$$r_1\lambda_3 + r_{0,u_1} = 0. \tag{3.32}$$

i) Let us substitute the function r_0 defined by formula (3.29) into (3.32):

$$\delta(u_2)\lambda_3 + \delta'(u_1) = 0.$$

A simple analysis of the last equation gives the contradiction to condition (1.4).

ii) Let us substitute the function r_0 defined by formula (3.31) into (3.32):

$$\left(c_4 e^{-\frac{2}{M(M-1)}\lambda u_1} + \psi(u_1, u_2)\right)\lambda_3 + \psi_{u_1}(u_0, u_1) = 0. \tag{3.33}$$

We apply the operator $\frac{\partial}{\partial u_2}$ to both sides of identity (3.33) $\psi_{u_2}(u_1, u_2)\lambda_3 = 0$. Studying (3.33) in this case, we arrive to contradiction to condition (1.4).

Otherwise, if $\lambda_3 \neq 0$, then the expression in the left hand side of identity (3.33), coinciding with r_1, is equal to zero. Thus, we obtain the contradiction to condition (1.4). The proof is complete. □

For further purposes, it is convenient to divide sequence (3.26) into three subsequences $\{Z_{3m}\}$, $\{Z_{3m+1}\}$, $\{Z_{3m+2}\}$

Lemma 3.4. *Operator* ad_{D_x} *acts on sequence* (3.26) *according the following formulae:*

$$\begin{aligned}
[D_x, Z_{3m}] &= -(r_0 + mr_1)Z_{3m} + \left(\frac{m - m^2}{2} Y_1(r_1) - mY_1(r_0)\right) Z_{3m-3} + \cdots,\\
[D_x, Z_{3m+1}] &= -(r_2 + mr_1)Z_{3m+1} + \left(\frac{m - m^2}{2} Y_1(r_1) - mY_1(r_2)\right) Z_{3m-2} + \cdots,\\
[D_x, Z_{3m+2}] &= -(r_0 + mr_1 + r_2)Z_{3m+2} + Y_0(r_1)Z_{3m+1} - Y_2(r_1)Z_{3m}\\
&\quad - (m-1)\left(\frac{m}{2}Y_1(r_1) + Y_1(r_0 + r_2)\right) Z_{3m-1} + \cdots
\end{aligned}$$

Lemma 3.4 can be easily proved by induction.

Theorem 3.1. *Assume that* Z_{3k+2} *is a linear combination*

$$Z_{3k+2} = \lambda_k Z_{3k+1} + \mu_k Z_{3k} + \nu_k Z_{3k-1} + \cdots \tag{3.34}$$

of the previous terms in sequence (3.26) *and none of the operators* Z_{3j+2} *for* $j < k$ *is a linear combination of operators* Z_s *with* $s < 3j + 2$. *Then the coefficient* ν_k *satisfies the equation*

$$D_x(\nu_k) = -r_1\nu_k - \frac{k(k-1)}{2} Y_1(r_1) - (k-1)Y_1(r_0 + r_2). \tag{3.35}$$

Lemma 3.5. *Suppose that the assumptions of Theorem 3.1 are satisfied and the operator* Z_{3k} *(the operator* Z_{3k+1}*) is linearly expressed in terms of the operators* Z_i, $i < 3k$. *Then in this decomposition the coefficient at* Z_{3k-1} *vanishes.*

Proof. We argue by contradiction. Suppose that

$$Z_{3k} = \lambda Z_{3k-1} + \cdots \tag{3.36}$$

and $\lambda \neq 0$. We apply the operator ad_{D_x} to both sides of identity (3.36). Using formulae from Lemma 3.4, we get

$$-(r_0 + kr_1)\lambda Z_{3k-1} + \cdots = D_x(\lambda)Z_{3k-1} - \lambda\big(r_0 + (k-1)r_1 + r_2\big)Z_{3k-1} + \cdots$$

Collecting coefficients at Z_{3k-1}, we obtain

$$D_x(\lambda) = \lambda(r_2 - r_1)k.$$

This equation implies that λ is a constant and $\lambda(r_2 - r_1)k = 0$. Then $r_2 = r_1 = \mathit{const}$ that contradictions condition (1.4). The proof is complete. □

In order to prove Theorem 3.1, we apply the operator ad_{D_x} to both sides of the identity (3.34). Then we simplify a obtained identity using formulae from Lemma 3.4. Collecting coefficients at Z_{3k-1}, we obtain equation (3.35).

The next step of our work is studying equation (3.35) as r_0 is defined by formulae (3.10) or (3.11) under condition (1.4) and for $M \geqslant 2$, $k \geqslant 2$.

We find exact values of coefficients in equation (3.35) and substitute them into (3.35):

$$D_x(\nu_k) = -r_1\nu_k - \frac{k(k-1)}{2} r_{1,u_1} - (k-1)(r_{0,u_1} + r_{2,u_1}).$$

This equation implies that ν_k is a constant and, hence,

$$\nu_k r_1 + \frac{k(k-1)}{2} r_{1,u_1} + (k-1)(r_{0,u_1} + r_{2,u_1}) = 0. \tag{3.37}$$

Lemma 3.6. *If relations* (3.9), (3.34) *hold true for some* $M \geqslant 2$, $k \geqslant 2$, *and condition* (1.4) *holds true, then*

i) if $\lambda = 0$, $\nu_k = 0$, *then*

$$r_n(u_{n+1}, u_n, u_{n-1}) = \alpha(u_{n-1}) - \frac{2}{M-1}\alpha(u_n) + \left(\frac{k}{M-1} - 1\right)\alpha(u_{n+1}) + c_1; \tag{3.38}$$

ii) if $\lambda \neq 0$, $\nu_k = 0$, $k = 2$, $M \neq 3$, *then*

$$r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n - hu_{n-1}} + ce^{u_{n+1} - hu_n}; \tag{3.39}$$

iii) if $\lambda \neq 0$, $\nu_k \neq 0$, *then* r_n *is defined by formula* (3.39).

The proof of this lemma is rather complicated and is presented in Appendix.

We proceed to relations (3.9), (3.34).

We need another one test sequence:

$$Y_0, \quad Y_1, \quad \overline{W}_1 = [Y_1, Y_0], \quad \overline{W}_2 = \left[Y_1, \overline{W}_1\right], \dots \overline{W}_{k+1} = \left[Y_1, \overline{W}_k\right] \dots .$$

The following commutation relation hold:

$$\left[D_x, \overline{W}_1\right] = -(r_1 + r_0)\overline{W}_1 + Y_0(r_1)Y_1 - Y_1(r_0)Y_0, \tag{3.40}$$

$$\left[D_x, \overline{W}_2\right] = -(2r_1 + r_0)\overline{W}_2 - Y_1(r_1 + 2r_0)\overline{W}_1 + Y_1Y_0(r_1)Y_1 - Y_1^2(r_0)Y_0, \tag{3.41}$$

$$\begin{aligned}\left[D_x, \overline{W}_3\right] = &-(3r_1 + r_0)\overline{W}_3 - Y_1(3r_1 + 3r_0)\overline{W}_2 - Y_1^2(r_1 + 3r_0)\overline{W}_1 \\ &+ Y_1^2Y_0(r_1)Y_1 - Y_1^3(r_0)Y_0,\end{aligned} \tag{3.42}$$

$$\begin{aligned}\left[D_x, \overline{W}_4\right] = &-(4r_1 + r_0)\overline{W}_4 - Y_1(6r_1 + 4r_0)\overline{W}_3 - Y_1^2(4r_1 + 6r_0)\overline{W}_2 \\ &- Y_1^3(r_1 + 4r_0)\overline{W}_1 + Y_1^3Y_0(r_1)Y_1 - Y_1^4(r_0)Y_0.\end{aligned} \tag{3.43}$$

It is easy to prove that

$$\left[D_x, \overline{W}_k\right] = \overline{a}_k\overline{W}_k + \overline{b}_k\overline{W}_{k-1} + \overline{s}_k\overline{W}_{k-2} + \cdots, \tag{3.44}$$

for $k \geqslant 3$, where

$$\overline{a}_k = -(kr_1 + r_0), \quad \overline{b}_k = \frac{k-k^2}{2}Y_1(r_1) - Y_1(r_0)k,$$

$$\overline{s}_k = -Y_1^2(r_1 + 3r_0) + \frac{1}{2}(k-3)Y_1(\overline{q}_3 + \overline{q}_{k-1}).$$

We observe that the first terms Y_0, Y_1, $\overline{W}_1 = -W_1$ obey Lemma 3.1.

We suppose that $\mathcal{L}(y, N)$ is finitely-dimensional, that is, each sequence of its elements terminates at some step. Consequently, there exists N such that:

$$\overline{W}_N = \Lambda\overline{W}_{N-1} + \cdots, \tag{3.45}$$

where the operators Y_0, Y_1, $\overline{W}_1$, ..., $\overline{W}_{N-1}$ are linearly independent, and the dots stand for linear combination of the operators Y_0, Y_1, $\overline{W}_1$, ..., $\overline{W}_{N-2}$.

3.3. Case $M = 2$. Suppose that relation (3.9) holds true for $M = 2$:

$$W_2 = \lambda W_1 + \varepsilon Y_1 + \eta Y_0. \tag{3.46}$$

We apply the operator ad_{D_x} to both sides of identity (3.46) and we get:

$$-(r_1 + 2r_0)(\lambda W_1 + \varepsilon Y_1 + \eta Y_0) - Y_0(2r_1 + r_0)W_1 - Y_0^2(r_1)Y_1 + Y_0Y_1(r_0)Y_0$$
$$= \lambda\big(-(r_1 + r_0)W_1 - Y_0(r_1)Y_1 + Y_1(r_0)Y_0\big) - \varepsilon r_1 Y_1 - \eta r_0 Y_0.$$

Collecting the coefficients at independent operators W_1, Y_1, Y_0, we obtain the system

$$r_0\lambda + 2r_{1,u_0} + r_{0,u_0} = 0, \tag{3.47}$$
$$2r_0\varepsilon + r_{1,u_0u_0} - \lambda r_{1,u_0} = 0, \tag{3.48}$$
$$-(r_1 + r_0)\eta + r_{0,u_0u_1} - \lambda r_{0,u_1} = 0. \tag{3.49}$$

3.3.i) Let us consider the case when thefunction r_n is described by formula (3.38) and $\lambda = 0$. We substitute function (3.38) and $\lambda = 0$ into system (3.47)–(3.49). Then equation (3.47) becomes identity and we arrive to the system:

$$2\big(\alpha(u_{-1}) - 2\alpha(u_0) + (k-1)\alpha(u_1) + c_1\big)\varepsilon + \frac{d^2\alpha(u_0)}{du_0^2} = 0,$$
$$\big(\alpha(u_0) + 2\alpha(u_1) - (k-1)\alpha(u_2) + 2c_1 + \alpha(u_{-1}) + (k-1)\alpha(u_1)\big)\eta = 0.$$

This system yields that

$$\varepsilon = \eta = 0, \qquad \alpha(u_0) = C_1u_0 + C_2,$$
$$r_0(u_1, u_0, u_{-1}) = (k-1)C_1u_1 - 2C_1u_0 + C_1u_{-1} + C_3,$$

where $C_3 = -2C_2 + kC_2 + c_1$. We will study the lattice corresponding to this function, in Section 3.5.i, see (3.62).

3.3.ii) Let us consider the case when the function r_n is described by formula (3.39) and $\lambda \neq 0$. System (3.47)–(3.49) casts into the form:

$$(\lambda + 1)e^{u_0 - hu_{-1}} + (-ch + \lambda c - 2h)e^{u_1 - hu_0} = 0,$$
$$2\varepsilon e^{u_0 - hu_{-1}} + (2\varepsilon c + h^2 + \lambda h)e^{u_1 - hu_0} = 0,$$
$$-\eta e^{u_0 - hu_{-1}} + (-\eta - \eta c - ch - \lambda c)e^{u_1 - hu_0} - \eta c e^{u_2 - hu_1} = 0.$$

A simple analysis of the last system leads us to the identities $\lambda = -1$, $h = 1$, $c = -1$, $\varepsilon = \eta = 0$. We get that r_n has the following form:

$$r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}. \tag{3.50}$$

And $W_2 = -W_1$, $\overline{W}_2 = \overline{W}_1$.

Now let us substitute (3.50) into (3.37):

$$\left(\nu_k + \frac{1}{2}k^2 - \frac{3}{2}k + 1\right)e^{u_1 - u_0} + \left(-\nu_k + \frac{1}{2}k^2 - \frac{3}{2}k + 1\right)e^{u_2 - u_1} = 0,$$

which implies $\nu_k = 0$, $k = 2$.

Thus, relation (3.34) is of the form

$$Z_8 = \rho Z_4 + \sigma Z_3 + \tau Z_2 + \phi Z_1 + \pi Z_0, \tag{3.51}$$

and

$$Z_6 = [Y_1, [Y_1, Y_0]] = \overline{W}_2 = \overline{W}_1 = [Y_1, Y_0] = Z_3,$$
$$Z_7 = [Y_1, [Y_2, Y_1]] = D_n\,[Y_0, [Y_1, Y_0]] = -D_nW_2 = D_nW_1 = -Z_4.$$

Commutation relation (3.24) become

$$\begin{aligned}[D_x, Z_8] = &-(r_0+2r_1+r_2)Z_8 - Y_1(r_0+r_1+r_2)Z_5 + \big(Y_1Y_0(r_1) - Y_0(r_1)\big)Z_4 \\ &- \big(Y_2(r_0+r_1) + Y_1Y_2(r_1)\big)Z_3 + Y_1Y_2Y_0(r_1)Y_1 - Y_1Y_2Y_1(r_0)Y_0.\end{aligned} \tag{3.52}$$

We apply the operator ad_{D_x} to both sides of identity (3.51) and take into consideration the formulae (3.20)–(3.23), (3.52), then we collect coefficients at independent operators Z_4, Z_3, Z_2, Z_1, Z_0:

$$\begin{aligned}&-(e^{u_0-u_{-1}} - e^{u_2-u_1})\rho = 0, \quad (-e^{u_3-u_2} + e^{u_1-u_0})\sigma = 0,\\ &-(e^{u_0-u_{-1}} + e^{u_1-u_0} - 2e^{u_2-u_1})\tau + \rho e^{u_2-u_1} = 0,\\ &-(e^{u_0-u_{-1}} - e^{u_3-u_2})\phi - \rho e^{u_2-u_1} + \sigma e^{u_1-u_0} = 0.\\ &-(2e^{u_1-u_0} - e^{u_2-u_1} - e^{u_3-u_2})\pi - \sigma e^{u_1-u_0} = 0.\end{aligned}$$

It is clear that $\rho = \sigma = \tau = \phi = \pi = 0$. Hence, $Z_8 = 0$.

3.4. Case $M = 3$**.** Suppose that relation (3.9) holds true for $M = 3$:

$$W_3 = \lambda W_2 + \rho W_1 + \varepsilon Y_1 + \eta Y_0. \tag{3.53}$$

We apply the operator ad_{D_x} to both sides of identity (3.53) and use formulae (3.2), (3.3), (3.4), (3.5). Collecting coefficients at the independent operators, we obtain the system

$$r_0\lambda + 3r_{1,u_0} + 3r_{0,u_0} = 0, \tag{3.54}$$

$$-2r_0\rho + \lambda(2r_{1,u_0} + r_{0,u_0}) - 3r_{1,u_0u_0} - r_{0,u_0u_0} = 0, \tag{3.55}$$

$$-3r_0\varepsilon + \lambda r_{1,u_0u_0} + \rho r_{1,u_0} - r_{1,u_0u_0u_0} = 0, \tag{3.56}$$

$$-(r_1 + 2r_0)\eta - \lambda r_{0,u_0u_1} - \rho r_{0,u_1} + r_{0,u_0u_0u_1} = 0. \tag{3.57}$$

3.4.i) Let us consider case (3.38), $\lambda = 0$. It follows from equations (3.54)–(3.57) that

$$\alpha(u_n) = C_1u_n + C_2, \qquad r_n(u_{n+1}, u_n, u_{n-1}) = \frac{k-2}{2}C_1u_{n+1} - C_1u_n + C_1u_{n-1} + C_3,$$

where $C_3 = \frac{1}{2}C_2k - C_2 + c_1$. Further study of the lattice with r_n defined by this formula is provided in 3.5.i, see (3.62).

3.4.ii) Let us consider case (3.39) $\lambda \neq 0$. We substitute r_n into (3.47)–(3.49). Studying this system, we obtain that $\lambda = -3$, $\rho = -2$, $\varepsilon = \eta = 0$, $h = 1$, $c = -\frac{1}{2}$. The function (3.39) becomes

$$r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n-u_{n-1}} - \frac{1}{2}e^{u_{n+1}-u_n}.$$

We substitute this function into equation (3.37) and we get $k = 1$ or $k = \frac{5}{2}$. These identities contradict condition $k \geqslant 2$.

3.5. Case $M > 3$**.** Let the following relation be true for $M > 3$

$$W_M = \lambda W_{M-1} + \rho W_{M-2} + \kappa W_{M-3} + \cdots \tag{3.58}$$

Taking into account formula (3.7), we apply the operator ad_{D_x} to both sides of the above identity:

$$\begin{aligned}a_M(\lambda W_{M-1} + \rho W_{M-2} + \kappa W_{M-3} + \cdots) + b_MW_{M-1} + s_MW_{M-2} + t_MW_{M-3} + \cdots\\ = \lambda(a_{M-1}W_{M-1} + b_{M-1}W_{M-2} + s_{M-1}W_{M-3} + \cdots)\\ + \rho(a_{M-2}W_{M-2} + b_{M-2}W_{M-3} + \cdots) + \kappa(a_{M-3}W_{M-3} + \cdots)\end{aligned}$$

We collect coefficients at the independent operators W_{M-1}, W_{M-2}, W_{M-3}:

$$\lambda(a_M - a_{M-1}) + b_M = 0, \tag{3.59}$$

$$\rho(a_M - a_{M-2}) + s_M - \lambda b_{M-1} = 0, \tag{3.60}$$

$$\kappa(a_M - a_{M-3}) + t_M - \lambda s_{M-1} - \rho b_{M-2} = 0. \tag{3.61}$$

3.5.i) By system (3.59)–(3.61) we obtain that $\alpha(u_n) = C_1 u_n + C_2$ and

$$r_n(u_{n+1}, u_n, u_{n-1}) = \frac{k-(M-1)}{M-1} C_1 u_{n+1} - \frac{2C_1}{M-1} u_n + C_1 u_{n-1} + C_3,$$

where

$$C_3 = \frac{c_1 M - c_1 - 2C_2 + kC_2}{M-1}.$$

Now we consider the function

$$r_n(u_{n+1}, u_n, u_{n-1}) = c_1 u_{n+1} + c_2 u_n + c_3 u_{n-1} + c_4. \tag{3.62}$$

Commutation relations (3.7), (3.44) become

$$[D_x, W_k] = a_k W_k + b_k W_{k-1}, \quad \left[D_x, \overline{W}_k\right] = \overline{a}_k \overline{W}_k + \overline{b}_k \overline{W}_{k-1}, \tag{3.63}$$

where

$$a_k = -(r_1 + k r_0), \qquad b_k = \frac{k-k^2}{2} c_2 - c_3 k, \tag{3.64}$$

$$\overline{a}_k = -(k r_1 + r_0), \qquad \overline{b}_k = \frac{k-k^2}{2} c_2 - c_1 k. \tag{3.65}$$

Assume that sequence $\{W_n\}$ is terminated at the step M:

$$W_M = \sum_{k=1}^{M-1} \Lambda_{M-k} W_{M-k} + \phi_1 Y_1 + \phi_0 Y_0. \tag{3.66}$$

We apply the operator $\mathrm{ad}_{\mathrm{D_x}}$ to both sides of identity (3.66)

$$a_M \left(\sum_{k=1}^{M-1} \Lambda_{M-k} W_{M-k} + \phi_1 Y_1 + \phi_0 Y_0 \right) + b_M W_{M-1}$$
$$= \sum_{k=1}^{M-2} \Lambda_{M-k} (a_{M-k} W_{M-k} + b_{M-k} W_{M-k-1})$$
$$+ \Lambda_1(-(r_1 + r_0) W_1 - c_3 Y_1 + c_1 Y_0) - \phi_1 r_1 Y_1 - \phi_0 r_0 Y_0.$$

We collect the coefficients at W_{M-1} in this identity:

$$\Lambda_{M-1}(a_M - a_{M-1}) + b_M = 0.$$

We substitute formulae (3.64),(3.65) into the last equation:

$$-\Lambda_{M-1}\big(c_1 u_1 + c_2 u_0 + c_3 u_{-1} + c_4\big) + \frac{M-M^2}{2} c_2 - c_3 M = 0.$$

A simple analysis of this equation shows that $\Lambda_{M-1} = 0$ and $c_3 = \frac{1-M}{2} c_2$. Then, collecting coefficients before W_{M-k}, $k = 2, \ldots, M-2$, we arrive at the equations

$$\Lambda_{M-k}(a_M - a_{M-k}) = 0, \quad k = 2, \ldots, M-2,$$

which implies $\Lambda_{M-k} = 0$, $k = 2, \ldots, M-2$. The coefficient at W_1 is $\Lambda_1(a_M + r_1 + r_0) = 0$. Then $\Lambda_1 = 0$. The coefficients at Y_1 and Y_0 read as $(a_M + r_1)\phi_1 = 0$, $(a_M + r_0)\phi_0 = 0$ and hence, $\phi_1 = \phi_0 = 0$. Thus, $W_M = 0$.

Similarly, if sequence $\{\overline{W}_k\}$ is terminated at step N, then $c_1 = \frac{1-N}{2} c_2$ and $\overline{W}_N = 0$. As a result, we obtain:

$$r_n(u_{n+1}, u_n, u_{n-1}) = \frac{1-N}{2} c_2 u_{n+1} + c_2 u_n + \frac{1-M}{2} c_2 u_{n-1} + c_4.$$

By rescaling $\frac{c_2}{2} u_i \to v_i$, the original lattice is reduced to a lattice of the same form with function r_n defined by the formula

$$r_n(u_{n+1}, u_n, u_{n-1}) = (1-N) u_{n+1} + 2u_n + (1-M) u_{n-1} + c,$$

where c is an arbitrary constant. If $4-M-N\neq 0$, then we exclude constant c by the shift transformation $u\to u-\frac{c}{4-M-N}$. If $M+N=4$, then $M=N=2$, and c is excluded by the transformation $u_n\to u_n+\frac{c}{2}n^2$. Thus, the function r_n becomes:

$$r_n(u_{n+1},u_n,u_{n-1})=(1-N)u_{n+1}+2u_n+(1-M)u_{n-1} \tag{3.67}$$

and, in particular,

$$r_n(u_{n+1},u_n,u_{n-1})=-u_{n+1}+2u_n-u_{n-1}. \tag{3.68}$$

We substitute (3.67) into (3.37), and we get that $k=M+N-2$. We substitute (3.68) into (3.37), and we get that $k=2$.

Let us consider lattice (2.6) when r_n is defined by (3.67). We impose cut-off conditions $u_0=0$, $u_{L+1}=0$ and we reduce this lattice to the following hyperbolic system:

$$\begin{aligned} &u_{1,xy}=(2u_1+pu_2)u_{1,y},\\ &u_{k,xy}=(qu_{k-1}+2u_k+pu_{k+1})u_{k,y},\quad 2\leqslant k\leqslant L-1,\\ &u_{L,xy}=(qu_{L-1}+2u_L)u_{L,y}, \end{aligned} \tag{3.69}$$

where $p=1-N$, $q=1-M$; we recall that $N>1$, $M>1$. This system is reduced by differential substitution $v_i=\ln u_{i,y}$ to the exponential system:

$$\begin{aligned} &v_{1,xy}=2e^{v_1}+pe^{v_2},\\ &v_{k,xy}=qe^{v_{k-1}}+2e^{v_k}+pe^{v_{k+1}},\quad 2\leqslant k\leqslant L-1,\\ &v_{L,xy}=qe^{v_{L-1}}+2e^{v_L}.\\ &\Big(\mathbf{v}_{xy}=Ae^{\mathbf{v}}\Big). \end{aligned} \tag{3.70}$$

We denote by A the matrix of coefficients before exponents in the right hand side of the system and we denote by $\mathbf{v}=(v_1,v_2,\dots,v_K)^T$, $e^{\mathbf{v}}=(e^{v_1},e^{v_2},\dots,e^{v_K})^T$ the column vectors. System (3.70) is related with the system

$$\begin{aligned} &w_{1,xy}=e^{2w_1+pw_2},\\ &w_{k,xy}=e^{qw_{k-1}+2w_k+pw_{k+1}},\quad 2\leqslant k\leqslant L-1,\\ &w_{L,xy}=e^{qw_{L-1}+2w_L}.\\ &\Big(\mathbf{w}_{xy}=e^{A\mathbf{w}}\Big). \end{aligned} \tag{3.71}$$

by the following point change of variables

$$\begin{aligned} &v_1=2w_1+pw_2,\\ &v_k=qw_{k-1}+2w_k+pw_{k+1},\quad 2\leqslant k\leqslant L-1,\\ &v_L=qw_{L-1}+2w_L.\\ &\Big(\mathbf{v}=A\mathbf{w}\Big). \end{aligned} \tag{3.72}$$

System (3.71) is reduced to system (3.69) by differential substitution

$$u_i=w_{i,x}. \tag{3.73}$$

It is shown in [11, 25] (see also [14]) that if A is the Cartan matrix of a simple Lie algebra, then the system (3.70) ((3.71)) is integrated in quadratures. Comparing the Cartan matrix and matrix A, one can see that $p=q=-1$. Thus, we have that $M=N=2$. In this case we find that $W_2=0$, $\overline{W}_2=0$, $Z_8=0$.

Let us show that if systems (3.70), (3.71) is integrable in the sense of Darboux then system (3.69) is integrable in the sense of Darboux, too. Suppose that $\overline{I}(\mathbf{w}_x,\mathbf{w}_{xx},\cdots)$ is an y-integral of system (3.71). We change variables by the rule $w_{i,x}=u_i$, $w_{i,xx}=u_{i,x}$ and so on, due to

(3.73), then we obtain an y-integral $\overline{I}(\mathbf{u}, \mathbf{u}_x, \cdots)$ of system (3.69). Assume that $I(\mathbf{v}_y, \mathbf{v}_{yy}, \cdots)$ is an x-integral of system (3.70). Using (3.73) and (3.72), we derive:

$$\mathbf{u} = \mathbf{w}_x = A^{-1}\mathbf{v}_x.$$

Hence, by virtue (3.70)

$$\mathbf{u}_y = A^{-1}\mathbf{v}_{xy} = A^{-1}Ae^{\mathbf{v}} = e^{\mathbf{v}}.$$

We change variables in the function $I(\mathbf{v}_y, \mathbf{v}_{yy}, \ldots)$ by the rule $v_i = \ln u_{i,y}$, $v_{i,y} = (\ln u_{i,y})_y$ and so on. Thus, we get an x-integral $I((\ln u_{i,y})_y, (\ln u_{i,y})_{yy}, \ldots)$ of system (3.69).

3.5.ii) Let us consider case (3.39) and $\lambda \neq 0$. We substitute r_n into system (3.59)–(3.61) and into equation (3.37), we get the following system:

$$\begin{aligned} A_1 e^{u_0 - hu_{-1}} + B_1 e^{u_1 - hu_0} &= 0, \\ A_2 e^{u_0 - hu_{-1}} + B_2 e^{u_1 - hu_0} &= 0, \\ A_3 e^{u_0 - hu_{-1}} + B_3 e^{u_1 - hu_0} &= 0, \\ A_4 e^{u_1 - hu_0} + B_4 e^{u_2 - hu_1} &= 0. \end{aligned}$$

Obviously, the coefficients A_i, B_i at independent exponent functions have to be equal to zero. Thus, we obtain a system of 8 algebraic equations in 8 unknowns $c, h, M, \lambda, \rho, \kappa, k, \nu_k$. Studying this system, we get the following possible variants:

$$M = 4, \quad k = \frac{10}{3}; \qquad M = 5, \quad k = \frac{17}{4}; \qquad M = 2, \quad k = 2.$$

All of these variants contradict our assumptions about values of k, M.

Thus, we have proved the following statement.

Lemma 3.7. *If relations* (3.9), (3.34), (3.45) *hold true for some* $M \geqslant 2$, $k \geqslant 2$, $N \geqslant 2$, *then the function* r_n *casts into one of the forms* (3.50) *or* (3.68) *up to point transformations.*

Lemma 2.2 is implied immediately by Lemma 3.7.

Summarizing the rezults of this section, we observe that we have lattice (1.2) for further study, where ther function r_n is defined by one of the formulae $r_n = r_n(u_n)$, (3.50), (3.68). Similarly, function p_n is defined by one of the following formulae:

$$\begin{aligned} &p_n = p_n(u_n), \\ &p_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}, \\ &p_n(u_{n+1}, u_n, u_{n-1}) = -u_{n+1} + 2u_n - u_{n-1}. \end{aligned}$$

4. Function q_n

We recall that the operator Y can be represented as follows, see formula (2.3):

$$Y = \sum_i u_{i,y} Y_i + R,$$

where

$$\begin{aligned} Y_i &= \frac{\partial}{\partial u_i} + r_i \frac{\partial}{\partial u_{i,x}} + \left(D_x(r_i) + r_i^2\right) \frac{\partial}{\partial u_{i,xx}} + \cdots \\ R &= \sum_i (u_{i,x} p_i + q_i) \frac{\partial}{\partial u_{i,x}} + \left(D_x(u_{i,x} p_i + q_i) + (u_{i,x} p_i + q_i) r_i\right) \frac{\partial}{\partial u_{i,xx}} + \cdots \end{aligned}$$

We shall determine the function q_n by using the operator R. We define a sequence of operators in the characteristic algebra $\mathcal{L}(y, N)$ by the following recurrent formula:

$$\begin{gathered} Y_{-1}, \quad Y_0, \quad Y_1, \quad Y_{0,-1} = [Y_0, Y_{-1}], \quad Y_{1,0} = [Y_1, Y_0], \\ R_0 = [Y_0, R], \quad R_1 = [Y_0, R_0], \quad R_2 = [Y_0, R_1], \quad \ldots \quad R_{k+1} = [Y_0, R_k]. \end{gathered} \tag{4.1}$$

For elements of the sequence the following commutation relations hold:

$$[D_x, Y_{-1}] = -r_{-1}Y_{-1}, \qquad [D_x, Y_0] = -r_0Y_0, \qquad [D_x, Y_1] = -r_1Y_1,$$
$$[D_x, Y_{0,-1}] = -(r_{-1}+r_0)Y_{0,-1} - Y_0(r_{-1})Y_{-1} + Y_{-1}(r_0)Y_0,$$
$$[D_x, Y_{1,0}] = -(r_0+r_1)Y_{1,0} - Y_1(r_0)Y_0 + Y_0(r_1)Y_1,$$
$$[D_x, R] = -\sum_i h_iY_i, \quad h_i = p_iu_{i,x} + q_i,$$
$$[D_x, R_0] = - r_0R_0 + h_1Y_{1,0} - h_{-1}Y_{0,-1} - Y_0(h_1)Y_1 + (R(r_0) - Y_0(h_0))Y_0 - Y_0(h_{-1})Y_{-1},$$
$$[D_x, R_1] = -2r_0R_1 - Y_0(r_0)R_0 + \cdots ,$$
$$[D_x, R_2] = -3r_0R_2 - 3Y_0(r_0)R_1 - Y_0^2(r_0)R_0 + \cdots ,$$
$$[D_x, R_3] = -4r_0R_3 - 6Y_0(r_0)R_2 - 4Y_0^2(r_0)R_1 - Y_0^3(r_0)R_0 + \cdots ,$$

where the dots stand for a linear combinations of operators $Y_{1,0}$, $Y_{0,-1}$, Y_1, y_0, Y_{-1}. By induction we prove that the following formula holds for all $n \geqslant 2$:

$$[D_x, R_n] = a_nR_n + b_nR_{n-1} + \cdots ,$$

where

$$a_n = -(n+1)r_0, \quad b_n = -\frac{n^2+n}{2}Y_0(r_0),$$

and the dots stand for a linear combinations of the operators R_k, $k < n-1$, $Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$.

Lemma 4.1. *If the operator R_0 is linearly expressed in terms of operators* (4.1)

$$R_0 = \mu Y_{1,0} + \tilde{\mu}Y_{0,-1} + \nu Y_1 + \eta Y_0 + \varepsilon Y_{-1}, \tag{4.2}$$

then chain (1.2) *is reduced to one of forms* (2.7), (2.8) *by point transformations.*

Proof. We apply the operator ad_{D_x} to both sides of identity (4.2). Collecting the coefficients at independent operators $Y_{1,0}$, $Y_{0,-1}$, Y_1, Y_0, Y_{-1}, we get the system of equations

$$D_x(\mu) = r_1\mu + h_1, \tag{4.3}$$
$$D_x(\tilde{\mu}) = r_{-1}\tilde{\mu} - h_{-1}, \tag{4.4}$$
$$D_x(\nu) = (r_1 - r_0)\nu - Y_0(h_1) - \mu Y_0(r_1), \tag{4.5}$$
$$D_x(\eta) = R(r_0) - Y_0(h_0) + \mu Y_1(r_0) - \tilde{\mu}Y_{-1}(r_0), \tag{4.6}$$
$$D_x(\varepsilon) = (r_{-1} - r_0)\varepsilon - Y_0(h_{-1}) + \tilde{\mu}Y_0(r_{-1}). \tag{4.7}$$

We consider equation (4.3):

$$r_1(u_2, u_1, u_0)\mu + p_1(u_2, u_1, u_0)u_{1,x} + q_1(u_2, u_1, u_0) = D_x(\mu).$$

A simple analysis of this equation shows that $\mu = \mu(u_1)$ and, hence, this equation splits into two equations

$$\mu'(u_1) = p_1(u_2, u_1, u_0), \quad r_1(u_2, u_1, u_0)\mu(u_1) + q_1(u_2, u_1, u_0) = 0.$$

Hence,

$$p_n(u_{n+1}, u_n, u_{n-1}) = \mu'(u_n), \quad q_n(u_{n+1}, u_n, u_{n-1}) = -r_n(u_{n+1}, u_n, u_{n-1})\mu(u_n). \tag{4.8}$$

Using equation (4.4), we obtain that $\tilde{\mu} = \tilde{\mu}(u_{-1})$, $\tilde{\mu}(v) = -\mu(v)$.

We simplify identity (4.5) using (4.8) and we get

$$D_x(\nu) = (r_1 - r_0)\nu.$$

It easy to see that $\nu = 0$. Similarly, it follows from (4.7) that $\varepsilon = 0$.

We simplify identity (4.6) as follows:

$$D_x(\eta) = -p_{0,u_0}u_{0,x} - q_{0,u_0} - r_0p_0 + \mu r_{0,u_1} - \tilde{\mu}r_{0,u_{-1}}.$$

A simple analysis of this equation shows that $\eta = \eta(u_0)$ and, hence, this equation splits into two equations

$$\eta'(u_0) = -p_{0,u_0}, \qquad -q_{0,u_0} - r_0 p_0 + \mu r_{0,u_1} - \tilde{\mu} r_{0,u_{-1}} = 0. \tag{4.9}$$

We substitute formulae (4.8) into identities (4.9) and we obtain $\eta'(u_0) = -\mu''(u_0)$,

$$r_{0,u_0}\mu(u_0) + \mu(u_1) r_{0,u_1} + \mu(u_{-1}) r_{0,u_{-1}} = 0. \tag{4.10}$$

We substitute the function r_n defined by formula (3.50) into (4.10), and we get that $\mu = c$ is an arbitrary constant. Therefore, $p_n = 0$, $q_n = -cr_n$, and lattice (1.2) becomes

$$u_{n,xy} = (e^{u_n - u_{n-1}} - e^{u_{n+1}-u_n}) u_{n,y} - c(e^{u_n - u_{n-1}} - e^{u_{n+1}-u_n}).$$

The transformation $u_n - cy \to u_n$ reduces this lattice to (2.7).

If r_n is defined by (3.68), then (4.10) implies $\mu = c$, where c is an arbitrary constant. Hence, $p_n = 0$, $q_n = -cr_n$, and lattice (1.2) takes the following form:

$$u_{n,xy} = (-u_{n+1} + 2u_n - u_{n-1}) u_{n,y} + c(-u_{n+1} + 2u_n - u_{n-1}).$$

The transformation $u_n - cy \to u_n$ reduces this lattice to (2.8).

If $r_n = r_n(u_n)$, then it follows from (4.10) that $\mu = 0$ or $r_{0,u_0} = 0$. In the first case formulae (4.8) imply $p_n = 0$, $q_n = 0$. Then chain (1.2) becomes $u_{n,xy} = r_n(u_n) u_{n,y}$. In the second case $r_0 = c_1$, where c_1 is an arbitrary constant, hence, by (4.8)), $p_n = \mu'(u_n)$, $q_n = -c_1\mu(u_n)$, and chain (1.2) casts into the form $u_{n,xy} = \mu'(u_n) u_{n,x} + c_1 u_{n,y} - c_1 \mu(u_n)$. The proof is complete. □

Suppose that $R_n$depends linearly on R_k, $k < n$, $Y_{1,0}$, $Y_{0,-1}$, Y_1, Y_0, Y_{-1} for some n:

$$R_n = \lambda R_{n-1} + \cdots, \quad n > 0. \tag{4.11}$$

Lemma 4.2. *If function r_n has one of forms* (3.50), (3.68)*, then case* (4.11) *is not realized.*

Proof. We apply the operator ad_{D_x} to both sides of identity (4.11). Collecting coefficients at R_{n-1} in obtained relation, we get the equation:

$$D_x(\lambda) = -r_0\lambda - \frac{n^2+n}{2} r_{0,u_0}.$$

A simple analysis of this equation shows that λ is a constant, hence

$$r_0\lambda + \frac{n^2+n}{2} r_{0,u_0} = 0. \tag{4.12}$$

Substituting formulae (3.50), (3.68) into (4.12), we get that $\lambda = 0$ and $n^2 + n = 0$. Hence, $n = 0$ or $n = -1$. Both solutions contradict the assumption $n > 0$. The proof is complete. □

heorem 1.1 is implied Lemma 4.1, 4.2.

5. Appendix. Proof of Lemma 3.6

The proof is a study of equation (3.37):

$$\nu_k r_1 + \frac{k(k-1)}{2} r_{1,u_1} + (k-1)(r_{0,u_1} + r_{2,u_1}) = 0 \tag{5.1}$$

in different cases (3.10) and (3.11) under conditions (1.4), $M \geqslant 2$, $k \geqslant 2$. We denote $\nu_k = \nu$.

i) We substitute function r_0 defined by formula (3.10) into (5.1)

$$\nu\left(\alpha(u_0) - \frac{2}{M-1}\alpha(u_1) + \delta(u_2)\right) - \frac{k(k-1)}{M-1}\alpha'(u_1) + (k-1)\big(\delta'(u_1) + \alpha'(u_1)\big) = 0. \tag{5.2}$$

We apply the operator $\frac{\partial}{\partial u_2}$ to this identity, and we get $\nu\delta'(u_2) = 0$. It is easy to show that the case $\nu \neq 0$ leads us to a contradiction to (1.4). Assume that $\nu = 0$, then from (5.2) we obtain that the function r_0 becomes

$$r_0(u_1, u_0, u_{-1}) = \alpha(u_{-1}) - \frac{2}{M-1}\alpha(u_0) + \left(\frac{k}{M-1} - 1\right)\alpha(u_1) + c_1.$$

ii) We substitute the function r_0 defined by formula (3.11) into equation (5.1).

$$\begin{aligned}&\frac{\beta(u_0)(\nu M^2-\nu M+k\lambda-k^2\lambda)}{M(M-1)}e^{-\frac{2\lambda u_1}{M(M-1)}}+(k-1)\beta'(u_1)e^{-\frac{2\lambda u_2}{M(M-1)}}\\&+\nu\psi(u_1,u_2)+(k-1)\frac{\partial\psi(u_0,u_1)}{\partial u_1}+\frac{1}{2}(k-1)k\frac{\partial\psi(u_1,u_2)}{\partial u_1}=0.\end{aligned}\tag{5.3}$$

We apply the operator $\frac{\partial}{\partial u_0}$ to both sides of identity (5.3)

$$\frac{\beta'(u_0)(\nu M^2-\nu M+k\lambda-k^2\lambda)}{M(M-1)}e^{-\frac{2\lambda u_1}{M(M-1)}}+(k-1)\frac{\partial^2\psi(u_0,u_1)}{\partial u_0\partial u_1}=0.$$

This equation has the following solution::

$$\psi(u_0,u_1)=\frac{\beta(u_0)(\nu M^2-\nu M+k\lambda-k^2\lambda)}{2(k-1)\lambda}e^{-\frac{2\lambda u_1}{M(M-1)}}+F_1(u_0)+F_2(u_1).\tag{5.4}$$

We substitute function (5.4) into equation (5.3), then we differentiate an obtained identity with respect to u_2, and we multiple both sides of the obtained identity by $e^{\frac{2\lambda u_2}{M(M-1)}}$:

$$\begin{aligned}&-\frac{\nu(\nu M^2-\nu M-k^2\lambda+k\lambda)\beta(u_1)}{M(k-1)(M-1)}\\&-\frac{1}{2}\frac{(\lambda k^3+k^2\lambda+kM^2\nu-kM\nu+4k\lambda-4\lambda)\beta'(u_1)}{(M-1)M}+F_2'(u_2)e^{\frac{2\lambda u_2}{M(M-1)}}\nu=0.\end{aligned}\tag{5.5}$$

Let us consider two different cases $\nu=0$ and $\nu\neq 0$.

ii.1) If $\nu=0$, then (5.5) becomes

$$\frac{1}{2}\frac{\lambda(k-1)(k-2)(k+2)\beta'(u_1)}{M(M-1)}=0\tag{5.6}$$

It follows from this identity that $k=2$ or $\beta'(u_1)=0$.

ii.1.1) If $k=2$, then equation (5.3) casts into the form $F_1'(u_1)+F_2'(u_1)=0$. It is clear that $F_2(u_1)=-F_1(u_1)+c_1$. Equation (3.12) becomes

$$\begin{aligned}&\left(-\lambda\beta(u_0)-\frac{1}{2}\beta'(u_0)M^2+\frac{3}{2}M\beta'(u_0)\right)e^{-\frac{2\lambda u_1}{M(M-1)}}\\&+\frac{1}{2}M(M-1)F_1'(u_0)+\lambda(F_1(u_0)-F_1(u_1)+c_1)=0.\end{aligned}\tag{5.7}$$

We apply the operator $\frac{\partial^2}{\partial u_1\partial u_0}$ to both sides of identity (5.7)

$$\lambda e^{-\frac{2\lambda u_1}{M(M-1)}}\frac{2\lambda\beta'(u_0)+M(M-3)\beta''(u_0)}{M(M-1)}=0.$$

By the condition $\lambda\neq 0$ we see that

$$2\lambda\beta'(u_0)+M(M-3)\beta''(u_0)=0.\tag{5.8}$$

ii.1.1.1) If $M=3$, then $\beta(u_0)=c_0$, where c_0 is an arbitrary constant. The function r_0 defined by formula (3.11) becomes

$$r_0(u_1,u_0,u_{-1})=c_0e^{-\frac{1}{3}\lambda u_0}-c_0e^{-\frac{1}{3}\lambda u_1}+F_1(u_0)-F_1(u_1)+c_1,\tag{5.9}$$

and equation (3.12) reads as

$$-\lambda c_0e^{-\frac{1}{3}\lambda u_1}-\lambda F_1(u_1)+\lambda F_1(u_0)+\lambda c_1+3F_1'(u_0)=0.\tag{5.10}$$

We apply the operator $\frac{\partial}{\partial u_1}$ to identity (5.10) :

$$\frac{1}{3}\lambda^2c_0e^{-\frac{1}{3}\lambda u_1}-\lambda F_1'(u_1)=0,$$

hence,

$$F_1(u_1) = -c_0 e^{-\frac{1}{3}\lambda u_1} + C_1.$$

We substitute F_1 into (5.9) and we get $r_0 = C_1$. This contradicts condition (1.4).

ii.1.1.2) If $M \neq 3$, then equation (5.8) has the solution

$$\beta(u_0) = C_1 + C_2 e^{-\frac{2\lambda u_0}{M(M-3)}}.$$

We differentiate equation (3.12) with respect to u_1 and, since $\lambda \neq 0$, this equation gives

$$F_1(u_1) = -C_1 e^{-\frac{2\lambda u_1}{M(M-1)}} + C_2.$$

Equation (3.12) becomes $\lambda c_1 = 0$, hence, $c_1 = 0$, and, finally,

$$r_0(u_1, u_0, u_{-1}) = C_2 e^{-\frac{2\lambda}{M(M-1)}u_0 - \frac{2\lambda}{M(M-3)}u_{-1}} - C_2 e^{-\frac{2\lambda}{M(M-3)}u_0 - \frac{2\lambda}{M(M-1)}u_1}. \tag{5.11}$$

We return back to equation (5.6) and consider the following case.

ii.1.2) If $\beta'(u_1) = 0$, then $\beta(u_1) = c_3$, where c_3 is an arbitrary constant. By equation (5.1) we find

$$F_2(u_1) = -\frac{1}{2}kF_1(u_1) + c_4.$$

Equation (3.12) is transformed as

$$\lambda F_1(u_0) + \frac{1}{2}M(M-1)F_1'(u_0) - \frac{1}{2}\lambda\left(c_3 k e^{-\frac{2\lambda u_1}{M(M-1)}} + kF_1(u_1) - 2c_4\right) = 0. \tag{5.12}$$

We apply the operator $\frac{\partial}{\partial u_1}$ to both sides of identity (5.12)

$$-\frac{1}{2}\frac{k\lambda}{M(M-1)}\left(-2c_3\lambda e^{-\frac{2\lambda u_1}{M(M-1)}} + M(M-1)F_1'(u_1)\right) = 0.$$

This equation has the solution

$$F_1(u_1) = -c_3 e^{-\frac{2\lambda u_1}{M(M-1)}} + c_5.$$

We substitute F_1 into (5.12), and we find c_4: $c_4 = \frac{1}{2}c_5(k-2)$. We substitute the found functions and constants into (3.11) and we get $r_0(u_1, u_0, u_{-1}) = 0$, which contradicts condition (1.4).

We return back to equation (5.5).

ii.2) If $\nu \neq 0$, then $F_2'(u_2)e^{\frac{2\lambda u_2}{M(M-1)}} = c_1$ and, hence,

$$F_2(u_2) = -\frac{1}{2}\frac{M(M-1)c_1 e^{-\frac{2\lambda u_2}{M(M-1)}}}{\lambda} + c_2.$$

Equation (5.5) reads as

$$\begin{aligned} &-\frac{\nu(\nu M^2 - \nu M - k^2\lambda + k\lambda)\beta(u_1)}{M(k-1)(M-1)} \\ &\quad -\frac{1}{2}\frac{(\lambda k^3 + k^2\lambda + kM^2\nu - kM\nu + 4k\lambda - 4\lambda)\beta'(u_1)}{(M-1)M} + c_1\nu = 0. \end{aligned} \tag{5.13}$$

We denote:

$$A = \nu M^2 - \nu M - k^2\lambda + k\lambda, \tag{5.14}$$

$$B = \lambda k^3 + k^2\lambda + kM^2\nu - kM\nu + 4k\lambda - 4\lambda. \tag{5.15}$$

We shall consider the following different cases:

ii.2.1) $A = 0$, $B = 0$;
ii.2.2) $A = 0$, $B \neq 0$;
ii.2.3) $A \neq 0$, $B = 0$;
ii.2.4) $A \neq 0$, $B \neq 0$.

In case ii.2.1), that is, as

$$\nu M^2 - \nu M - k^2\lambda + k\lambda = 0, \qquad \lambda k^3 + k^2\lambda + kM^2\nu - kM\nu + 4k\lambda - 4\lambda = 0.$$

Then we express ν from the first equation and we substitute this function into the second equation, and we get $4(k-1)\lambda = 0$, which contradicts to $k \geqslant 2$, $\lambda \neq 0$.

ii.2.2) Assume that

$$A = \nu M^2 - \nu M - k^2\lambda + k\lambda = 0.$$

We express ν from this identity and we substitute ν into (5.13). This equation has the solution $\beta(u_1) = \frac{1}{2}kc_1u_1 + c_3$. Equation (5.3) becomes

$$\frac{1}{2}k(k-1)\frac{dF_1(u_1)}{du_1} + \frac{k(k-1)\lambda}{M(M-1)}F_1(u_1) + c_1(k-1)e^{-\frac{2\lambda u_1}{M(M-1)}} + \frac{k(k-1)c_2\lambda}{M(M-1)} = 0.$$

This equation has the solution

$$F_1(u_1) = -\frac{(2c_1u_1 - c_4k)}{k}e^{-\frac{2\lambda u_1}{M(M-1)}} - c_2.$$

Equation (3.12) casts into the form

$$-\frac{c_1M(M-1)}{k}e^{-\frac{2\lambda u_0}{M(M-1)}} - \frac{c_1M(M-k-1)}{2}e^{-\frac{2\lambda u_1}{M(M-1)}} = 0.$$

It is clear that this identity holds true only if $c_1 = 0$ (we are working under the condition $M \geqslant 2$). Hence, we have

$$r_0(u_1, u_0, u_{-1}) = (c_3 + c_4)e^{-\frac{2\lambda u_0}{M(M-1)}},$$

which contradicts condition (1.4).

ii.2.3) Suppose that

$$B = \lambda k^3 + k^2\lambda + kM^2\nu - kM\nu + 4k\lambda - 4\lambda = 0.$$

We express ν:

$$\nu = \frac{\lambda(k-1)(k-2)(k+2)}{kM(M-1)}.$$

Since $\nu \neq 0$, then $k \neq 2$. Equation (5.13) becomes

$$\frac{\lambda(k-1)(k-2)(k+2)}{k^2M^2(M-1)^2}(c_1kM^2 - c_1kM + 4\lambda\beta(u_1)) = 0.$$

We find the function β:

$$\beta(u_1) = -\frac{1}{4}\frac{c_1kM(M-1)}{\lambda}. \tag{5.16}$$

Taking into consideration the obtained function, we simplify equation (5.3):

$$\frac{1}{2}k(k-1)\frac{dF_1(u_1)}{du_1} + \frac{\lambda(k-1)(k-2)(k+2)}{kM(M-1)}F_1(u_1) + (k-1)c_1e^{-\frac{2\lambda u_1}{M(M-1)}} + \frac{c_2\lambda(k-2)(k-1)(k+2)}{kM(M-1)} = 0.$$

This equation has the solution:

$$F_1(u_1) = \frac{c_1kM(M-1)}{4\lambda}e^{-\frac{2\lambda u_1}{M(M-1)}} + C_1e^{-\frac{2\lambda(k-2)(k+2)}{k^2M(M-1)}u_1} - c_2.$$

Let us transform equation (3.12)

$$\frac{4\lambda C_1}{k^2}e^{-\frac{2\lambda(k-2)(k+2)}{k^2M(M-1)}u_1} = 0.$$

It follows from this identity that $C_1 = 0$. Substitution found functions and constants into (3.11), we obtain that $r_0 = 0$, that contradicts condition (1.4).

ii.2.4) If $A \neq 0$ and $B \neq 0$, then equation (5.13) has the solution

$$\beta(u_1) = c_3 e^{-\frac{2\nu(\nu M^2 - \nu M - k^2\lambda + k\lambda)u_1}{(k-1)(-\lambda k^3 + k^2\lambda + 4k\lambda + kM^2\nu - kM\nu - 4\lambda)}} + \frac{Mc_1(k-1)(M-1)}{\nu M^2 - \nu M - k^2\lambda + k\lambda}. \tag{5.17}$$

We substitute (5.17) into (5.3) and we obtain

$$\frac{1}{2}k(k-1)\frac{dF_1(u_1)}{du_1} + \nu F_1(u_1) + c_1(k-1)e^{-\frac{2\lambda u_1}{M(M-1)}} + \nu c_2. \tag{5.18}$$

Equation (5.18) has the solution

$$F_1(u_1) = -c_2 + c_4 e^{-\frac{2\nu u_1}{k(k-1)}} - \frac{c_1 M(M-1)(k-1)}{\nu M^2 - \nu M - k^2\lambda + k\lambda} e^{-\frac{2\lambda u_1}{M(M-1)}}.$$

Function (3.11) becomes

$$r_0(u_1, u_0, u_{-1}) = c_4 e^{-\frac{2\nu u_0}{k(k-1)}} + c_3 e^{-\frac{2\lambda u_0}{M(M-1)} - \frac{2\nu A u_{-1}}{B(k-1)}} + \frac{Ac_3}{2(k-1)\lambda} e^{-\frac{2\lambda u_1}{M(M-1)} - \frac{2\nu A u_0}{B(k-1)}}.$$

Here A, B are defined by formulae (5.14), (5.15). We substitute these functions into (3.12)

$$-\frac{Ac_4}{k(k-1)} e^{-\frac{2\nu u_0}{k(k-1)}} - \frac{Ac_3(\lambda B + \nu M^2 A - \nu MA - \lambda kB - 4M\nu\lambda + 4M\nu k\lambda)}{2Bk\lambda(k-1)^2} e^{-\frac{2\lambda u_1}{M(M-1)} - \frac{2\nu A u_0}{B(k-1)}} = 0.$$

Since $A \neq 0$, $\nu \neq 0$, it follows from the last identity that $c_4 = 0$ and

$$\lambda B + \nu M^2 A - \nu MA - \lambda kB - 4M\nu\lambda + 4M\nu k\lambda = 0.$$

Thus, we have specified the function r_0:

$$r_0(u_1, u_0, u_{-1}) = c_3 e^{-\frac{2\lambda u_0}{M(M-1)} - \frac{2\nu A u_{-1}}{B(k-1)}} + \frac{Ac_3}{2(k-1)\lambda} e^{-\frac{2\lambda u_1}{M(M-1)} - \frac{2\nu A u_0}{B(k-1)}}.$$

We can rewrite r_0 in the following form:

$$r_0(u_1, u_0, u_{-1}) = C_1 e^{h_1 u_0 - h_2 u_{-1}} + C_2 e^{h_1 u_1 - h_2 u_0},$$

where $C_1 C_2 \neq 0$, $h_1 h_2 \neq 0$ are some constants.

Lattice (1.2) is reduced to one with r_n of the following form

$$r_n(u_{n+1}, u_n, u_{n-1}) = e^{u_n - hu_{n-1}} + ce^{u_{n+1} - hu_n}$$

by rescaling $h_1 u_n \to u_n$, $c_1 h_1 x \to x$. Similarly transformations one can apply to the lattice in case ii.1.1.2 (see (5.11)). The proof of Lemma 3.6 is complete.

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