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ON SERIES OF DARBOUX INTEGRABLE DISCRETE EQUATIONS ON SQUARE LATTICE

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Abstract. We present a series of Darboux integrable discrete equations on a square lattice. The equations in the series are numbered by natural numbers M. All the equations possess a first order first integral in one of directions of the two-dimensional lattice. The minimal order of a first integral in the other direction is equal to $3M$ for an equation with the number M. First integrals in the second direction are defined by a simple general formula depending on the number M.

In the cases $M = 1, 2, 3$ we show that the equations are integrable by quadrature. More precisely, we construct their general solutions in terms of the discrete integrals.

We also construct a modified series of Darboux integrable discrete equations having the first integrals of the minimal orders 2 and $3M - 1$ in different directions, where M is the equation number in series. Both first integrals are not obvious in this case.

A few similar series of integrable equations were known before; however, they were of Burgers or sine-Gordon type. A similar series of the continuous hyperbolic type equations was discussed by A.V. Zhiber and V.V. Sokolov in 2001.

Keywords: discrete quad-equation, Darboux integrability, general solution.

Mathematics Subject Classification: 39A14, 39A10, 35L10

1. Introduction

We consider discrete equations of the form:

$$(u_{n+1,m+1} + 1)(u_{n,m+1} - 1) = \theta(u_{n+1,m} + 1)(u_{n,m} - 1), \tag{1}$$

where $n, m \in \mathbb{Z}$ and θ is a constant coefficient. Two integrable equations of this form are known. The case $\theta = 1$ was presented in [12] and the case $\theta = -1$ was found in [2]. In both the cases the equations are Darboux integrable. Both equations have the first order first integral in the n-direction, while in the m-direction the minimal orders of first integrals are 3 and 6, respectively.

We present a series of equations of the form (1) with special coefficients $\theta = \theta_M$, $M \in \mathbb{N}$, including two above examples. All equations are Darboux integrable and possess a first order first integral in the n-direction. The minimal order of a first integral $W_{2,M}$ in the m-direction is equal to $3M$ for an equation with the number M and hence, these equations may have first integrals of an arbitrarily high minimal order in the m-direction.

A few similar series of integrable equations were known before. In [5], a series of Darboux integrable discrete equations was discussed, which however were of Burgers type. The minimal orders of first integrals in both directions could be arbitrarily high there. An analogous series of the continuous hyperbolic type equations was studied in [14]. In [7], a series of sine-Gordon type autonomous discrete equations was presented. Autonomous generalized symmetries and

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conservation laws in both directions could have arbitrarily high minimal orders in this case. Some series of non-autonomous discrete equations of sine-Gordon type were studied in [3].

It is interesting to construct the general solutions for considered equations of the form (1) with such a high minimal order of first integrals. Following [6, 8, 9], we succeed to do this in the cases $M = 1, 2, 3$, where the first integrals $W_{2,M}$ have the minimal orders 3, 6 and 9. In the case $M = 1$ the general solution is constructed explicitly and coincides with the known solution [12]. In the cases $M = 2, 3$ we show that the equations are integrable by quadrature. This means that one can construct general solutions in terms of the discrete integrals.

Using non-point transformations invertible on the solutions of discrete equations [13, 12], we construct one more series of Darboux integrable discrete equations. Equations of this series possess first integrals of the minimal orders 2 and $3M - 1$ in different directions, where M is the equation number in the series. Both first integrals are not obvious in this case.

In Section 2 we introduce a series of discrete equations, prove their Darboux integrability, and discuss the minimal orders of first integrals. The general solutions in the cases $M = 1, 2, 3$ are constructed in Section 3. A modified series of integrable discrete equations is discussed in Section 4.

2. Darboux integrability

We are going to study the following series of discrete equations:

$$(u_{n+1,m+1} + 1)(u_{n,m+1} - 1) = \theta_M(u_{n+1,m} + 1)(u_{n,m} - 1), \tag{2}$$

where θ_M is the primitive root of unit of the degree $M \in \mathbb{N}$. More precisely, $\theta_1 = 1$ and for $M > 1$ one has:

$$\theta_M^M = 1, \qquad \theta_M^j \neq 1, \qquad 1 \leqslant j \leqslant M - 1. \tag{3}$$

For example, all the primitive roots of the degree $M \leqslant 4$ are

$$\theta_1 = 1, \quad \theta_2 = -1, \quad \theta_3 = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \quad \theta_4 = \pm i. \tag{4}$$

For every $M > 2$ we have at least two values of θ_M:

$$\theta_M = \exp\left(\pm 2\pi \frac{i}{M}\right) = \cos\frac{2\pi}{M} \pm i \sin\frac{2\pi}{M}.$$

An equation of the form

$$F(u_{n+1,m+1}, u_{n+1,m}, u_{n,m+1}, u_{n,m}) = 0 \tag{5}$$

is called Darboux integrable if it has two first integrals W_1, W_2 such that

$$(T_n - 1)W_2 = 0, \quad W_2 = w_{n,m}^{(2)}(u_{n,m+l}, u_{n,m+l-1}, \ldots, u_{n,m}), \tag{6}$$

$$(T_m - 1)W_1 = 0, \quad W_1 = w_{n,m}^{(1)}(u_{n+k,m}, u_{n+k-1,m}, \ldots, u_{n,m}). \tag{7}$$

Here l, k are some positive integers, and T_n, T_m are operators of the shift in the n- and m-directions, respectively: $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$. We suppose that relations (6), (7) hold identically on the solutions of the corresponding equation (5).

The functions W_1 and W_2 are called the first integrals in the n- and m-directions, respectively. We assume here that each of the conditions

$$\frac{\partial W_1}{\partial u_{n,m}} \neq 0, \qquad \frac{\partial W_1}{\partial u_{n+k,m}} \neq 0, \qquad \frac{\partial W_2}{\partial u_{n,m}} \neq 0, \qquad \frac{\partial W_2}{\partial u_{n,m+l}} \neq 0 \tag{8}$$

is satisfied for at least some n, m. The numbers k, l are called the orders of these first integrals W_1, W_2, respectively.

The case $M = 1$ is known, see equation (4.6) in [12]; equation (4.6) is obtained from (2) with $\theta_M = 1$ by the point transformation

$$v_{n,m} = \frac{1 - u_{n,m}}{1 + u_{n,m}}.$$

By constructing the first integrals in both directions, it is shown in [12] that this equation is Darboux integrable and its general solution was found.

The case $M = 2$ is also known, see equation (51a) in [2]. The first integrals in both directions were found for this equation, see relations (53) in [2].

For each M, equation (2) has the following first integral in the n-direction:

$$W_{1,M} = (\theta_M)^{-m}(u_{n+1,m} + 1)(u_{n,m} - 1). \tag{9}$$

This is true since equation (2) is equivalent to the identity

$$(T_m - \theta_M)[(u_{n+1,m} + 1)(u_{n,m} - 1)] = 0. \tag{10}$$

Moreover, formula (9) with θ_M replaced by θ provides the first integral for equation (1) for each θ. As for the m-direction, we succeed to find the following formula:

$$W_{2,M} = \frac{(u_{n,m+3M} - u_{n,m+M})(u_{n,m+2M} - u_{n,m})}{(u_{n,m+3M} - u_{n,m+2M})(u_{n,m+M} - u_{n,m})}, \tag{11}$$

which provides first integrals for all equations (2). It is easy to see that these integrals (9) and (11) have the orders 1 and $3M$, respectively. Conditions (8) are satisfied for all n, m in this case. The fact that formula (11) defines a first integral in the case $M = 1$ is checked by straightforward calculations.

Theorem 1. *The function $W_{2,M}$ defined by* (11) *is the first integral of equation (2), (3) in the m-direction for each $M > 1$.*

Proof. We denote

$$\Psi_{n,m} = (u_{n+1,m} + 1)(u_{n,m} - 1). \tag{12}$$

By (2) we have

$$T_m \Psi_{n,m} = \theta_M \Psi_{n,m}, \tag{13}$$

and therefore,

$$T_m^M \Psi_{n,m} = \Psi_{n,m} \tag{14}$$

for equations (2) satisfying condition (3). Using the notation

$$u^{(j)}_{n,k} = u_{n,Mk+j}, \qquad 1 \leqslant j \leqslant M, \tag{15}$$

we rewrite (14) as a system:

$$(u^{(j)}_{n+1,k+1} + 1)(u^{(j)}_{n,k+1} - 1) = (u^{(j)}_{n+1,k} + 1)(u^{(j)}_{n,k} - 1), \qquad 1 \leqslant j \leqslant M. \tag{16}$$

We see that all equations in (16) are independent and each of them coincides with (2), in which $M = 1$ and m is replaced by k.

For this reason, for each equation of system (16) we can use the first integral $W_{2,1}$ of equation (2). As a result we get:

$$W_2^{(j)} = \frac{(u^{(j)}_{n,k+3} - u^{(j)}_{n,k+1})(u^{(j)}_{n,k+2} - u^{(j)}_{n,k})}{(u^{(j)}_{n,k+3} - u^{(j)}_{n,k+2})(u^{(j)}_{n,k+1} - u^{(j)}_{n,k})}. \tag{17}$$

Taking into consideration (15) we are led to the identity

$$W_2^{(j)} = \frac{(u_{n,M(k+3)+j} - u_{n,M(k+1)+j})(u_{n,M(k+2)+j} - u_{n,Mk+j})}{(u_{n,M(k+3)+j} - u_{n,M(k+2)+j})(u_{n,M(k+1)+j} - u_{n,Mk+j})}. \tag{18}$$

Denoting $m = Mk + j$, we see that the relation $T_n W_2^{(j)} = W_2^{(j)}$ implies $T_n W_{2,M} = W_{2,M}$ for all n, m. Since (16) is equivalent to (14), the last relation is satisfied on each solution of equation (14) and hence of (13). □

The first integral $W_{1,M}$ obviously has the lowest possible order for each M. As it is known from [12, 2], in the cases $M = 1$ and $M = 2$ the integral $W_{2,M}$ also has the lowest possible order in its direction. The same is true for the case $M = 3$ as it follows from the next theorem.

Theorem 2. *Equation (2), (3) with $M = 3$ has no first integral of the order $l < 9$ in the m-direction.*

In order to prove this theorem, we apply a method described in detail in [4, Sect. 2.2]. That method uses so-called annihilation operators introduced in [10] and allows one to find the first integrals. In the framework of this method, the proof reduces to straightforward but cumbersome calculations.

The following conjecture seems to be true: first integral (11) of equation (2), (3) has the lowest possible order for each $M > 1$.

3. General solutions

We use and improve a method developed in [6, Section 5.2], [9, Section 4], [8]. We construct the general solutions for equations (2) with $M = 1, 2, 3$. In the case $M = 1$, the solution is explicit and coincides with a solution of [12] up to the Möbius transformation

$$v_{n,m} = \frac{1 - u_{n,m}}{1 + u_{n,m}}.$$

Further calculations are however needed in the cases $M = 2, 3$. Here the general solutions are given in terms of discrete integrals in terminology of [8], i.e., equations (2) with $M = 2, 3$ are solved by quadrature.

Let us consider an ordinary discrete equation

$$a_{n+1} - a_n = A_n, \tag{19}$$

where a_n is an unknown function and A_n is given. We say that a_n is found by a discrete integration similar to the ordinary differential equation $a'(x) = A(x)$, and the solution a_n of equation (19) is called the discrete integral of A_n.

The explicit general solution of equation (5) is called a function of the form $u_{n,m} = \Phi_{n,m}[a_n, b_m]$, where a_n, b_m are arbitrary functions of one variable. Here the square brackets mean that the function $\Phi_{n,m}$ depends on finitely many shifts a_{n+j}, b_{m+j}. Such a solution is to satisfy identically equation (5) for all values of the functions a_n, b_m. For example, the discrete wave equation

$$u_{n+1,m+1} - u_{n+1,m} - u_{n,m+1} + u_{n,m} = 0$$

has the following general solution:

$$u_{n,m} = a_n + b_m.$$

Equation (5) is solved by quadrature if it has a solution of the form:

$$u_{n,m} = \Phi_{n,m}\left[a_n, b_m, a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(j_1)}, b_m^{(1)}, b_m^{(2)}, \ldots, b_m^{(j_2)}\right], \tag{20}$$

where a_n, b_m are arbitrary functions and the square brackets mean, as above, that the function $\Phi_{n,m}$ depends on a finite number of the shifts of its arguments. The functions $a_n^{(j)}$ are obtained from a_n by a finite number of applications of the shift operator T_n, of the functions of many variables, and of the discrete integrations. The functions $b_m^{(j)}$ are obtained from b_m analogously. So, the functions $a_n^{(j)}, b_m^{(j)}$ and therefore solution (20) are implicit in a sense.

3.1. Case $M=1$. Equation (2) is equivalent to

$$(u_{n+1,m}+1)(u_{n,m}-1)=\lambda_n, \tag{21}$$

where λ_n is an arbitrary function. This is a discrete Riccati equation, and we need to know its particular solution in order to linearize and then to solve it. We cannot solve this equation for a given λ_n in general case. We use the fact that the function λ_n is arbitrary and replace λ_n by another arbitrary function α_n which plays the role of particular solution:

$$\lambda_n=(\alpha_{n+1}+1)(\alpha_n-1). \tag{22}$$

In accordance with the known method of solving the Riccati equation, we use the transformation

$$u_{n,m}=\alpha_n+\frac{\alpha_n-1}{v_{n,m}} \tag{23}$$

to get a linear equation for $v_{n,m}$:

$$\frac{\alpha_{n+1}+1}{\alpha_{n+1}-1}v_{n+1,m}+v_{n,m}+1=0. \tag{24}$$

To solve this equation, it is convenient to introduce a new arbitrary function β_n instead of α_n as follows:

$$\frac{\alpha_{n+1}+1}{\alpha_{n+1}-1}=-\frac{\beta_{n+2}-\beta_{n+1}}{\beta_{n+1}-\beta_n}. \tag{25}$$

Here we follow [7], see (39), (40). Representing equation (24) in the form

$$(T_n-1)[(\beta_n-\beta_{n+1})v_{n,m}+\beta_n]=0, \tag{26}$$

we find

$$v_{n,m}=\frac{\beta_n+\omega_m}{\beta_{n+1}-\beta_n}, \tag{27}$$

where ω_m is another arbitrary function. Finally, using (23), (25), (27), we find $u_{n,m}$:

$$u_{n,m}=\frac{\beta_{n+1}-2\beta_n+\beta_{n-1}}{\beta_{n+1}-\beta_{n-1}}-2\frac{(\beta_{n+1}-\beta_n)(\beta_n-\beta_{n-1})}{(\beta_{n+1}-\beta_{n-1})(\beta_n+\omega_m)}. \tag{28}$$

It is easy to check that function (28) satisfies equation (2) with $M=1$ for all values of the arbitrary functions β_n, ω_m. Thus, we have found explicitly the general solution to (2) with $M=1$.

3.2. Cases $M=2$ and $M=3$. Equation (2) is equivalent to

$$(u_{n+1,m}+1)(u_{n,m}-1)=\theta_M^m\lambda_n, \tag{29}$$

where λ_n is an arbitrary function. It is convenient to pass from (29) to an equivalent system by means of transformation (15):

$$(u^{(j)}_{n+1,k}+1)(u^{(j)}_{n,k}-1)=\theta_M^j\lambda_n, \quad 1\leqslant j\leqslant M. \tag{30}$$

We note that j is a number of the function $u^{(j)}_{n,k}$, while n and k are discrete variables. Unlike (29), the right hand side of equations (30) depends on one discrete variable n only, as in the case of (21).

By analogy with the previous case $M=1$, we can introduce functions $\alpha_n^{(j)}$ so that:

$$\theta_M^j\lambda_n=(\alpha^{(j)}_{n+1}+1)(\alpha^{(j)}_n-1), \qquad 1\leqslant j\leqslant M. \tag{31}$$

Now we can apply the transformations

$$u^{(j)}_{n,k}=\alpha^{(j)}_n+\frac{\alpha^{(j)}_n-1}{v^{(j)}_{n,k}} \tag{32}$$

to get the linear equations for $v_{n,k}^{(j)}$:

$$\frac{\alpha_{n+1}^{(j)}+1}{\alpha_{n+1}^{(j)}-1}v_{n+1,k}^{(j)}+v_{n,k}^{(j)}+1=0. \tag{33}$$

As above, we introduce functions $\beta_n^{(j)}$ such that

$$\frac{\alpha_{n+1}^{(j)}+1}{\alpha_{n+1}^{(j)}-1}=-\frac{\beta_{n+2}^{(j)}-\beta_{n+1}^{(j)}}{\beta_{n+1}^{(j)}-\beta_n^{(j)}}, \tag{34}$$

and we obtain

$$v_{n,k}^{(j)}=\frac{\beta_n^{(j)}+\omega_k^{(j)}}{\beta_{n+1}^{(j)}-\beta_n^{(j)}}, \tag{35}$$

where $\omega_k^{(j)}$ are arbitrary functions on k.

For $u_{n,k}^{(j)}$, we can write formulae similar to (28). The problem is that, instead of one n-dependent arbitrary function β_n in the case $M=1$, we have now M functions $\beta_n^{(j)}$ with a complex relationship between them defined by (31) and (34). We can solve this problem for $M=2$ and $M=3$ in terms of the discrete integrals.

Case $M=2$**.** Excluding λ_n from system (31), we obtain an identity relating the functions $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$:

$$(\alpha_{n+1}^{(1)}+1)(\alpha_n^{(1)}-1)=-(\alpha_{n+1}^{(2)}+1)(\alpha_n^{(2)}-1). \tag{36}$$

If one of these functions is known, then the second function is found by the Riccati equation. If we replace $\alpha_n^{(j)}$ by $\beta_n^{(j)}$ by the identities

$$\alpha_n^{(j)}=\frac{\beta_{n+1}^{(j)}-2\beta_n^{(j)}+\beta_{n-1}^{(j)}}{\beta_{n+1}^{(j)}-\beta_{n-1}^{(j)}}, \tag{37}$$

then a new relation between the functions $\beta_n^{(1)}$ and $\beta_n^{(2)}$ is even more complex.

In order to solve this problem, we rewrite (36) as

$$\frac{\alpha_n^{(1)}-1}{\alpha_n^{(2)}-1}=-\frac{\alpha_{n+1}^{(2)}+1}{\alpha_{n+1}^{(1)}+1}. \tag{38}$$

Denoting the left hand side by γ_{n+1}, we get a system for $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$:

$$\frac{\alpha_n^{(1)}-1}{\alpha_n^{(2)}-1}=\gamma_{n+1}, \qquad \frac{\alpha_n^{(2)}+1}{\alpha_n^{(1)}+1}=-\gamma_n. \tag{39}$$

Its solution reads as follows:

$$\alpha_n^{(1)}=-\frac{\gamma_{n+1}\gamma_n+2\gamma_{n+1}-1}{\gamma_{n+1}\gamma_n+1}, \qquad \alpha_n^{(2)}=\frac{\gamma_{n+1}\gamma_n-2\gamma_n-1}{\gamma_{n+1}\gamma_n+1}. \tag{40}$$

Now we treat γ_n as a new arbitrary function, then the functions $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$ are found explicitly by (40). The functions $\beta_n^{(1)}$ and $\beta_n^{(2)}$ are found from (34) by two discrete integrations, as relations (34) can be rewritten in the form:

$$(T_n-1)\log(\beta_{n+1}^{(j)}-\beta_n^{(j)})=\log\frac{1+\alpha_{n+1}^{(j)}}{1-\alpha_{n+1}^{(j)}}.$$

We employ (32) and (35) to get a formula for the solution $u_{n,m}$:

$$u_{n,m}=\chi_{m+1}\left(\alpha_n^{(1)}+\frac{(\alpha_n^{(1)}-1)(\beta_{n+1}^{(1)}-\beta_n^{(1)})}{\beta_n^{(1)}+\omega_m}\right)+\chi_m\left(\alpha_n^{(2)}+\frac{(\alpha_n^{(2)}-1)(\beta_{n+1}^{(2)}-\beta_n^{(2)})}{\beta_n^{(2)}+\omega_m}\right), \tag{41}$$

where

$$\chi_m = \frac{1+(-1)^m}{2}, \quad \omega_{2k+1} = \omega_k^{(1)}, \quad \omega_{2k+2} = \omega_k^{(2)}. \tag{42}$$

In formula (41) we have two arbitrary functions γ_n and ω_m, and the functions $\alpha_n^{(j)}$ and $\beta_n^{(j)}$ are defined as explained above. The functions $\alpha_n^{(j)}$ are found explicitly, while the functions $\beta_n^{(j)}$ are found by quadrature.

Case $M = 3$. In this case, the relations for the functions $\alpha_n^{(j)}$, $j = 1, 2, 3$, are implied by system (31):

$$(\alpha_{n+1}^{(2)} + 1)(\alpha_n^{(2)} - 1) = \theta_3(\alpha_{n+1}^{(1)} + 1)(\alpha_n^{(1)} - 1),$$
$$(\alpha_{n+1}^{(3)} + 1)(\alpha_n^{(3)} - 1) = \theta_3(\alpha_{n+1}^{(2)} + 1)(\alpha_n^{(2)} - 1).$$

We rewrite these relations to introduce new functions $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$:

$$\frac{\alpha_{n+1}^{(2)} + 1}{\alpha_{n+1}^{(1)} + 1} = \theta_3 \frac{\alpha_n^{(1)} - 1}{\alpha_n^{(2)} - 1} = \gamma_{n+1}^{(1)},$$
$$\frac{\alpha_{n+1}^{(3)} + 1}{\alpha_{n+1}^{(2)} + 1} = \theta_3 \frac{\alpha_n^{(2)} - 1}{\alpha_n^{(3)} - 1} = \gamma_{n+1}^{(2)}.$$

Using the shift operator T_n, we get two systems for three functions $\alpha_n^{(j)}$. The solutions of these systems read as:

$$\alpha_n^{(1)} = \frac{2(\gamma_{n+1}^{(1)} - \theta_3)}{\gamma_{n+1}^{(1)}\gamma_n^{(1)} - \theta_3} - 1, \quad \alpha_n^{(2)} = \frac{2\theta_3(1 - \gamma_n^{(1)})}{\gamma_{n+1}^{(1)}\gamma_n^{(1)} - \theta_3} + 1, \tag{43}$$

$$\alpha_n^{(2)} = \frac{2(\gamma_{n+1}^{(2)} - \theta_3)}{\gamma_{n+1}^{(2)}\gamma_n^{(2)} - \theta_3} - 1, \qquad \alpha_n^{(3)} = \frac{2\theta_3(1 - \gamma_n^{(2)})}{\gamma_{n+1}^{(2)}\gamma_n^{(2)} - \theta_3} + 1. \tag{44}$$

Two different formulae for the function $\alpha_n^{(2)}$ should be made compatible. It is convenient to do this for the following function of $\alpha_n^{(2)}$:

$$\frac{\alpha_n^{(2)} + 1}{\alpha_n^{(2)} - 1} = \frac{\gamma_n^{(1)}(\theta_3 - \gamma_{n+1}^{(1)})}{\theta_3(\gamma_n^{(1)} - 1)} = \frac{\theta_3 - \gamma_{n+1}^{(2)}}{\gamma_{n+1}^{(2)}(\gamma_n^{(2)} - 1)}.$$

We rewrite the last identity in the form

$$\frac{\gamma_n^{(1)}(\gamma_n^{(2)} - 1)}{\gamma_n^{(1)} - 1} = \theta_3 \frac{\gamma_{n+1}^{(2)} - \theta_3}{\gamma_{n+1}^{(2)}(\gamma_{n+1}^{(1)} - \theta_3)} \tag{45}$$

and denote the left hand side by $\delta_{n+1} - 1$. Using the shift T_n, we get a system for $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$, which can be expressed as

$$\gamma_n^{(1)} = \frac{\delta_{n+1} - 1}{\delta_{n+1} - \gamma_n^{(2)}}, \tag{46}$$

$$\theta_3\delta_n(\gamma_n^{(2)})^2 - [(\theta_3 - 1)\delta_{n+1}\delta_n + \delta_{n+1} + \delta_n + \theta_3^2 - 1]\gamma_n^{(2)} + \theta_3^2\delta_{n+1} = 0. \tag{47}$$

Now we consider δ_n as a new arbitrary function. All the other n-dependent functions are expressed via this function. The functions $\gamma_n^{(1)}$, $\gamma_n^{(2)}$ are given by (46,47), the functions $\alpha_n^{(1)}$, $\alpha_n^{(2)}$ and $\alpha_n^{(3)}$ are found from (43), (44), and the functions $\beta_n^{(1)}$, $\beta_n^{(2)}$ and $\beta_n^{(3)}$ are found from (34). Let us note that the functions $\alpha_n^{(j)}$ and $\gamma_n^{(1)}$ are found explicitly, while the functions $\beta_n^{(j)}$ are found by two discrete integrations, and $\gamma_n^{(2)}$ is defined implicitly by the quadratic equation. Solutions $u_{n,k}^{(1)}$, $u_{n,k}^{(2)}$ and $u_{n,k}^{(3)}$ of the system (30) are given by (32), (35).

These solutions depend on the arbitrary functions δ_n and $\omega_k^{(1)}$, $\omega_k^{(2)}$, $\omega_k^{(3)}$, see (35). We return back to the solution $u_{n,m}$ of equation (29) with $M = 3$ equivalent to equation (2), which is given by transformation (15). From the viewpoint of this solution we have two arbitrary functions δ_n and ω_m, where

$$\omega_{3k+1} = \omega_k^{(1)}, \qquad \omega_{3k+2} = \omega_k^{(2)}, \qquad \omega_{3k+3} = \omega_k^{(3)}.$$

4. Modified series

Here we use a transformation theory developed in [12, 13].

Let us rewrite equation (2) in the form:

$$\frac{u_{n,m+1} - 1}{u_{n,m} - 1} = \theta_M \frac{u_{n+1,m} + 1}{u_{n+1,m+1} + 1}. \tag{48}$$

This allows us to introduce a new function $v_{n,m}$ such that:

$$v_{n,m} = \theta_M \frac{u_{n,m} + 1}{u_{n,m+1} + 1}, \qquad v_{n+1,m} = \frac{u_{n,m+1} - 1}{u_{n,m} - 1}. \tag{49}$$

The resulting relations are rewritten as

$$u_{n,m} = \frac{v_{n+1,m}v_{n,m} - 2v_{n,m} + \theta_M}{v_{n+1,m}v_{n,m} - \theta_M} \qquad u_{n,m+1} = -\frac{v_{n+1,m}v_{n,m} - 2\theta_M v_{n+1,m} + \theta_M}{v_{n+1,m}v_{n,m} - \theta_M}. \tag{50}$$

Rewriting these formulae at the same point $u_{n,m+1}$, we get an equation for $v_{n,m}$:

$$(v_{n+1,m+1} - 1)(v_{n,m} - \theta_M) = \theta_M(1 - v_{n+1,m}^{-1})(1 - \theta_M v_{n,m+1}^{-1}), \tag{51}$$

where θ_M is a primitive root of unit.

For all equations of form (2) we have got transformation (49) invertible on the solutions of (2). Two series of equations (2) and (51) are equivalent up to this transformation. The particular case $M = 2$ of equation (51) is presented in [2, (3.31)] up to $v_{n,m} \to -v_{n,m}$ together with the first integrals. In the case $M = 1$ we can apply the point transformation $v_{n,m} = 1 + w_{n,m}^{-1}$ and get the following equation:

$$w_{n+1,m+1}w_{n,m} = (w_{n+1,m} + 1)(w_{n,m+1} + 1). \tag{52}$$

This is nothing but the discrete Liouville equation found in [11]. Its first integrals and general solution were constructed in [1, (19)]. In the general case we can rewrite the first integrals by using transformation (49).

Theorem 3. *For each $M \geqslant 1$, equation (51) has the following first integrals in the n- and m-directions, respectively:*

$$W_{1,M} = \theta_M^{-m} \frac{(v_{n+2,m} - 1)v_{n+1,m}(v_{n,m} - \theta_M)}{(v_{n+2,m}v_{n+1,m} - \theta_M)(v_{n+1,m}v_{n,m} - \theta_M)}, \tag{53}$$

$$W_{2,M} = \frac{\left(V_{n,m+2M}^{(M)}V_{n,m+M}^{(M)} - 1\right)\left(V_{n,m+M}^{(M)}V_{n,m}^{(M)} - 1\right)}{\left(V_{n,m+2M}^{(M)} - 1\right)V_{n,m+M}^{(M)}\left(V_{n,m}^{(M)} - 1\right)}, \tag{54}$$

$$V_{n,m}^{(M)} = v_{n,m}v_{n,m+1}\dots v_{n,m+M-1}.$$

Proof. First integral (53) is obtained from (9) by straightforward calculations using transformation (50).

In order to prove (54), we need an auxiliary relation. It follows from the first identity of (49) that for each $k \geqslant 1$ the identities

$$\frac{u_{n,m} + 1}{u_{n,m+k} + 1} = \frac{u_{n,m} + 1}{u_{n,m+1} + 1}\frac{u_{n,m+1} + 1}{u_{n,m+2} + 1}\dots\frac{u_{n,m+k-1} + 1}{u_{n,m+k} + 1} = \theta_M^{-k}v_{n,m}v_{n,m+1}\dots v_{n,m+k-1}$$

hold true. Now first integral (11) can be rewritten as

$$\begin{aligned}W_{2,M} =&\frac{[(u_{n,m+3M}+1)-(u_{n,m+M}+1)][(u_{n,m+2M}+1)-(u_{n,m}+1)]}{[(u_{n,m+3M}+1)-(u_{n,m+2M}+1)][(u_{n,m+M}+1)-(u_{n,m}+1)]}\\ =&\frac{\left(\frac{u_{n,m+3M}+1}{u_{n,m+M}+1}-1\right)\left(1-\frac{u_{n,m}+1}{u_{n,m+2M}+1}\right)}{\left(\frac{u_{n,m+3M}+1}{u_{n,m+2M}+1}-1\right)\left(1-\frac{u_{n,m}+1}{u_{n,m+M}+1}\right)}\\ =&\frac{\left(\frac{\theta_M^{2M}}{V^{(M)}_{n,m+M}V^{(M)}_{n,m+2M}}-1\right)\left(1-\theta_M^{-2M}V^{(M)}_{n,m}V^{(M)}_{n,m+M}\right)}{\left(\frac{\theta_M^{M}}{V^{(M)}_{n,m+2M}}-1\right)\left(1-\theta_M^{-M}V^{(M)}_{n,m}\right)}.\end{aligned}$$

As $\theta_M^M=1$, we are led to first integral (54). $\square$

The order of first integral (53) is two. Let us show that this order is the minimal possible. If there exists a first integral $\widetilde{W}_{1,n,m}(v_{n+1,m},v_{n,m})$ for equation (51) in the n-direction, we can use transformation (49) and get a first integral for equation (2) of a nonstandard form $\widehat{W}_{1,n,m}(u_{n,m+1},u_{n,m})$ satisfying the relation $(T_m-1)\widehat{W}_{1,n,m}=0$. It is easy to check that this is impossible.

The order of first integral (54) is equal to $3M-1$. In the cases $M=1$ and $M=2$, the fact that this order $3M-1$ is the minimal possible follows from [1] and [2], respectively. In the case $M=3$ we can prove the same using the fact that the order 9 of corresponding first integral (11) of equation (2) is minimal, see Theorem 2.

In the case $M=3$, let us suppose that equation (51) has a first integral in the m-direction

$$\widetilde{W}_{2,n,m}(v_{n,m+k},v_{n,m+k-1},\dots,v_{n,m})$$

of an order $1\leqslant k\leqslant 7$. This means that for some n, m

$$\frac{\partial\widetilde{W}_{2,n,m}}{\partial v_{n,m}}\neq 0,\qquad \frac{\partial\widetilde{W}_{2,n,m}}{\partial v_{n,m+k}}\neq 0.$$

Employing the first relation of (49), we rewrite $\widetilde{W}_{2,n,m}$ in terms of $u_{n,m+j}$ and get a first integral for equation (2) of the following form:

$$\widehat{W}_{2,n,m}(u_{n,m+k+1},u_{n,m+k},\dots,u_{n,m}).$$

It easy to prove that its order is equal to $k+1$, where $2\leqslant k+1\leqslant 8<9$, but this is impossible.

Finally we note that using the results of Section 3 and the first of transformations (49), we can construct the general solutions for equations (51) with $M=1,2,3$.

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