

# RENORMALIZATIONS OF MEASURABLE OPERATOR IDEAL SPACES AFFILIATED TO SEMI-FINITE VON NEUMANN ALGEBRA

A.M. BIKCHENTAEV

**Abstract.** This work is devoted to non-commutative analogues of classical methods of constructing functional spaces. Let a von Neumann algebra  $\mathcal{M}$  of operators act in a Hilbert space  $\mathcal{H}$ ,  $\tau$  be a faithful normal semi-finite trace  $\mathcal{M}$ . Let  $\widetilde{\mathcal{M}}$  be an  $*$ -algebra of  $\tau$ -measurable operators,  $|X| = \sqrt{X^*X}$  for  $X \in \widetilde{\mathcal{M}}$ . A lineal  $\mathcal{E}$  in  $\widetilde{\mathcal{M}}$  is called ideal space on  $(\mathcal{M}, \tau)$  if

- 1)  $X \in \mathcal{E}$  implies  $X^* \in \mathcal{E}$ ;
- 2)  $X \in \mathcal{E}$ ,  $Y \in \widetilde{\mathcal{M}}$  and  $|Y| \leq |X|$  imply  $Y \in \mathcal{E}$ .

Let  $\mathcal{E}, \mathcal{F}$  be ideal spaces on  $(\mathcal{M}, \tau)$ . We propose a method of constructing a mapping  $\tilde{\rho}: \mathcal{E} \rightarrow [0, +\infty]$  with nice properties by employing a mapping  $\rho$  on a positive cone  $\mathcal{E}^+$ . At that, if  $\mathcal{E} = \mathcal{M}$  and  $\rho = \tau$ , then  $\tilde{\rho}(X) = \tau(|X|)$  and if the trace  $\tau$  is finite, then  $\tilde{\rho}(X) = \|X\|_1$  for all  $X \in \mathcal{M}$ . We study the case as  $\tilde{\rho}(X)$  is equivalent to the original mapping  $\rho(|X|)$ . Employing mappings on  $\mathcal{E}$  and  $\mathcal{F}$ , we construct a new mapping with nice properties on the sum  $\mathcal{E} + \mathcal{F}$ . We provide examples of such mappings. The results are new also for  $*$ -algebra  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  of all bounded linear operators in  $\mathcal{H}$  equipped with a canonical trace  $\tau = \text{tr}$ .

**Keywords:** Hilbert space, linear operator, von Neumann algebra, normal trace, measurable operators, ideal space, renormalization.

**Mathematics Subject Classification:** 46L10; 47C15; 46L51

## 1. INTRODUCTION

The work is devoted to non-commutative analogues of classical methods of constructing functional spaces. The beginning of developing a corresponding aspect of the theory of non-commutative integration is due to I.E. Sigal and J. Dixmier, who in the beginning of 1950s created the theory of non-commutative integration with respect to the trace on a semi-finite von Neumann algebra [1]. The results of their studies were effectively applied in the duality theory for unimodular groups and stimulated a progress in “non-commutative probability theory”. The theory of algebras of measurable and locally measurable operator is being intensively developed and has interesting applications in various fields of functional analysis, mathematical physics, statistical mechanics, quantum field theory.

Till the mid of 1980s, the ideal spaces of measurable operators served mostly as an object of studying, see [2] and the references therein. Recently there appeared publications in which they served as a tool, see, for instance, [3]. This demonstrates a topicality of finding new methods for constructing new methods for constructing ideal spaces and developing a general theory of

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these spaces. In [4], [5] new methods were proposed for constructing ideal spaces on semi-finite von Neumann algebras and the geometric and topological spaces of the obtained spaces were studied.

Let a von Neumann algebra  $\mathcal{M}$  of operators act in a Hilbert space  $\mathcal{H}$ ,  $\tau$  be a faithful normal semi-infinite trace on  $\mathcal{M}$ . Let  $\mathcal{E}, \mathcal{F}$  be ideal spaces on  $(\mathcal{M}, \tau)$ . In Section 3 we propose a method of constructing a mapping  $\tilde{\rho}: \mathcal{E} \rightarrow [0, +\infty]$  with nice properties by means of a given mapping  $\rho$  on a positive cone  $\mathcal{E}^+$ . At that, if  $\mathcal{E} = \mathcal{M}$  and  $\rho = \tau$ , then  $\tilde{\rho}(X) = \tau(|X|)$  and if the trace  $\tau$  is finite, then  $\tilde{\rho}(X) = \|X\|_1$  for each  $X \in \mathcal{M}$ . We also study the case when  $\tilde{\rho}(X)$  is equivalent to the initial mapping  $\rho(|X|)$ . Employing mappings onto  $\mathcal{E}$  and  $\mathcal{F}$ , in Section 4 we construct a new mapping with nice properties on the sum  $\mathcal{E} + \mathcal{F}$ . The results are new also for  $*$ -algebra  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  of all bounded linear operators in  $\mathcal{H}$  equipped by a canonical trace  $\tau = \text{tr}$ .

## 2. NOTATIONS AND DEFINITIONS

Let  $\mathcal{M}$  be a von Neumann algebra in a Hilbert space  $\mathcal{H}$ ,  $\mathcal{M}^{\text{pr}}$  be a lattice of projectors ( $P = P^2 = P^*$ ) in  $\mathcal{M}$ ,  $I$  be the identity mapping  $\mathcal{M}$ ,  $P^\perp = I - P$  for  $P \in \mathcal{M}^{\text{pr}}$ ,  $\mathcal{M}^+$  be a cone of positive elements in  $\mathcal{M}$ , and  $\mathcal{M}_1 = \{X \in \mathcal{M} : \|X\| \leq 1\}$ .

The mapping  $\varphi: \mathcal{M}^+ \rightarrow [0, +\infty]$  is called *trace* if  $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ ,  $\varphi(\lambda X) = \lambda\varphi(X)$  for each  $X, Y \in \mathcal{M}^+$ ,  $\lambda \geq 0$  (at that  $0 \cdot (+\infty) \equiv 0$ ) and  $\varphi(Z^*Z) = \varphi(ZZ^*)$  for each  $Z \in \mathcal{M}$ . A trace  $\varphi$  is called *faithful* if  $\varphi(X) > 0$  for each  $X \in \mathcal{M}^+$ ,  $X \neq 0$ ; and it is *semi-infinite* if

$$\varphi(X) = \sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$$

for each  $X \in \mathcal{M}^+$ . A trace is *normal* if

$$X_i \nearrow X \quad (X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = \sup \varphi(X_i),$$

see [6, Ch. V, Sect. 2].

An operator in  $\mathcal{H}$  (not necessarily bounded or densely defined) is called *affiliated to a von Neumann algebra*  $\mathcal{M}$  if it commutes with each unitary operator in the commutator subgroup  $\mathcal{M}'$  of the algebra  $\mathcal{M}$ . Hereafter  $\tau$  is a faithful normal semi-infinite trace on  $\mathcal{M}$ . A closed operator  $X$  affiliated to  $\mathcal{M}$  with a domain  $\mathcal{D}(X)$  dense in  $\mathcal{H}$  is called  $\tau$ -*measurable* if for each  $\varepsilon > 0$  there exists  $P \in \mathcal{M}^{\text{pr}}$ , such that  $P\mathcal{H} \subset \mathcal{D}(X)$  and  $\tau(P^\perp) < \varepsilon$ . A set  $\widetilde{\mathcal{M}}$  of all  $\tau$ -measurable operators is an  $*$ -algebra with respect to the operations of taking the adjoint operator, multiplication by a scalar and the operators of strong addition and multiplication obtained by closure of usual operators [1], [7]. For a family  $\mathcal{L} \subset \widetilde{\mathcal{M}}$  by  $\mathcal{L}^+$  and  $\mathcal{L}^{\text{sa}}$  we denote its positive and Hermitian part, respectively. A partial order in  $\widetilde{\mathcal{M}}^{\text{sa}}$  generated by a proper cone  $\widetilde{\mathcal{M}}^+$  is denoted by  $\leq$ . If  $X \in \widetilde{\mathcal{M}}$  and  $X = U|X|$  is the polar decomposition of  $X$ , then  $U \in \mathcal{M}_1$  and  $|X| = \sqrt{X^*X} \in \widetilde{\mathcal{M}}^+$ .

The  $*$ -algebra  $\widetilde{\mathcal{M}}$  is equipped with the topology  $t_\tau$  of convergence measure [7], whose fundamental system of neighbourhood is formed by the sets

$$U(\varepsilon, \delta) = \{X \in \widetilde{\mathcal{M}} : \exists P \in \mathcal{M}^{\text{pr}} (\|XP\| \leq \varepsilon \text{ and } \tau(P^\perp) \leq \delta)\}, \quad \varepsilon > 0, \quad \delta > 0.$$

It is known that  $(\widetilde{\mathcal{M}}, t_\tau)$  is a complete metrizable topological  $*$ -algebra and  $\mathcal{M}$  is dense in  $(\widetilde{\mathcal{M}}, t_\tau)$ .

By  $\mu_t(X)$  we denote a *permutation* of an operator  $X \in \widetilde{\mathcal{M}}$ , that is, a non-decreasing right-continuous function  $\mu(X): (0, \infty) \rightarrow [0, \infty)$  defined by the formula

$$\mu_t(X) = \inf\{\|XP\| : P \in \mathcal{M}^{\text{pr}}, \tau(P^\perp) \leq t\}, \quad t > 0.$$

The set of  $\tau$ -compact operators  $\widetilde{\mathcal{M}}_0 = \{X \in \widetilde{\mathcal{M}} : \lim_{t \rightarrow \infty} \mu_t(X) = 0\}$  is a  $t_\tau$ -closed ideal in  $\widetilde{\mathcal{M}}$  [8]. Let  $m$  be a Lebesgue linear measure on  $\mathbb{R}$ . An associated with  $(\mathcal{M}, \tau)$  non-commutative Lebesgue  $L_p$ -space ( $0 < p < \infty$ ) can be defined as  $L_p(\mathcal{M}, \tau) = \{X \in \widetilde{\mathcal{M}} : \mu_t(X) \in L_p(\mathbb{R}^+, m)\}$  with  $F$ -norm (norm for  $1 \leq p < \infty$ )  $\|X\|_p = \|\mu_t(X)\|_p$ ,  $X \in L_p(\mathcal{M}, \tau)$ .

A lineal  $\mathcal{E}$  in  $\widetilde{\mathcal{M}}$  is called *ideal space on*  $(\mathcal{M}, \tau)$ , see [9], [3], [2], if

- 1)  $X \in \mathcal{E}$  implies  $X^* \in \mathcal{E}$ ;
- 2)  $X \in \mathcal{E}$ ,  $Y \in \widetilde{\mathcal{M}}$  and  $|Y| \leq |X|$  imply  $Y \in \mathcal{E}$ .

This is, for instance, the algebra  $\mathcal{M}$ , the set of elementary operators  $\mathcal{F}(\mathcal{M})$ ,  $\widetilde{\mathcal{M}}_0$ ,  $(L_1 + L_\infty)(\mathcal{M}, \tau)$  and  $L_p(\mathcal{M}, \tau)$  as  $0 < p < +\infty$ . For each ideal space  $\mathcal{E}$  on  $(\mathcal{M}, \tau)$  we have  $\mathcal{M}\mathcal{E}\mathcal{M} \subseteq \mathcal{E}$  [2, Lm. 5].

If  $\mathcal{M} = \mathcal{B}(\mathcal{H})$  is  $*$ -algebra of all bounded linear operators in  $\mathcal{H}$  and  $\tau = \text{tr}$  is the canonical trace, then  $\widetilde{\mathcal{M}}$  and  $\widetilde{\mathcal{M}}_0$  coincide with  $\mathcal{B}(\mathcal{H})$  and the ideal of compact operators in  $\mathcal{H}$ , respectively. We have

$$\mu_t(X) = \sum_{n=1}^{\infty} s_n(X) \chi_{[n-1, n)}(t), \quad t > 0,$$

where  $\{s_n(X)\}_{n=1}^{\infty}$  is the sequence of  $s$ -numbers of a compact operator  $X$  [10];  $\chi_A$  is the indicator of a set  $A \subset \mathbb{R}$ . Then the space  $L_p(\mathcal{M}, \tau)$  is the Schatten-von Neumann ideal  $\mathfrak{S}_p$ ,  $0 < p < \infty$ .

**Lemma 2.1.** [11]. *If  $X, Y \in \widetilde{\mathcal{M}}^+$  and  $X \leq Y$ , then there exists an operator  $Z \in \mathcal{M}_1$  such that  $\sqrt{X} = Z\sqrt{Y}$  and  $X = ZYZ^*$ .*

**Lemma 2.2.** [12] *If  $X, Y \in \widetilde{\mathcal{M}}^{\text{sa}}$  and  $Z \in \widetilde{\mathcal{M}}$ , then the inequality  $X \leq Y$  implies  $ZXZ^* \leq ZYZ^*$ .*

### 3. RENORMALIZATION OF IDEAL SPACES

Let  $\tau$  be a faithful normal semi-infinite trace on a von Neumann algebra  $\mathcal{M}$ ,  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$ . If  $A \in \widetilde{\mathcal{M}}$  and  $A^*A \in \mathcal{E}$ , then  $AA^* \in \mathcal{E}$  [2, Lm. 5]. We assume the following conditions for a mapping  $\rho: \mathcal{E}^+ \rightarrow [0, +\infty]$ :

- (i) if  $X, Y \in \mathcal{E}^+$  and  $X \leq Y$ , then  $\rho(X) \leq \rho(Y)$ ;
- (ii)  $\rho(X^*X) = \rho(XX^*)$  for each  $X \in \widetilde{\mathcal{M}}$  with  $X^*X \in \mathcal{E}$ ;
- (iii)  $\rho(X + Y) \leq \rho(X) + \rho(Y)$  for each  $X, Y \in \mathcal{E}^+$ .

We assume the following conditions for a mapping  $\rho: \mathcal{E} \rightarrow [0, +\infty]$ :

- (iv)  $\rho(X) = \rho(|X|) = \rho(X^*)$  for each  $X \in \mathcal{E}$ ;
- (v)  $\rho(X + Y) \leq \rho(X) + \rho(Y)$  for each  $X, Y \in \mathcal{E}$ ;
- (vi)  $\rho(\lambda X) = |\lambda| \rho(X)$  for each  $\lambda \in \mathbb{C}$  and  $X, Y \in \mathcal{E}$  (at that,  $0 \cdot +\infty = 0$ ).

**Example 3.1.** *Let  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  be a normalized ideal space on  $(\mathcal{M}, \tau)$  [9]. Then the restriction of the norm  $\|\cdot\|_{\mathcal{E}}$  on  $\mathcal{E}^+$  satisfies conditions (i)–(iii), while  $\|\cdot\|_{\mathcal{E}}$  satisfies conditions (iv)–(vi). The examples are  $(\mathcal{M}, \|\cdot\|)$  and  $(L_p(\mathcal{M}, \tau), \|\cdot\|_p)$  for  $p \geq 1$ .*

**Example 3.2.** *Let  $\rho = \langle (\cdot)\xi, \xi \rangle$  ( $\xi \in \mathcal{H}$ ,  $\|\xi\| = 1$ ) be a vector state on the algebra  $\mathcal{B}(\mathcal{H})$ . The restriction  $\rho|_{\mathcal{B}(\mathcal{H})^+}$  satisfies conditions (i) and (iii). For  $\rho_1 = |\rho|$ , conditions (v) and (vi) are satisfied.*

**Example 3.3.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and  $Y \in \mathcal{E}^+$ . We let*

$$\rho(X) = \inf\{\lambda > 0 : X \leq \lambda Y\}$$

*for each  $X \in \mathcal{E}^+$  assuming that inf over  $\emptyset$  is equal to  $+\infty$ . Then  $\rho$  satisfies conditions (i) and (iii).*

**Proposition 3.1.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E} \rightarrow [0, +\infty]$  satisfy conditions (iv)–(vi) (or (i), (iv) and (v)). Then  $\mathcal{F} = \{X \in \mathcal{E} : \rho(X) < +\infty\}$  is an ideal space on  $(\mathcal{M}, \tau)$ .*

**Proposition 3.2.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E} \rightarrow [0, +\infty]$  satisfy conditions (i)–(iii). Then*

$$\rho(X) \leq \sum_{k=1}^n \rho(Y_k X Y_k^*)$$

for each  $X \in \mathcal{E}^+$  and  $\{Y_k\}_{k=1}^n \subset \mathcal{M}$  with  $\sum_{k=1}^n Y_k^* Y_k \geq I$ .

*Proof.* By Lemma 2.2 we have

$$X \leq \sqrt{X} \left( \sum_{k=1}^n Y_k^* Y_k \right) \sqrt{X} = \sum_{k=1}^n \sqrt{X} Y_k^* Y_k \sqrt{X}.$$

This is why

$$\rho(X) \leq \sum_{k=1}^n \rho(\sqrt{X} Y_k^* Y_k \sqrt{X}) = \sum_{k=1}^n \rho(Y_k X Y_k^*)$$

and this completes the proof.  $\square$

**Lemma 3.1.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E} \rightarrow [0, +\infty]$  satisfies conditions (i), (iv). If  $X, Y \in \mathcal{M}_1$ , then  $\rho(XZY) \leq \rho(Z)$  for each  $Z \in \mathcal{E}$ .*

*Proof.* If for some operators we have  $A, B \in \mathcal{M}$ , then  $AZB \in \mathcal{E}$  for each  $Z \in \mathcal{E}$ . For each  $X \in \mathcal{M}_1$  and  $Z \in \mathcal{E}$ , by the operator monotonicity of the function  $\lambda \mapsto \sqrt{\lambda}$  on  $\mathbb{R}^+$  and Lemma 2.2 we have

$$\rho(XZ) = \rho(|XZ|) = \rho(\sqrt{Z^* X^* X Z}) \leq \rho(\sqrt{Z^* Z}) = \rho(|Z|) = \rho(Z).$$

If  $Y \in \mathcal{M}_1$ , then  $Y^* \in \mathcal{M}_1$  and  $\rho(ZY) = \rho((ZY)^*) = \rho(Y^* Z^*) \leq \rho(Z^*) = \rho(Z)$ . The proof is complete.  $\square$

**Proposition 3.3.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E} \rightarrow [0, +\infty]$  satisfies conditions (i), (iv). Then  $\rho$  satisfies condition (ii).*

*Proof.* Let  $X = U|X|$  be the polar decomposition of the operator  $X \in \widetilde{\mathcal{M}}$ . Then  $U, U^* \in \mathcal{M}_1$  and  $XX^* = UX^*XU^* \in \mathcal{E}$ . By Lemma 3.1 we have

$$\rho(XX^*) = \rho(UX^*XU^*) \leq \rho(X^*X).$$

Replacing  $X$  by  $X^*$ , we obtain  $\rho(X^*X) \leq \rho(XX^*)$  and this proves (ii).  $\square$

**Theorem 3.1.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E}^+ \rightarrow [0, +\infty]$  be given. We denote*

$$\tilde{\rho}(X) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq Z|X|Z^*} \rho(A) \quad \text{for each } X \in \mathcal{E}. \quad (1)$$

Then  $\tilde{\rho}$  satisfies conditions (i), (ii), (iv) and  $\tilde{\rho}(X) \geq \rho(|X|)$  for each  $X \in \mathcal{E}$ .

*Proof.* Let  $X, Y \in \mathcal{E}^+$  and  $X \leq Y$ . By Lemma 2.1, there exists an operator  $U \in \mathcal{M}_1$  such that  $X = UYU^*$ . This is why

$$\tilde{\rho}(X) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq ZXZ^*} \rho(A) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq ZUYU^*Z^*} \rho(A) \leq \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq ZYZ^*} \rho(A) = \tilde{\rho}(Y)$$

and this proves (i).

Let  $X = U|X|$  be the polar decomposition of an operator  $X \in \widetilde{\mathcal{M}}$ . Then  $|X^*| = U|X|U^*$  and  $|X^*|^2 = U|X|^2U^*$ . We have

$$\tilde{\rho}(X^*) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq Z|X^*|Z^*} \rho(A) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq ZU|X|U^*Z^*} \rho(A) \leq \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq Z|X|Z^*} \rho(A) = \tilde{\rho}(X).$$

Thanks to the identity  $(X^*)^* = X$ , this proves (iv). Now (ii) follows Proposition 3.3. The proof is complete.  $\square$

**Theorem 3.2.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$  and a mapping  $\rho: \mathcal{E}^+ \rightarrow [0, +\infty]$  satisfy condition (iii). Then the mapping  $\tilde{\rho}: \mathcal{E} \rightarrow [0, +\infty]$  defined by formula (1) satisfies condition (v).*

*Proof.* Let  $X, Y \in \mathcal{E}$  and  $\alpha = \tilde{\rho}(X + Y)$ . Then for each number  $\varepsilon > 0$  there exist operators  $Z_\varepsilon \in \mathcal{M}_1$  and  $A_\varepsilon \in \mathcal{E}^+$  such that

$$A_\varepsilon \leq Z_\varepsilon|X + Y|Z_\varepsilon^*, \quad \alpha \geq \rho(A_\varepsilon) > \alpha - \varepsilon.$$

There exist partial isometries  $V, W \in \mathcal{M}_1$  such that  $|X + Y| \leq V|X|V^* + W|Y|W^*$ , see [13, Thm. 2.2], [14]. This is why, cf. also Lemma 2.2, for each number  $\varepsilon > 0$  there exist operators  $Z_\varepsilon \in \mathcal{M}_1$  and  $A_\varepsilon \in \mathcal{E}^+$  such that

$$A_\varepsilon \leq Z_\varepsilon V|X|V^* Z_\varepsilon^* + Z_\varepsilon W|Y|W^* Z_\varepsilon^*, \quad \alpha \geq \rho(A_\varepsilon) > \alpha - \varepsilon.$$

By Lemma 2.1, for each number  $\varepsilon > 0$  there exists an operator  $U_\varepsilon \in \mathcal{M}_1$  such that

$$A_\varepsilon = U_\varepsilon Z_\varepsilon V|X|V^* Z_\varepsilon^* U_\varepsilon^* + U_\varepsilon Z_\varepsilon W|Y|W^* Z_\varepsilon^* U_\varepsilon^*.$$

The operators  $U_\varepsilon Z_\varepsilon V$ ,  $U_\varepsilon Z_\varepsilon W$  belong to  $\mathcal{M}_1$  and

$$\rho(A_\varepsilon) \leq \rho(U_\varepsilon Z_\varepsilon V|X|V^* Z_\varepsilon^* U_\varepsilon^*) + \rho(U_\varepsilon Z_\varepsilon W|Y|W^* Z_\varepsilon^* U_\varepsilon^*) \leq \tilde{\rho}(X) + \tilde{\rho}(Y).$$

By the arbitrary choice of the number  $\varepsilon > 0$  we obtain  $\alpha \leq \tilde{\rho}(X) + \tilde{\rho}(Y)$  and this completes the proof.  $\square$

**Remark 3.1.** *Under the assumptions of Theorem 3.1, if  $\rho(A) = 0 \Leftrightarrow A = 0$  ( $A \in \mathcal{E}^+$ ), then  $\tilde{\rho}(X) = 0 \Leftrightarrow X = 0$  ( $X \in \mathcal{E}$ ); if  $\rho(\lambda A) = \lambda \rho(A)$  for each  $\lambda \in \mathbb{R}^+$  and  $A \in \mathcal{E}^+$ , then  $\tilde{\rho}$  satisfies condition (iv). If  $\rho$  satisfies condition (i), then  $\tilde{\rho}(I) = \rho(I)$  and*

$$\tilde{\rho}(X) = \sup_{Z \in \mathcal{M}_1} \rho(Z|X|Z^*) \quad \text{for each } X \in \mathcal{E}.$$

**Proposition 3.4.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$ ,  $\rho: \mathcal{E}^+ \rightarrow [0, +\infty]$ , and a mapping  $\tilde{\rho}: \mathcal{E} \rightarrow [0, +\infty]$  is defined by formula (1). If  $\rho$  satisfies conditions (i) and (ii), then  $\tilde{\rho}(X) = \rho(|X|)$  for each  $X \in \mathcal{E}$ .*

*Proof.* For each  $Z \in \mathcal{M}_1$  we have  $Z^*Z \leq I$  and  $\sqrt{|X|}Z^*Z\sqrt{|X|} \leq |X|$  holds for each  $X \in \mathcal{E}$  by Lemma 2.2. Then

$$\tilde{\rho}(X) = \sup_{Z \in \mathcal{M}_1} \sup_{0 \leq A \leq Z|X|Z^*} \rho(A) = \sup_{Z \in \mathcal{M}_1} \rho(Z|X|Z^*) = \sup_{Z \in \mathcal{M}_1} \rho(\sqrt{|X|}Z^*Z\sqrt{|X|}) = \rho(|X|).$$

Thus, the restriction  $\tilde{\rho}|_{\mathcal{E}^+}$  coincides with  $\rho$ . The proof is complete.  $\square$

**Example 3.4.** *Let  $\mathcal{E} = \mathcal{M}$  and  $\rho = \tau$ . Then  $\tilde{\rho}(X) = \tau(|X|)$  and if the trace  $\tau$  is finite, then  $\tilde{\rho}(X) = \|X\|_1$  for each  $X \in \mathcal{M}$ .*

**Proposition 3.5.** *Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$ ,  $\rho: \mathcal{E}^+ \rightarrow [0, +\infty]$ , and let a mapping  $\tilde{\rho}: \mathcal{E} \rightarrow [0, +\infty]$  is defined by formula (1). If the following conditions hold:*

(vii) *there exists  $C_1 > 0$  such that  $\rho(X^*X) \leq C_1\rho(XX^*)$  for each  $X \in \widetilde{\mathcal{M}}$  with  $X^*X \in \mathcal{E}$ ;*

(viii) *there exists  $C_2 > 0$  such that  $\rho(X) \leq C_2\rho(Y)$  for each  $X, Y \in \mathcal{E}^+$  with  $X \leq Y$ ,*

*then  $\rho(|X|) \leq \tilde{\rho}(X) \leq 2C_1C_2^2\rho(|X|)$  for each  $X \in \mathcal{E}$ .*

*Proof.* Let  $X \in \mathcal{E}$ ,  $Z \in \mathcal{M}_1$  and  $0 \leq A \leq Z|X|Z^*$  be such that  $\tilde{\rho}(X) \leq 2\rho(A)$ . By Lemma 2.2 we have  $\sqrt{|X|Z^*Z}\sqrt{|X|} \leq |X|$  and

$$\tilde{\rho}(X) \leq 2\rho(A) \leq 2C_2\rho(Z|X|Z^*) \leq 2C_1C_2\rho(\sqrt{|X|Z^*Z}\sqrt{|X|}) \leq 2C_1C_2^2\rho(|X|).$$

Thus,  $\rho(|X|) \leq \tilde{\rho}(X) \leq 2C_1C_2^2\rho(|X|)$  for each  $X \in \mathcal{E}$ . The proof is complete.  $\square$

**Remark 3.2.** Let  $\mathcal{E}$  be an ideal space on  $(\mathcal{M}, \tau)$ , where  $\mathcal{M}$  is a finite von Neumann algebra, that is,  $0 \leq U \in \mathcal{M}$  and  $U^*U = I$  imply  $UU^* = I$ . Assume that a mapping  $\rho: \mathcal{E} \rightarrow \mathbb{R}^+$  satisfies condition (v). We let

$$\rho_1(X) = \sup_{Z, T \in \mathcal{M}_1} \rho(ZXT) \quad \text{for each } X \in \mathcal{E}.$$

It was shown in [15, Thm. 2] that a mapping  $\rho_1: \mathcal{E} \rightarrow [0, +\infty]$  satisfies conditions (i), (iv) and (v). For a mapping  $\rho$  with property (vi), the mapping  $\rho_1$  also satisfies (vi).

**Proposition 3.6.** Let  $\rho_2 = \widetilde{\rho|_{\mathcal{E}^+}}$ . Then  $\rho_2(X) \leq \rho_1(X)$  for each  $X \in \mathcal{E}$ .

*Proof.* Let  $X \in \mathcal{E}$  and  $\alpha = \rho_2(X)$ . Then for each number  $\varepsilon > 0$  there exist operators  $Z_\varepsilon \in \mathcal{M}_1$  and  $A_\varepsilon \in \mathcal{E}^+$  such that

$$A_\varepsilon \leq Z_\varepsilon|X|Z_\varepsilon^*, \quad \alpha \geq \rho(A_\varepsilon) > \alpha - \varepsilon.$$

By Lemma 2.1, for each number  $\varepsilon > 0$ , there exists an operator  $U_\varepsilon \in \mathcal{M}_1$  such that

$$A_\varepsilon = U_\varepsilon Z_\varepsilon|X|Z_\varepsilon^* U_\varepsilon^*.$$

Hence,

$$\rho_1(X) = \rho_1(|X|) = \sup_{Z, T \in \mathcal{M}_1} \rho(Z|X|T) \geq \rho(A_\varepsilon) > \alpha - \varepsilon.$$

By an arbitrary choice of the number  $\varepsilon > 0$  we have  $\rho_1(X) \geq \alpha$  and this completes the proof.  $\square$

**Example 3.5.** Let a mapping  $\rho: \mathcal{M}^{\text{pr}} \rightarrow \mathbb{R}^+$  is monotone and unitarily invariant. Then the mapping  $\rho_s: \widetilde{\mathcal{M}} \rightarrow \mathbb{R}^+$  defined by the formula  $\rho_s(A) = \rho(s(|A|))$ , where  $A \in \widetilde{\mathcal{M}}$  and  $s(|A|)$  is the support of the operator  $|A|$  satisfies conditions (i), (ii) and (iv).

#### 4. NORMALIZATION OF SUMS OF IDEAL SPACES

If  $\mathcal{E}, \mathcal{F}$  are ideal spaces on  $(\mathcal{M}, \tau)$ , then the sets  $\mathcal{E} \cap \mathcal{F}$  and  $\mathcal{E} + \mathcal{F} = \{A + B : A \in \mathcal{E}, B \in \mathcal{F}\}$  are also ideal spaces on  $(\mathcal{M}, \tau)$  [2, Thm. 2]. The structure of ideal spaces is modular: if  $\mathcal{E}, \mathcal{F}$  and  $\mathcal{G}$  are ideal spaces on  $(\mathcal{M}, \tau)$  and  $\mathcal{E} \subset \mathcal{G}$ , then  $(\mathcal{E} + \mathcal{F}) \cap \mathcal{G} = \mathcal{E} + (\mathcal{F} \cap \mathcal{G})$  [2, Thm. 3]. In the theory of non-commutative integration, the space  $(L_1 + L_\infty)(\mathcal{M}, \tau) = L_1(\mathcal{M}, \tau) + \mathcal{M}$  plays an important role [12].

**Theorem 4.1.** Let  $\mathcal{E}, \mathcal{F}$  be ideal spaces on  $(\mathcal{M}, \tau)$  and  $\mathcal{G} = \mathcal{E} + \mathcal{F}$ . If mappings  $\rho_1: \mathcal{E} \rightarrow [0, +\infty]$  and  $\rho_2: \mathcal{F} \rightarrow [0, +\infty]$  satisfy conditions (i), (iv), then the mapping  $\rho: \mathcal{G} \rightarrow [0, +\infty]$  defined by the formula

$$\rho(Z) = \inf\{\rho_1(X) + \rho_2(Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } Z = X + Y\}, \quad (2)$$

also satisfies conditions (i), (iv).

*Proof.* Let  $Z = U|Z|$  be the polar decomposition of the operator  $Z \in \mathcal{G}$ . Then  $|Z| = U^*Z$ . To check (iv), we first observe that  $\rho(Z^*) = \rho(Z)$ . By Lemma 3.1 we have

$$\begin{aligned} \rho(|Z|) &= \inf\{\rho_1(X) + \rho_2(Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } |Z| = X + Y\} \\ &\geq \inf\{\rho_1(X) + \rho_2(Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } U|Z| = UX + UY\} \\ &\geq \inf\{\rho_1(UX) + \rho_2(UY) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } U|Z| = UX + UY\} \geq \end{aligned}$$

$$\begin{aligned}
&\geq \underline{\rho}(Z) = \inf\{\rho_1(X) + \rho_2(Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } Z = X + Y\} \\
&\geq \inf\{\rho_1(X) + \rho_2(Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } |Z| = U^*Z = U^*X + U^*Y\} \\
&\geq \inf\{\rho_1(U^*X) + \rho_2(U^*Y) : X \in \mathcal{E}, Y \in \mathcal{F} \text{ and } |Z| = U^*X + U^*Y\} \\
&\geq \inf\{\rho_1(T) + \rho_2(S) : T \in \mathcal{E}, S \in \mathcal{F} \text{ and } |Z| = T + S\} = \underline{\rho}(|Z|)
\end{aligned}$$

and this proves (iv).

In order to check (i), we choose  $A, B \in \mathcal{E}^+$ ,  $A \leq B$ . By Lemma 2.1, the identity  $A = VBV^*$  holds for some  $V \in \mathcal{M}_1$ . Let  $\alpha = \rho(B)$ . Then for each  $\varepsilon > 0$  there exist operators  $X_\varepsilon \in \mathcal{E}$  and  $Y_\varepsilon \in \mathcal{F}$  such that

$$B = X_\varepsilon + Y_\varepsilon, \quad \alpha \leq \rho_1(X_\varepsilon) + \rho_2(Y_\varepsilon) < \alpha + \varepsilon.$$

Then  $A = V(X_\varepsilon + Y_\varepsilon)V^*$  and by Lemma 3.1 we have

$$\rho(A) \leq \rho_1(VX_\varepsilon V^*) + \rho_2(VY_\varepsilon V^*) \leq \rho_1(X_\varepsilon) + \rho_2(Y_\varepsilon).$$

By an arbitrary choice of the number  $\varepsilon > 0$  this completes the proof.  $\square$

**Proposition 4.1.** *Let  $\mathcal{E}, \mathcal{F}$  be ideal spaces on  $(\mathcal{M}, \tau)$  and  $\mathcal{G} = \mathcal{E} + \mathcal{F}$ . If the mappings  $\rho_1: \mathcal{E} \rightarrow [0, +\infty]$  and  $\rho_2: \mathcal{F} \rightarrow [0, +\infty]$  satisfy condition (v) (respectively, (vi)), then the mapping  $\rho: \mathcal{G} \rightarrow [0, +\infty]$  defined by formula (2) also satisfies condition (v) (respectively, (vi)).*

**Example 4.1.** *The identity  $\mathcal{E} + \mathcal{F} = \widetilde{\mathcal{M}}$  holds for  $\mathcal{E} = \mathcal{M}$  and  $\mathcal{F} = \widetilde{\mathcal{M}}_0$  [16]. The topology  $t_\tau$  can be defined by means of an ideal  $F$ -norm*

$$\rho_\tau(X) = \inf_{t>0} \max\{t, \mu_t(X)\}, \quad X \in \widetilde{\mathcal{M}}.$$

For  $Z \in \widetilde{\mathcal{M}}$  we define (cf. formula (2))

$$\rho(Z) = \inf\{\|X\| + \rho_\tau(Y) : X \in \mathcal{M}, Y \in \widetilde{\mathcal{M}}_0 \text{ and } Z = X + Y\}.$$

Then  $\rho$  satisfies conditions (iv) and (v). The restriction  $\rho|_{\widetilde{\mathcal{M}}^+}$  satisfies conditions (i)–(iii). Thus, the mapping  $\rho: \widetilde{\mathcal{M}} \rightarrow \mathbb{R}^+$  is an ideal  $F$ -norm majorizing  $\rho_\tau$ . Since

$$\|X\| = \lim_{t \rightarrow 0^+} \mu_t(X) = \sup_{t>0} \mu_t(X) \geq \rho_\tau(X) \quad \text{for each } X \in \mathcal{M},$$

we have  $\rho(Z) \leq 2\rho_\tau(Z)$  for each  $Z \in \widetilde{\mathcal{M}}$ .

**Remark 4.1.** *Let  $\mathcal{E}, \mathcal{F}$  be ideal spaces on  $(\mathcal{M}, \tau)$  and  $\mathcal{G} = \mathcal{E} \cap \mathcal{F}$ . If mappings  $\rho_1: \mathcal{E} \rightarrow [0, +\infty]$  and  $\rho_2: \mathcal{F} \rightarrow [0, +\infty]$  satisfy one of conditions (i)–(vi), then the mapping  $\rho: \mathcal{G} \rightarrow [0, +\infty]$  defined by the formula  $\rho(Z) = \max\{\rho_1(Z), \rho_2(Z)\}$  for each  $Z \in \mathcal{G}$  satisfies this condition as well.*

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