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AZARIN LIMITING SETS OF FUNCTIONS AND ASYMPTOTIC REPRESENTATION OF INTEGRALS

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Abstract. In the paper we consider integrals of form

$$\int_a^b f(t)\exp[i\varphi(rt)\ln(rt)]\,dt\,,$$

where $\varphi(r)$ is a smooth increasing function on the semi-axis $[0,\infty)$ such that $\lim\limits_{r\to+\infty}\varphi(r)=\infty$. We find a precise information on their asymptotic behavior and we prove an analogue of Riemann-Lebesgue lemma for trigonometric integrals. By applying this lemma, we succeed to obtain the asymptotic formulae for integrals with an absolutely continuous function. The proposed method of obtaining asymptotic formulae differs from classical method like Laplace method, applications of residua, saddle-point method, etc. To make the presentation more solid, we mostly restrict ourselves by the kernels $\exp[i\ln^p(rt)]$. Appropriate smoothness conditions for the function $f(t)$ allow us to obtain many-terms formulae. The properties of the integrals and the methods of obtaining asymptotic estimates differ in the cases $p\in(0,1)$, $p=1$, $p>1$. As $p\in(0,1)$, the asymptotic expansions are obtained by another method, namely, by expanding the kernel into a series. We consider the cases, when as an absolutely continuous function $f(t)$, we take a product of a power function t^ρ and the Poisson kernel or the conjugate Poisson kernel for the half-plane and as the integration set, the imaginary semi-axis serves. The real and imaginary parts of these integrals are harmonic functions in the complex plane cut along the positive semi-axis. We find the Azarin limiting sets for such functions.

Keywords: Riemann-Lebesgue lemma, trigonometric integral, asymptotic formula, Poisson kernel, harmonic function, Azarin limiting set.

Mathematics Subject Classification: 30E15, 31C05

1. Introduction

In the paper we consider the integrals of the form:

$$\int_a^b f(t)\exp[i\varphi(rt)\ln(rt)]\,dt,$$

where $\varphi(r)$ is a smooth increasing function on the half-line $[0,\infty)$ such that

$$\lim_{r\to+\infty}\varphi(r)=\infty,$$

and obtain sharp description of their asymptotic behavior.

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The kernel $\exp[i\varphi(rt)\ln(rt)]$ is rather peculiar. For instance, in three-volumes monograph by E. Riestinš [1]–[3], where, in particular the integrals $\int_a^b f(t)K(rt)\,dt$ were considered, the asymptotic expansions of the integrals with the above kernels were not studied. We prove an analogue of Riemann-Lebesgue lemma for trigonometric integrals, see Lemma 1. This lemma allows us to obtain single-term asymptotic formulae for the integrals with an absolutely continuous function $f(t)$, see Theorems 1, 2, 3. We note that a proposed method for obtaining asymptotic formulae differs from classical method like the Laplace method, application of residues, saddle-point method and others described in monograph by M.A. Evgrafov [4].

To achieve a better completeness of the presentation, we mostly restrict ourselves by the kernels $\exp[i\ln^p(rt)]$. The corresponding smoothness conditions for the function $f(t)$ allow us to obtain many-terms formulae, see Theorem 4. The properties of the integrals and the ways of obtaining asymptotic estimates differ in the cases $p \in (0,1)$, $p = 1$, $p > 1$. As $p \in (0,1)$, the asymptotic expansions are obtained by expanding the kernel into an asymptotic series, Theorem 5. Then we study particular functions taking the product of t^ρ by the Poisson kernel or by the adjoint Poisson kernel for the half-plane as $f(t)$, and as interval of integration we take the ray $[0,\infty)$. The real and imaginary parts of these integrals are denoted by $u_k(z)$, $k = 3,4,5,6$, are harmonic functions in the complex plane cut along the positive semi-axis.

An important characteristics of the growth of a subharmonic and, in particular, harmonic function $u(z)$ is its Azarin limiting set$\mathrm{Fr}\,u$ [5]. This is a limiting set of the family of the functions $u_t(z) = u(tz)/t^\rho$ (ρ is the order of u) as $t \to +\infty$ in the topology of the Schwarz space of tempered distributions. In the case $p \in (0,1)$, we find the Azarin limiting of the functions $u_k(z)$. Azaring limiting set possesses more information on the behavior of subharmonic functions in the vicinity of the infinity than the indicator. A general theory provides a theorem on characterization of Azarin limiting sets and ensures the existence of the subharmonic functions with a prescribed limiting set. However, we provide examples of particular functions of irregular growth, for which we calculate Azarin limiting set. Up to now, asymptotic estimates have been constructed for the functions of completely regular growth. Such functions approximate well some subharmonic functions with zeroes on a single ray. We provide such functions. In this way, we describe a wide class of subharmonic functions of irregular growth with a known asymptotic behavior. We mention, that asymptotic formulae for entire functions of completely regular growth were provided in book by B.Ya. Levin [6]. Entire functions of completely regular growth are studied quite well, they arise in many works and have various applications. However, modern studies on completeness and representations by series in functional spaces, on problems in the theory of analytic continuation, on problems in theory of infinite order differential operators and operators of convolution type require a thorough study of entire functions possessing no regular behavior. This is why the developing of methods for solving problems related with finding extremal values for main asymptotic characteristics for the growth of entire functions in very general and natural classes is topical. Recently, there appeared many papers devoted to this subject; here we mention a joint work by G.G. Braichev and V.B. Sherstykov [7] and a work by V.B. Sherstykov [8].

2. On analogue of Riemann-Lebesgue theorem

We begin with an analogue of Riemann–Lebesgue lemma [9] on tending to the zero of the Fourier coefficients of an arbitrary integrable function.

Lemma 1. *Let $f(t) \in L_1([a,b])$, $0 \leqslant a < b < +\infty$, and let $\varphi(r)$ be an increasing twice differentiable function on the half-line $[0,+\infty)$ such that*

$$\lim_{r\to+\infty} \varphi(r) = \infty \qquad \lim_{r\to+\infty} \frac{r^2\varphi''(r)\ln r}{(r\varphi'(r)\ln r + \varphi(r))^2} = 0\,. \tag{1}$$

Then

$$\lim_{r\to+\infty}\int\limits_a^b f(t)\exp[i\varphi(rt)\ln(rt)]\,dt=0\,.$$

Proof. We observe that if $f(t)\in L_1([a,b])$, then the inequality holds:

$$\left|\int\limits_a^b f(t)\exp[i\varphi(rt)\ln(rt)]\,dt\right|\leqslant\int\limits_a^b |f(t)|\,dt=\|f\|_1\,. \tag{2}$$

By E we denote the set of functions $f(t)\in L_1([a,b])$, for which the lemma holds true. It follows from (1) and (2) that E is a closed subset in $L_1([a,b])$ in the topology defined by the norm $\|\cdot\|_1$. It is also obvious that E is a linear subspace in the space $L_1([a,b])$. This is why, to prove the lemma, it is sufficient to show that E contains at least a set of functions, whose linear combinations are dense in $L_1([a,b])$. We can find many sets of such kind. As an example we take the set of the functions $C_1([a,b])$ continuously differentiable on the segment $[a,b]$.

Let $f\in C_1([a,b])$ and assume first that $a>0$. We observe that as $r>1/a$,

$$\exp[i\varphi(rt)\ln(rt)]\,dt=\frac{t\,d(\exp[i\varphi(rt)\ln(rt)])}{i(rt\varphi'(rt)\ln(rt)+\varphi(rt))}\,.$$

Integrating by parts, we obtain:

$$\begin{aligned}\int\limits_a^b f(t)\exp[i\varphi(rt)\ln(rt)]\,dt&=\left.\frac{f(t)t\exp[i\varphi(rt)\ln(rt)]}{i(rt\varphi'(rt)\ln(rt)+\varphi(rt))}\right|_a^b\\&-\frac{1}{i}\int\limits_a^b\left[\frac{f'(t)t+f(t)}{rt\varphi'(rt)\ln(rt)+\varphi(rt)}\right.\\&-\left.\frac{f(t)t(r\varphi'(rt)\ln(rt)+r^2t\varphi''(rt)\ln(rt)+2r\varphi'(rt))}{(rt\varphi'(rt)\ln(rt)+\varphi(rt))^2}\right]\\&\cdot\exp[i\varphi(rt)\ln(rt)]\,dt\,.\end{aligned}$$

It follows from (1) that the right hand of the latter identity tends to zero as $r\to+\infty$.

If $a=0$, $f\in L_1([0,b])$, then given $\varepsilon>0$, we choose $\delta>0$ so that the identity

$$\left|\int\limits_0^\delta f(t)\,dt\right|\leqslant\varepsilon$$

is satisfied. Then, taking into consideration identity $|\exp[i\varphi(rt)\ln(rt)]|=1$, we get:

$$\left|\int\limits_0^b f(t)\exp[i\varphi(rt)\ln(rt]|\,dt\right|=\left|\int\limits_0^\delta+\int\limits_\delta^b\right|\leqslant\varepsilon+\left|\int\limits_\delta^b f(t)\exp[i\varphi(rt)\ln(rt]|\,dt\right|,$$

and by the first of the proof we arrive at the statement of the lemma. □

Remark 1. *As in Riemann-Lebesgue lemma, in Lemma 1, we do not estimate the rate of the vanishing. This is impossible under the assumption $f\in L_1([a,b])$. Under additional assumptions for the smoothness of the function f, by means of integrating by parts, one can obtain a more detailed information on the asymptotic behavior of the integral.*

Remark 2. *A rather wide class of functions obeys restrictions (1). For instance, they hold for the functions $\varphi(r)=(\ln r)^\sigma$, $\varphi(r)=r^\sigma$, $\varphi(r)=\exp(r^\sigma)$ $(\sigma>0)$ and so forth.*

3. Asymptotic formulae for integrals

Theorem 1. *Let $f(t)$ be an absolutely continuous function on the segment $[a,b]$, $0<a<b<\infty$, $\rho\in\mathbb{R}$, and a function $\varphi(r)$ satisfies condition (1). Then*

$$\begin{aligned}\int_a^b f(t)\,d(t^\rho\cos[\varphi(rt)\ln(rt)]) &= b^\rho f(b)\cos[\varphi(br)\ln(br)]-a^\rho f(a)\cos[\varphi(ar)\ln(ar)]+o(1),\\ \int_a^b f(t)\,d(t^\rho\sin[\varphi(rt)\ln(rt)]) &= b^\rho f(b)\sin[\varphi(br)\ln(br)]-a^\rho f(a)\sin[\varphi(ar)\ln(ar)]+o(1)\,,\end{aligned} \tag{3}$$

as $r\to+\infty$.

Proof. Let us prove the first formula in (3). Integrating by parts, we obtain:

$$\int_a^b f(t)\,d(t^\rho\cos[\varphi(rt)\ln(rt)]) = f(t)\,t^\rho\cos[\varphi(rt)\ln(rt)]\Big|_a^b-\int_a^b f'(t)\,t^\rho\cos[\varphi(rt)\ln(rt)]\,dt\,.$$

Since $f'\in L_1([a,b])$, by Lemma 1, the integral in the right hand side converges to zero as $r\to+\infty$. The second formula in (3) can be proved in the same way. □

Let us consider a particular case $\varphi(r)=(\ln r)^\alpha$, where $\alpha>0$ is a constant. In this case we obtain formulae for the limiting set of the integrals in (3) in the direction $r\to+\infty$.

Theorem 2. *Let $f(t)$ be an absolutely continuous function on the segment $[a,b]$, $0<a<b<\infty$, and $\rho\in\mathbb{R}$, $p>1$ be constants. Then the limiting set of the integrals*

$$\int_a^b f(t)\,d(t^\rho\cos[(\ln(rt))^p]),\qquad \int_a^b f(t)\,d(t^\rho\sin[(\ln(rt))^p])$$

in the direction $r\to+\infty$ coincides with the segment

$$[-a^\rho|f(a)|-b^\rho|f(b)|,\ a^\rho|f(a)|+b^\rho|f(b)|]\ .$$

The proof of this theorem is based on the following lemma.

Lemma 2. *Let φ be a T-periodic continuous function, $p>1$, $A=\varphi([0,T))$, a, b be arbitrary numbers, $0<a<b<\infty$. Then the limiting set of the function $(\varphi((x+a)^p),\varphi((x+b)^p))$: $(-a,+\infty)\to\mathbb{R}^2$ as $x\to+\infty$ coincides with the Cartesian square $A\times A$.*

Proof. Let $f(x)=\left(x^{\frac1p}+c\right)^p$, then for $x>|c|^p$

$$f'(x)=\left(1+\frac{c}{x^{\frac1p}}\right)^{p-1}=1+\frac{c(p-1)}{x^{\frac1p}}+\dots\,.$$

We fix a number $x\in[0,T)$ and define a sequence x_k by the formula

$$x_k=(x+kT)^{\frac1p}-a,\quad (x_k+a)^p=x+kT,\quad k\in\mathbb{N}\,.$$

We denote

$$w_k=(x_k+b)^p=\left((x+kT)^{\frac1p}+b-a\right)^p=y_k+j(k)\,T\,,$$

where $j(k)\in\mathbb{N},\ y_k\in[0,T)$.

By the Lagrange formula we get:

$$w_{k+1}-w_k=\left((x+(k+1)T)^{\frac1p}+b-a\right)^p-\left((x+kT)^{\frac1p}+b-a\right)^p$$

$$=T\left(1+\frac{b-a}{(x+(k+\theta)T)^{\frac{1}{p}}}\right)^{p-1}=T\left(1+\frac{(b-a)(p-1)}{(x+(k+\theta)T)^{\frac{1}{p}}}+\dots\right)$$
$$=T+\Delta_k,\quad \theta\in(0,1)\,.$$

It is clear that

$$\Delta_k=T\left(\frac{(b-a)(p-1)}{(x+(k+\theta)T)^{\frac{1}{p}}}+\frac{(b-a)(p-1)(p-2)}{2(x+(k+\theta)T)^{2/p}}+\dots\right)\sim T\frac{(b-a)(p-1)}{(x+(k+\theta)T)^{\frac{1}{p}}},\quad k\to\infty\,.$$

Hence, the sequence Δ_k possesses the properties:

1) $\Delta_k\to 0$ as $k\to\infty$, $\Delta_k\geqslant 0$,

2) $\sum\limits_{k=1}^{\infty}\Delta_k=\infty$,

3) if $w_k=y_k+j(k)T,\ j(k)\in\mathbb{N},\ y_k\in[0,T)$, then

$$w_{k+1}=y_k+\Delta_k+(j(k)+1)\,T\,. \tag{4}$$

It follows from Properties 1), 2), 3) that the limiting set of the sequence y_k as $k\to\infty$ is the left-closed interval $[0,T)$.

Indeed, let $\varepsilon>0$ be an arbitrary number and $y\in[0,T)$. We denote $\varepsilon_1=\min\{\varepsilon,\,T-y\}$. Then $\varepsilon_1>0$. Let k_0 be a number such that as $k\geqslant k_0$, the inequality holds $\Delta_k<\varepsilon_1$. By the definition, $y_{k_0}\in[0,T)$. It follows from (4) that

$$w_{k_0+m}=y_{k_0}+\Delta_{k_0}+\Delta_{k_0+1}+\dots+\Delta_{k_0+m-1}+(j(k_0)+m)\,T\,,$$

for each $m\in\mathbb{N}$. The divergence of the series $\Delta_{k_0}+\Delta_{k_0+1}+\dots$ implies that there exists a minimal m such that the inequality $y_{k_0}+\Delta_{k_0}+\dots+\Delta_{k_0+m-1}>T+y$ holds. Then

$$y_{k_0}+\Delta_{k_0}+\dots+\Delta_{k_0+m-1}=T+y+\delta\,,$$

where $0<\Delta_{k_0+m-1}<\varepsilon_1$.

It is obvious that $0<y+\delta<T$. Then

$$w_{k_0+m}=y_{k_0+m}+(j(k_0)+m+1)\,T,\qquad y_{k_0+m}=y+\delta\,.$$

Since $0<\delta<\varepsilon_1<\varepsilon$, and $\varepsilon>0$ is an arbitrary number, then y belongs to the limiting set of the sequence y_k. Since the number y is arbitrary chosen in the left-closed interval $[0,T)$, this completes the proof. □

Theotrm 2 follows Lemma 2 if we let $x=\ln r$ and observe that $\ln(ra)=\ln r+\ln a$.

Theorem 3. *Let $f(t)$ be an absolutely continuous function on the segment $[a,b]$, $0<a<b<\infty$, and $p>1$. Then as $r\to+\infty$,*

$$\begin{aligned}\int\limits_a^b f(t)\cos[\ln^p(rt)]\,dt&=\frac{1}{p\ln^{p-1}r}[bf(b)\sin[\ln^p(br)]-af(a)\sin[\ln^p(ar)]]+\frac{o(1)}{\ln^{p-1}r},\\ \int\limits_a^b f(t)\sin[\ln^p(rt)]\,dt&=\frac{1}{p\ln^{p-1}r}[af(a)\cos[\ln^p(ar)]-bf(b)\cos[\ln^p(br)]]+\frac{o(1)}{\ln^{p-1}r}\,.\end{aligned} \tag{5}$$

In particular, the limiting set of the functions

$$p\ln^{p-1}r\int\limits_a^b f(t)\cos[\ln^p(rt)]\,dt,\quad p\ln^{p-1}r\int\limits_a^b f(t)\sin[\ln^p(rt)]\,dt$$

in the direction $r\to+\infty$ is the segment

$$[-a|f(a)|-b|f(b)|,\ a|f(a)|+b|f(b)|].$$

This theorem can be proved in the same way as Theorems 1, 2.

If $f(t)$ possesses several derivatives, we can repeat integration by parts and we obtain the following theorem.

Theorem 4. *Let a function $f(t)$ possesses an absolutely continuous $(k-1)th$ derivative $f^{(k-1)}(t)$ on $[a,b]$, $0<a<b<\infty$, $k \geqslant 1$, and let $p>1$. Then*

$$\begin{aligned}\int\limits_a^b f(t)\cos(\ln rt)^p\,dt = &\sum_{m=0}^{\lfloor\frac{k-1}{2}\rfloor}(-1)^m\,\frac{t\Phi_{2m}(t)}{p(\ln rt)^{p-1}}\sin(\ln rt)^p\Bigg|_a^b \\ &\sum_{m=0}^{\lfloor\frac{\lfloor k-2\rfloor}{2}\rfloor}(-1)^m\,\frac{t\Phi_{2m+1}(t)}{p(\ln rt)^{p-1}}\cos(\ln rt)^p\Bigg|_a^b+\frac{o(1)}{(\ln r)^{k(p-1)}},\qquad r\to+\infty\,,\end{aligned} \tag{6}$$

where

$$\Phi_m(t)=L^m[f(t)],\quad L[f(t)]=\frac{1}{p}\frac{d}{dt}\frac{tf(t)}{(\ln rt)^{p-1}}\,.$$

Proof. Let us prove that

$$\begin{aligned}\int\limits_a^b f(t)\cos(\ln rt)^p\,dt = &\sum_{m=0}^{\frac{k-1}{2}}(-1)^m\,\frac{t\Phi_{2m}(t)}{p(\ln rt)^{p-1}}\sin(\ln rt)^p\Bigg|_a^b \\ &+\sum_{m=0}^{\frac{k-2}{2}}(-1)^m\,\frac{t\Phi_{2m+1}(t)}{p(\ln rt)^{p-1}}\cos(\ln rt)^p\Bigg|_a^b \\ &+(-1)^{\frac{k-1}{2}}\int\limits_a^b\frac{t\Phi_k(t)}{p(\ln rt)^{p-1}}\,d\left[\cos(\ln rt)^p\right]\end{aligned} \tag{7}$$

for odd k, while for even k the latter term in the right hand side in (7) reads as

$$(-1)^{\frac{k}{2}}\int\limits_a^b\frac{t\Phi_k(t)}{p(\ln rt)^{p-1}}\,d\left[\sin(\ln rt)^p\right]\,.$$

Integrating by parts and applying the identity

$$\cos(\ln rt)^p\,dt=\frac{t\,d\left[\sin(\ln rt)^p\right]}{p(\ln rt)^{p-1}},\quad \sin(\ln rt)^p\,dt=-\frac{t\,d\left[\cos(\ln rt)^p\right]}{p(\ln rt)^{p-1}}\,, \tag{8}$$

we obtain

$$\int\limits_a^b f(t)\cos(\ln rt)^p\,dt=\frac{t\Phi_0(t)}{p(\ln rt)^{p-1}}\sin(\ln rt)^p\Bigg|_a^b-\int\limits_a^b\Phi_1(t)\sin(\ln rt)^p\,dt\,.$$

This proves the formula (7) as $k=1$. Suppose that formula (7) is valid for all odd $k\leqslant m$, while the function $f(t)$ possesses $(m+1)$th absolutely continuous derivative on the segment $[a,b]$. Integrating twice by parts and applying identity (8), we obtain:

$$\begin{aligned}(-1)^{\frac{m-1}{2}}\int\limits_a^b\frac{t\Phi_m(t)}{p(\ln rt)^{p-1}}\,d\left[\cos(\ln rt)^p\right]&=(-1)^{\frac{m-1}{2}}\frac{t\Phi_m(t)}{p(\ln rt)^{p-1}}\cos(\ln rt)^p\Bigg|_a^b \\ &+(-1)^{\frac{m+1}{2}}\int\limits_a^b\frac{t\Phi_{m+1}(t)}{p(\ln rt)^{p-1}}\,d\left[\sin(\ln rt)^p\right]\end{aligned}$$

$$= (-1)^{\frac{m+1}{2}} \frac{t\Phi_{m+1}(t)}{p(\ln rt)^{p-1}} \sin(\ln rt)^p \Big|_a^b + (-1)^{\frac{m-1}{2}} \frac{t\Phi_m(t)}{p(\ln rt)^{p-1}} \cos(\ln rt)^p \Big|_a^b$$

$$+ (-1)^{\frac{m+1}{2}} \int_a^b \frac{t\Phi_{m+2}(t)}{p(\ln rt)^{p-1}} \, d\left[\cos(\ln rt)^p\right] .$$

Thus, formula (7) is valid also for $k = m+2$. This proves it for all odd $k \geqslant 1$. The proof of even k is similar. Formula (6) follows (7) since

$$\Phi_k(t) = \frac{O(1)}{(\ln r)^{(k+1)(p-1)}}, \qquad r \to +\infty .$$

□

Remark 3. *It is clear that an infinitely differentiable function $f(t)$ satisfies asymptotic formula (6) for each k.*

In the case $p \in (0,1)$, asymptotic expansion for the integral is obtained not by integration by parts, but by expanding the kernel.

Theorem 5. *Let $f(t) \in L_1([a,b])$, $0 < a < b < \infty$ and $p \in (0,1)$. Then as $r > \max\{b,\, 1/a\}$, the expansion holds true:*

$$\int_a^b f(t) \exp(i\lambda(\ln rt)^p)\, dt = \exp(i\lambda \ln^p r) \left(\alpha_0 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha_{n,k}}{(\ln r)^{k(1-p)+n}} \right), \tag{9}$$

where

$$\alpha_0 = \int_a^b f(t)\, dt, \quad \alpha_{n,k} = \frac{i^k \lambda^k}{k!} c_{n,k} \int_a^b f(t)(\ln t)^{n+k}\, dt ,$$

and the coefficients $c_{n,k}$ are determined by the expansion

$$\left(\frac{1}{x}((1+x)^p - 1) \right)^k = \sum_{n=-k+1}^{\infty} c_{n,k}\, x^n .$$

The double series in formula (9) is absolutely convergent for sufficiently large r. Expansion (9) is true as an asymptotic expansion for $a = 0$ and

$$\int_0 t^{-\varepsilon} |f(t)|\, dt < \infty$$

for some $\varepsilon > 0$. This expansion holds as asymptotic expansion also in the case

$$b = \infty, \qquad \int^{\infty} t^{\varepsilon} |f(t)|\, dt < \infty .$$

Proof. We employ the expansion of the function e^x into the Taylor series and the identity

$$\exp\left(i\lambda(\ln rt)^p\right) = \exp\left(i\lambda \ln^p r\right) \exp\left(i\lambda \ln^{p-1} r \ln t \left(\frac{\ln r}{\ln t} \left(1 + \frac{\ln t}{\ln r} \right)^p - 1 \right) \right)$$

to obtain

$$\int_a^b f(t) \exp(i\lambda(\ln rt)^p)\, dt$$

$$= \exp\left(i\lambda \ln^p r\right)\left(\alpha_0 + \sum_{k=1}^{\infty} \frac{i^k \lambda^k}{\ln^{k(1-p)} r} \int\limits_a^b f(t) \ln^k t \left(\frac{\ln r}{\ln t}\left(1 + \frac{\ln t}{\ln r}\right)^p - 1\right)^k\right) dt\,.$$

Denoting $x = \ln t/\ln r$, we obtain expansion (9). The absolute convergence of the series is implied by the inequality $|\ln t| < |\ln r|$, which holds as $r > \max\{b,\, 1/a\}$. The final part of the theorem follows the fact that the convergence of the integral

$$\int\limits_0 t^{-\varepsilon}|f(t)|\,dt < \infty \qquad \left(\int\limits^{\infty} t^{\varepsilon}|f(t)|\,dt < \infty\right)$$

implies the convergence of the integral

$$\int\limits_0 f(t)\left(\ln t\right)^k\,dt < \infty \qquad \left(\int\limits^{\infty} f(t)\left(\ln t\right)^k\,dt < \infty\right)$$

for each $k > 0$. □

Remark 4. *The coefficients $c_{n,k}$ can be written in a closed form if we employ the binomial theorem for $((1+x)^p - 1)^k$ and then expand the function $(1+x)^{mp}$ into the Taylor series. As a result, for all integer $k \geqslant 1$ and $n \geqslant -k+1$ we get*

$$c_{n,k} = \sum_{m=1}^{k} (-1)^{k-m} \frac{k!}{m!(k-m)!} \frac{mp(mp-1)\cdots(mp-(n+k)+1)}{(n+k)!}\,.$$

4. Azarin limiting set for some functions

We consider the functions

$$u_1(z,p,\rho,\lambda) = \frac{r\sin\theta}{\pi}\int\limits_0^{\infty} \frac{\tau^\rho \exp(i\lambda|\ln\tau|^p)}{\tau^2 - 2\tau r\cos\theta + r^2}\,d\tau = \frac{r^\rho \sin\theta}{\pi}\int\limits_0^{\infty} \frac{t^\rho \exp(i\lambda|\ln tr|^p)}{t^2 - 2t\cos\theta + 1}\,dt\,, \tag{10}$$

$$\begin{aligned} u_2(z,p,\rho,\lambda) =& \frac{1}{\pi}\int\limits_0^{\infty} \frac{\tau^{\rho-1} r(r - \tau\cos\theta)}{\tau^2 - 2\tau r\cos\theta + r^2} \exp(i\lambda|\ln\tau|^p)\,d\tau \\ =& \frac{r^\rho}{\pi}\int\limits_0^{\infty} \frac{(1 - t\cos\theta)t^{\rho-1}}{t^2 - 2t\cos\theta + 1} \exp(i\lambda|\ln tr|^p)\,dt\,, \end{aligned} \tag{11}$$

$$u_3(z,p,\rho,\lambda) = \operatorname{Re} u_1(z,p,\rho,\lambda), \qquad u_4(z,p,\rho,\lambda) = \operatorname{Im} u_1(z,p,\rho,\lambda),$$
$$u_5(z,p,\rho,\lambda) = \operatorname{Re} u_2(z,p,\rho,\lambda), \qquad u_6(z,p) = \operatorname{Im} u_2(z,p,\rho,\lambda),$$

where $z = re^{i\theta}$, $\rho \in (0,1)$, $p > 0$, $\lambda \geqslant 0$. As $p = 1$, we omit the modulus sign to obtain a simpler function.

The following integrals are calculated by means of complex integration:

$$\frac{\sin\theta}{\pi}\int\limits_0^{\infty} \frac{t^\rho\,dt}{t^2 - 2t\cos\theta + 1} = \frac{\sin\rho(\pi-\theta)}{\sin\rho\pi}, \qquad \theta \in (0,2\pi), \quad \rho \in (0,1)\,, \tag{12}$$

$$\frac{1}{\pi}\int\limits_0^{\infty} \frac{(1 - t\cos\theta)t^{\rho-1}\,dt}{t^2 - 2t\cos\theta + 1} = \frac{\cos\rho(\pi-\theta)}{\sin\rho\pi}, \qquad \theta \in (0,2\pi), \quad \rho \in (0,1)\,. \tag{13}$$

Theorem 5 can be applied for the cases $a = 0$, $b = \infty$,

$$f(t) = \frac{\sin\theta}{\pi}\,\frac{t^\rho}{t^2 - 2t\cos\theta + 1} \quad \text{or} \quad f(t) = \frac{1}{\pi}\,\frac{(1 - t\cos\theta)t^{\rho-1}}{t^2 - 2t\cos\theta + 1}\,,$$

where $\theta \in (0, 2\pi)$, $\rho \in (0, 1)$. As $p \in (0, 1)$, $\theta \in (0, 2\pi)$, $\rho \in (0, 1)$, $\lambda \geqslant 0$ according this theorem we obtain the relations

$$\begin{aligned} u_3(z, p, \rho, \lambda) &= \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \cos(\lambda \ln^p r) + \frac{O(1)r^\rho}{\ln^{1-p} r}, \qquad r \to +\infty\,, \\ u_4(z, p, \rho, \lambda) &= \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \sin(\lambda \ln^p r) + \frac{O(1)r^\rho}{\ln^{1-p} r}, \qquad r \to +\infty\,, \end{aligned} \tag{14}$$

and similar formulae for $u_5(z, p, \rho, \lambda)$, $u_6(z, p, \rho, \lambda)$ with $\sin\rho(\pi - \theta)$ replaced by $\cos\rho(\pi - \theta)$. Namely,

$$\begin{aligned} u_5(z, p, \rho, \lambda) &= \frac{\cos\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \cos(\lambda \ln^p r) + \frac{O(1)r^\rho}{\ln^{1-p} r}, \qquad r \to +\infty\,, \\ u_6(z, p, \rho, \lambda) &= \frac{\cos\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \sin(\lambda \ln^p r) + \frac{O(1)r^\rho}{\ln^{1-p} r}, \qquad r \to +\infty\,. \end{aligned}$$

The Azarin limiting sets for the introduced functions are described by the relations

$$\begin{aligned} \operatorname{Fr} u_3 = \operatorname{Fr} u_4 &= \left\{ \alpha \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho :\ \alpha \in [-1, 1] \right\}\,, \\ \operatorname{Fr} u_5 = \operatorname{Fr} u_6 &= \left\{ \alpha \frac{\cos\rho(\pi - \theta)}{\sin\rho\pi} r^\rho :\ \alpha \in [-1, 1] \right\} \end{aligned} \tag{15}$$

for $\rho \in (0, 1)$, $p \in (0, 1)$, $\lambda > 0$.

As an example, let us prove relation (15). Let $\alpha \in [-1, 1]$ be a fixed number. We introduce the notations

$$u_t(z) = \frac{u_3(tz, p)}{t^\rho}, \qquad t_n = \exp\left(\left(\frac{1}{\lambda}(\arccos\alpha + 2\pi n) \right)^{\frac{1}{p}} \right)\,.$$

It follows from (14) that for each fixed $r > 0$ the asymptotic identity holds:

$$\begin{aligned} &\left| u_{t_n}(re^{i\theta}) - \alpha \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \right| \\ &\qquad = \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho \left| \cos\left(\lambda \ln^p t_n r\right) - \alpha + \frac{O(1)r^\rho}{\ln^{1-p} t_n r} \right|, \qquad n \to \infty\,. \end{aligned} \tag{16}$$

It follows from the definition of sequence t_n and an asymptotic identity

$$(a + x_n)^p = x_n^p + \frac{pa}{x_n^{1-p}} + O\left(\frac{1}{x_n^{2-p}} \right), \qquad x_n \to \infty\,,$$

that for each fixed $r > 0$

$$\lim_{n\to\infty} \left(\cos\left(\lambda \ln^p t_n r\right) - \alpha\right) = 0\,.$$

By (16) this implies that in the circle $\{z : |z| \leqslant R\}$, the sequence $u_{t_n}(z)$ converges uniformly to the function

$$w_\alpha(re^{i\theta}) = \alpha \frac{\sin\rho(\pi - \theta)}{\sin\rho\pi} r^\rho\,.$$

Hence, the convergence also holds in the sense of the topology of the space of tempered distributions on the plane. The definition of the limiting set implies that $\{w_\alpha(z) : \alpha \in [-1, 1]\} \subset \operatorname{Fr} u_3$.

Since in each sequence $t_n \to +\infty$ we can select a subsequence of the form

$$t_{n_k} = \exp\left(\left(\frac{\arccos \alpha_{n_k} + 2\pi n_k}{\lambda}\right)^{\frac{1}{p}}\right),$$

where n_k are natural, $\alpha_{n_k} \to \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, there are no other functions in the limiting set.

If $h_k(\theta)$ is the Phragmén-Lindelöf indicator of the function $u_k(z,p)$, that is,

$$h_k(\theta) = \limsup_{r\to+\infty} \frac{u_k(re^{i\theta},p)}{r^\rho},$$

then the identities hold:

$$h_3(\theta) = h_4(\theta) = \frac{|\sin\rho(\pi-\theta)|}{\sin\rho\pi}, \qquad h_5(\theta) = h_6(\theta) = \frac{|\cos\rho(\pi-\theta)|}{\sin\rho\pi}.$$

The structure of asymptotic formulae changes essentially while passing to the cases $p = 1$ or $p > 1$.

Theorem 6. *Let $p = 1$, $\rho \in (0,1)$, $\lambda \geqslant 0$. Then*

$$\begin{aligned}
u_3(z,1,\rho,\lambda) &= [A_\rho(\lambda,\theta)\cos(\lambda\ln r) - B_\rho(\lambda,\theta)\sin(\lambda\ln r)]r^\rho, \\
u_4(z,1,\rho,\lambda) &= [B_\rho(\lambda,\theta)\cos(\lambda\ln r) + A_\rho(\lambda,\theta)\sin(\lambda\ln r)]r^\rho, \\
u_5(z,1,\rho,\lambda) &= [C_\rho(\lambda,\theta)\cos(\lambda\ln r) - D_\rho(\lambda,\theta)\sin(\lambda\ln r)]r^\rho, \\
u_6(z,1,\rho,\lambda) &= [D_\rho(\lambda,\theta)\cos(\lambda\ln r) + C_\rho(\lambda,\theta)\sin(\lambda\ln r)]r^\rho,
\end{aligned} \tag{17}$$

where

$$A_\rho(\lambda,\theta) = \operatorname{Re}\frac{\sin(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi}, \qquad B_\rho(\lambda,\theta) = \operatorname{Im}\frac{\sin(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi},$$

while similar formulae for the quantities $C_\rho(\lambda,\theta)$ and $D_\rho(\lambda,\theta)$ are obtained via replacing $\sin(\rho+i\lambda)(\pi-\theta)$ *by* $\cos(\rho+i\lambda)(\pi-\theta)$*, that is,*

$$C_\rho(\lambda,\theta) = \operatorname{Re}\frac{\cos(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi}, \qquad D_\rho(\lambda,\theta) = \operatorname{Im}\frac{\cos(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi}.$$

Proof. First we observe that the written identities imply immediately that the function $u_5(z,1,\rho,\lambda)$ obeys the formulae:

$$\begin{aligned}
&\operatorname{Fr} u_5(z,1,\rho,\lambda) = \{(C_\rho(\lambda,\theta)\sin\varphi - D_\rho(\lambda,\theta)\cos\varphi)r^\rho : \ \varphi \in [0,2\pi]\}, \\
&h_5(\theta) = \sqrt{C_\rho^2(\lambda,\theta) + D_\rho^2(\lambda,\theta)}.
\end{aligned} \tag{18}$$

Similar formulae also hold for $u_3(z,1,\rho,\lambda)$, $u_4(z,1,\rho,\lambda)$, $u_6(z,1,\rho,\lambda)$.

Let us prove one of these identities. We have:

$$u_3(z,1,\rho,\lambda) = \frac{r^\rho\sin\theta}{\pi}\operatorname{Re} r^{i\lambda}\int_0^\infty \frac{t^{\rho+i\lambda}}{t^2-2t\cos\theta+1}\,dt.$$

In view of formula (12), by the method of analytic continuation in the parameter ρ we obtain the expression

$$\int_0^\infty \frac{t^{\rho+i\lambda}}{t^2-2t\cos\lambda+1}\,dt = \frac{\pi}{\sin\theta}\,\frac{\sin(\pi-\theta)(\rho+i\lambda)}{\sin\pi(\rho+i\lambda)},$$

and this implies formula (17). Identity (18) is obtained by the same arguing as for identity (15). □

Let us consider the functions $u_k(z,p,\rho,\lambda)$ as $p > 1$. We shall prove an asymptotic formula for the function $u_1(z,p,\rho,\lambda)$ only.

Theorem 7. *Let $p > 1$, $\rho \in (0,1)$, $\lambda > 0$. Then as $p \in (1,2]$ we have*

$$u_1(z,p,\rho,\lambda) = \frac{\sin\theta}{\pi r}\int_{-\infty}^{0}\frac{\exp\left[(\rho+1)\tau + i\lambda|\tau|^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau + \frac{\sin\theta}{\pi r}\int_{0}^{i\infty}\frac{\exp\left[(\rho+1)\tau + i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau$$
$$+ r^{\rho}e^{i\rho\theta}\sum_{n=0}^{\infty}\exp(2\pi i\rho n)\exp(i\lambda(\ln r + i(\theta+2\pi n))^p)$$
$$- r^{\rho}e^{-i\rho\theta}\sum_{n=0}^{\infty}\exp(2\pi i\rho(n+1))\exp(i\lambda(\ln r + i(2\pi(n+1)-\theta))^p)\,,$$

and each of the above series is convergent and asymptotic. As $p > 2$, if the straight line $\operatorname{Im}\tau = s$ *contains no poles of the denominator, the formula holds true:*

$$u_1(z,p,\rho,\lambda) = \frac{\sin\theta}{\pi r}\int_{-\infty}^{0}\frac{\exp\left[(\rho+1)\tau + i\lambda|\tau|^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau + \frac{\sin\theta}{\pi r}\int_{0}^{is}\frac{\exp\left[(\rho+1)\tau + i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau$$
$$+ \frac{\sin\theta}{\pi r}\int_{is}^{is+\infty}\frac{\exp\left[(\rho+1)\tau + i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau$$
$$+ r^{\rho}e^{i\rho\theta}\sum_{0\leqslant n<\frac{s-\theta}{2\pi}}\exp(2\pi i\rho n + i\lambda(\ln r + i(\theta+2\pi n))^p)$$
$$- r^{\rho}e^{i\rho\theta}\sum_{0\leqslant n<\frac{s+\theta}{2\pi}-1}\exp(2\pi i\rho(n+1) + i\lambda(\ln r + i(2\pi(n+1)-\theta))^p)\,,$$

and each term in the written sums tends to zero as $z \to \infty$ faster than any power of z. If in the above integrals we expand the kernel

$$\left(\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1\right)^{-1}$$

into the powers of r^{-1}, then a term-wise integration gives the expansion of the corresponding integral in an asymptotic series in powers of r^{-1}.

Proof. We make the change $t = \exp v$ in integral (10). Then

$$u_1(z,p,\rho,\lambda) = \frac{r^{\rho}\sin\theta}{\pi}\int_{-\infty}^{\infty}\frac{\exp\left[(\rho+1)v + i\lambda|v+\ln r|^p\right]}{e^{2v} - 2e^{v}\cos\theta + 1}\,dv\,.$$

After the change $v = \tau - \ln r$ we obtain

$$u_1(z,p,\rho,\lambda) = \frac{\sin\theta}{\pi r}\int_{-\infty}^{\infty}\frac{\exp\left[(\rho+1)\tau + i\lambda|\tau|^p\right]}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}\,d\tau\,.$$

The denominator in the latter integral vanishes at the points

$$\tau = \ln r \pm i\theta + 2n\pi, \quad n = 0, \pm1, \pm2, \ldots\,.$$

Let L_s be a rectangle with vertices at the points θ, R, $R+is$, is, and the number s is chosen so that the boundary of the rectangle contains no aforementioned zeroes. The function

$$f(\tau) = \frac{\exp\left((\rho+1)\tau + i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2} - 2\,\frac{e^{\tau}}{r}\,\cos\theta + 1}$$

is holomorphic inside the rectangle L_s except finitely many poles τ_k. By the theorems on the residues

$$\frac{1}{2\pi i}\int\limits_{\partial L_s}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^{\tau}}{r}\,\cos\theta+1}\,d\tau=\sum_{\tau_k\in L_s}\mathrm{Res}_{\tau_k}f(\tau)\,,$$

where ∂L_s is the boundary of the rectangle L_s passed in the positive direction. Since all poles of the function f are simple, then

$$\mathrm{Res}_{\tau_k}f(\tau)=\frac{\exp\left((\rho+1)\tau_k+i\lambda\tau_k^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^{\tau}}{r}\,\cos\theta+1}\,.$$

If $\tau_k=\tau_{1,n}=\ln r+i(\theta+2n\pi)$, then

$$\mathrm{Res}_{\tau_{1,n}}f(\tau)=\frac{r^{\rho+1}e^{i(\rho+1)\theta}\exp(2n\pi\rho i+i\lambda\tau_{1,n}^p)}{2e^{i\theta}(e^{i\theta}-\cos\theta)}=\frac{r^{\rho+1}e^{i\rho\theta}\exp(2n\pi\rho i+i\lambda\tau_{1,n}^p)}{2i\sin\theta}\,.$$

If $\tau_k=\tau_{2,n}=\ln r+i(2(n+1)\pi-\theta)$, then

$$\mathrm{Res}_{\tau_{2,n}}f(\tau)=\frac{r^{\rho+1}e^{-i(\rho+1)\theta}\exp(2(n+1)\pi\rho i+i\lambda\tau_{2,n}^p)}{2e^{-i\theta}(e^{-i\theta}-\cos\theta)}=\frac{r^{\rho+1}e^{-i\rho\theta}\exp(2(n+1)\pi\rho i+i\lambda\tau_{2,n}^p)}{2i\sin\theta}\,.$$

Hence,

$$\begin{aligned}\int\limits_{\partial L_s}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\cos\theta\frac{e^{\tau}}{r}+1}\,d\tau=&\pi\sum_{0\leqslant n<\frac{s-\theta}{2\pi}}\frac{r^{\rho+1}e^{i\rho\theta}\exp(2n\pi\rho i+i\lambda\tau_{1,n}^p)}{\sin\theta}\\&-\pi\sum_{0\leqslant n<\frac{s+\theta}{2\pi}-1}\frac{r^{\rho+1}e^{-i\rho\theta}\exp(2(n+1)\pi\rho i+i\lambda\tau_{2,n}^p)}{\sin\theta}\,,\end{aligned}$$

where the sum over the empty set is adopted to the zero. We denote by I_1 the segment $[0,R]$, by I_2 we denote $[R,R+is]$, by I_3 we denote the segment $[0,is]$, and by I_4 we denote the segment $[is,R+is]$. Then

$$\int\limits_{\partial L_s}f(\tau)\,d\tau=\int\limits_{I_1}f(\tau)\,d\tau+\int\limits_{I_2}f(\tau)\,d\tau-\int\limits_{I_3}f(\tau)\,d\tau-\int\limits_{I_4}f(\tau)\,d\tau\,.$$

If $\tau=R+iu$, where $0\leqslant u\leqslant s$, and the number R is large enough, then $\tau^p=R^p(1+iuR^{-1})^p$. This implies $\mathrm{Im}\,\tau^p\geqslant 0$. Moreover,

$$\left|\frac{e^{2\tau}}{r^2}-2\cos\theta\,\frac{e^{\tau}}{r}+1\right|\geqslant\left(\frac{e^R}{r}-1\right)^2\,.$$

This is why

$$\left|\int\limits_{I_2}f(\tau)\,d\tau\right|\leqslant\frac{e^{(\rho+1)R}}{\left(\frac{e^R}{r}-1\right)^2}\,s,\qquad\lim_{R\to+\infty}\int\limits_{I_2}f(\tau)\,d\tau=0\,.$$

This implies

$$\int\limits_0^\infty \frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau=\int\limits_0^{is}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau+\int\limits_{is}^{is+\infty}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau$$
$$+\pi r^{\rho+1}e^{i\rho\theta}\sum_{0\leqslant n<\frac{s-\theta}{2\pi}}\frac{\exp(2n\pi\rho i+i\lambda\tau_{1,n}^p)}{\sin\theta}$$
$$-\,r^{\rho+1}e^{-i\rho\theta}\pi\sum_{0\leqslant n<\frac{s+\theta}{2\pi}-1}\frac{\exp(2(n+1)\pi\rho i+i\lambda\tau_{2,n}^p)}{\sin\theta}\,,$$

$$\begin{aligned}u_1(z,p,\rho,\lambda)=&\frac{\sin\theta}{\pi}\int\limits_{-\infty}^{0}\frac{\exp\left[(\rho+1)\tau+i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau+\frac{\sin\theta}{\pi}\int\limits_0^{is}\frac{\exp\left[(\rho+1)\tau+i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau\\&+\frac{\sin\theta}{\pi}\int\limits_{is}^{is+\infty}\frac{\exp\left[(\rho+1)\tau+i\lambda\tau^p\right]}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^\tau}{r}\,\cos\theta+1}\,d\tau+\\&+\,r^\rho e^{i\rho\theta}\sum_{0\leqslant n<\frac{s-p}{2\pi}}\exp(2\pi i\rho n+i\lambda(\ln r+i(\theta+2\pi n))^p)\\&-\,r^\rho e^{-i\rho\theta}\sum_{0\leqslant n<\frac{s+p}{2\pi}-1}\exp(2\pi i\rho(n+1)+i\lambda(\ln r+i(2\pi(n+1)-\theta))^p)\,.\end{aligned}\tag{19}$$

First we consider the case $p>2$. If $\tau=u+is$, then

$$|\exp((\rho+1)\tau+i\lambda\tau^p)|=\exp((\rho+1)u+\varphi(u))\,,\tag{20}$$

where $\varphi(u)=\operatorname{Re}i\lambda(u+is)^p$. We have:

$$\begin{aligned}\varphi(u)=&\operatorname{Re}\left(i\lambda u^p\left(1+\frac{is}{u}\right)^p\right)\\=&\operatorname{Re}\left(i\lambda u^p\left(1+p\frac{is}{u}-\frac{p(p-1)}{2}\frac{s^2}{u^2}+\dots\right)\right)\sim-\lambda psu^{p-1},\qquad u\to+\infty\,.\end{aligned}\tag{21}$$

We also observe that if we choose s as follows: $s=\pi+2m\pi$ as $\cos\theta\geqslant0$ and $s=2m\pi$ as $\cos\theta<0$, then for $\tau=u+is$ the estimate holds:

$$\left|\frac{e^{2\tau}}{r^2}-2\cos\theta\,\frac{e^\tau}{r}+1\right|\geqslant\frac{e^{2u}}{r^2}+1\,.\tag{22}$$

We have:

$$\begin{aligned}\left(\frac{e^{2\tau}}{r^2}-2\cos\theta\frac{e^\tau}{r}+1\right)^{-1}=&\left(\left(\frac{e^\tau}{r}-e^{i\theta}\right)\left(\frac{e^\tau}{r}-e^{-i\theta}\right)\right)^{-1}\\=&\frac{1}{2i\sin\theta}\left(\frac{1}{e^{-i\theta}\left(1-\frac{e^{\tau+i\theta}}{r}\right)}-\frac{1}{e^{i\theta}\left(1-\frac{e^{\tau-i\theta}}{r}\right)}\right)\,.\end{aligned}\tag{23}$$

Employing the identity

$$\frac{1}{1-z}=1+z+\dots+z^n+\frac{z^{n+1}}{1-z}$$

and (23), we get:

$$\frac{1}{\frac{e^{2\tau}}{r^2}-2\cos\theta\frac{e^\tau}{r}+1}=\sum_{k=0}^{n}\frac{\sin((k+1)\theta)}{\sin\theta}\frac{e^{k\tau}}{r^k}$$

$$+\frac{1}{\sin\theta}\,\frac{e^{(n+1)\tau}}{r^{n+1}}\;\frac{\sin((n+2)\theta)-\sin((n+1)\theta)\frac{e^{\tau}}{r}}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^{\tau}}{r}\,\cos\theta+1}\,.$$

Employing this identity, we obtain:

$$\frac{\sin\theta}{\pi}\int\limits_{is}^{is+\infty}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^{\tau}}{r}\,\cos\theta+1}\,d\tau=\frac{1}{\pi}\sum_{k=0}^{n}\frac{\sin((k+1)\theta)}{r^k}\int\limits_{is}^{is+\infty}e^{k\tau}\exp((\rho+1)\tau+i\lambda\tau^p))\,d\tau$$

$$+\frac{1}{\pi r^{n+1}}\int\limits_{is}^{is+\infty}\frac{\left(\sin((n+2)\theta)-\frac{e^{\tau}}{r}\sin((n+1)\theta)\right)e^{(n+1)\tau}\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,\frac{e^{\tau}}{r}\,\cos\theta+1}\,d\tau\,.$$

In what follows, s is chosen in the above way. Then by relations (20), (21) and inequality (22) we conclude that all written above integrals converge. Moreover, if $r>1$ and

$$R_n=\frac{1}{\pi}\int\limits_{is}^{is+\infty}\frac{\left(\sin((n+2)\theta)-\frac{e^{\tau}}{r})\sin((n+1)\theta)\right)e^{(n+1)\tau}\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\frac{e^{\tau}}{r}\cos\theta+1}\,d\tau\,,$$

we apply the above relations to obtain:

$$R_n\leqslant\frac{1}{\pi}\int\limits_{0}^{\infty}(1+e^u)e^{(n+1)u}\exp((\rho+1)u+\varphi(u))\,du=c_n<\infty\,.$$

This yields that the series

$$\frac{1}{\pi}\sum_{k=0}^{n}\frac{\sin((k+1)\theta)}{r^k}\int\limits_{is}^{is+\infty}e^{k\tau}\exp((\rho+1)\tau+i\lambda\tau^p))\,d\tau$$

is asymptotic in powers of r^{-1} for the integral

$$\frac{\sin\theta}{\pi}\int\limits_{is}^{is+\infty}\frac{\exp\left((\rho+1)\tau+i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2}-2\,(\cos\theta)\frac{e^{\tau}}{r}+1}\,d\tau\,.$$

We note that the statements of the theorem on other integrals can be proved in the same way.

We consider the expression

$$b_{n,1}(z)=\exp(2\pi n\rho i+i\lambda(\ln r+i\theta+2\pi n)^p)\,.$$

We have:

$$\begin{aligned}|b_{n,1}(z)|&=\exp\left(\operatorname{Re}\,i\lambda(\ln r)^p\left(1+i\frac{\theta+2\pi n}{\ln r}\right)^p\right)\\&=\exp(-(1+o(1))\lambda p(\theta+2\pi n)(\ln r)^{p-1}),\qquad r\to+\infty\,.\end{aligned}$$

This implies that $b_{n+1,1}(z)=o(1)\,b_{n,1}(z)$.

In the case $p>2$, the quantity $b_{n,1}(z)$ decays faster than any power of z. Moreover,

$$|b_{n,1}(z)|=\exp\left(\operatorname{Re}\,ie^{\frac{i\pi p}{2}}\lambda(\theta+2\pi n)^p\left(1-i\frac{\ln r}{\theta+2\pi n}\right)^p\right)\,.$$

If $p\in(1,2)$, then for sufficiently large n the inequality

$$\operatorname{Re}\left(ie^{i\pi p/2}\left(1-i\frac{\ln r}{\theta+2\pi n}\right)^p\right)\leqslant\delta<0$$

holds true. This is why for such p the series $\sum\limits_{n=0}^{\infty}b_{n,1}(z)$ is convergent.

If $p = 2$, then

$$b_{n,1}(z) = \exp(2\pi n\rho i + i\lambda(\ln^2 r - (\theta + 2\pi n)^2) - 2\lambda \ln r(\theta + 2\pi n)),$$

and in this case the series $\sum_{n=0}^{\infty} b_{n,1}(z)$ is also convergent. Similar statements hold for the quantity

$$b_{n,2}(z) = \exp(2\pi(n+1)\rho i + i\lambda(\ln r + i(2\pi(n+1) - \theta))^p)\,.$$

We denote by γ_s the contour formed by the ray $(-\infty, 0)$, the segment $[0, is]$ and the ray $[is, is+\infty)$. In the case $p > 2$, we obtain an asymptotic expansion for the function $u_1(z,p)$ as follows. We expand the kernel

$$\frac{1}{\frac{e^{2\tau}}{r^2} - 2\frac{e^{\tau}}{r}\cos\theta + 1}$$

into the power series in r^{-1} (this series converges as $r > e^{\operatorname{Re}\tau}$) and we substitute it into the integral

$$\frac{\sin\theta}{\pi}\int\limits_{\gamma_s} \frac{\exp\left((\rho+1)\tau + i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2} - 2\left(\cos\theta\right)\frac{e^{\tau}}{r} + 1}\, d\tau\,.$$

Integrating then term-by-term, we obtain an asymptotic expansion for the function $u_1(z,p)$. We note that in the integrals for the coefficients the integration path can be replaced by the real axis. Therefore, to obtain an asymptotic expansion, we can avoid employing the contour γ_s. It is needed only to justify the asymptotic expansion. In particular, as $p > 2$, $\varepsilon < \arg z < 2\pi - \varepsilon$, where ε is an arbitrary positive number, the function $ru_1(z,p)$ tends to the constant

$$\frac{\sin\theta}{\pi}\int\limits_{-\infty}^{\infty} \exp\left((\rho+1)\tau + i\lambda|\tau|^p\right)\, d\tau\,.$$

uniformly as $z \to \infty$.

Let $p \in (1, 2]$. We are going to show that in this case

$$\lim_{s\to+\infty}\int\limits_{is}^{is+\infty} \frac{\exp\left((\rho+1)\tau + i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2} - 2\frac{e^{\tau}}{r}\cos\theta + 1}\, d\tau = 0\,,$$

as s tends to infinity passing the values, which we mentioned above.

Indeed, if $\tau = u + is$ and s takes the mentioned values, then

$$\begin{aligned}\left|\frac{\exp\left((\rho+1)\tau + i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2} - 2\left(\cos\theta\right)\frac{e^{\tau}}{r} + 1}\right| &\leqslant \frac{\exp\left((\rho+1)u + \operatorname{Re} i\lambda(u+is)^p\right)}{\frac{e^{2u}}{r^2} + 1}\\ &= \frac{\exp\left((\rho+1)u - \lambda(u^2+s^2)^{\frac{p}{2}}\sin\left(p\arctan\frac{s}{u}\right)\right)}{\frac{e^{2u}}{r^2} + 1}\,.\end{aligned}$$

We obtain:

$$\left|\int\limits_{is}^{is+\infty} \frac{\exp\left((\rho+1)\tau + i\lambda\tau^p\right)}{\frac{e^{2\tau}}{r^2} - 2\frac{e^{\tau}}{r}\cos\theta + 1}\, d\tau\right| \leqslant \int\limits_{0}^{\infty} \frac{e^{(\rho+1)u}}{\frac{e^{2\tau}}{r^2} + 1}\exp\left(-\lambda(u^2+s^2)^{\frac{p}{2}}\sin\left(p\arctan\frac{s}{u}\right)\right)\, du.$$

The first factor in the integral is an integrable function on the ray $[0,\infty)$, while the second factor is less than one and tends to zero as $s \to +\infty$. By the dominated convergence theorem, the integral tends to zero. Passing to the limit as $s \to \infty$ in identity (19), in the case $p \in (1,2]$ we obtain a formula for $u_1(z,p)$ given in the formulation of the theorem. One can also show that the obtained series are not only converging but also asymptotic. $\square$

Similar formulae can be written for other functions $u_k(z,p,\rho,\lambda)$. It is also possible to write more precise asymptotic formulae. Such formulae are obtained by applying a corresponding technique of transformation of the integrals in the complex plane. We note that to study the functions $u_k(z,p,\rho,\lambda)$ as $p>1$, Theorem 4 is not applicable since all non-integral terms vanish. We see that the structure of the asymptotic behavior of the functions $u_k(z,p,\rho,\lambda)$ differ for the cases $p\in(0,1)$, $p=1$, $p>1$. In particular, as $p>1$, the function $u_k(z,p,\rho,\lambda)$ have different growth rate on different rays θ_1, θ_2, $0<\theta_1<\theta_2\leqslant\pi$. This is not the case as $p\in(0,1]$. The behavior of the functions $u_k(z,p,\rho,\lambda)$ as $p\in(0,1)$ and as $p=1$ differs as well. This can be seen be the structure of the Azarin limiting sets of such functions.

5. Asymptotic formulae for irregularly growing entire functions

Let $f(z)$ be an entire function of order $\rho\in(0,1)$ with positive zeroes z_k, $k=1,2,\dots$. We shall consider the functions $\ln(1-\frac{z}{z_k})$ in the plane cut along the ray $[0,+\infty)$ fixing the branch by the positivity of the function for negative real z. Then

$$\ln f(z)=\sum_{k=1}^{\infty}\ln\left(1-\frac{z}{z_k}\right)=\int\limits_0^{\infty}\ln\left(1-\frac{z}{t}\right)\,dn(t)=\int\limits_0^{\infty}\frac{z}{z-t}\frac{n(t)}{t}\,dt\,,$$

where $n(t)=\sum\limits_{z_k\leqslant t}1$ is the counting functions of the zeroes of f. We have:

$$\ln|f(z)|=\int\limits_0^{\infty}\frac{r(r-t\cos\theta)}{t^2-2tr\cos\theta+r^2}\frac{n(t)}{t}\,dt,\qquad z=re^{i\theta}\,.$$

Let

$$\varphi(t)=t^{\rho}(a_0+a_1\cos(\lambda\ln t)+b_1\sin(\lambda\ln t)),\qquad t>0,$$

where $\rho\in(0,1)$, $a_0>0$, $\lambda\geqslant0$, and a_1, b_1 are arbitrary real numbers. If

$$a_0\geqslant\sqrt{1+\frac{\lambda^2}{\rho^2}}\sqrt{a_1^2+b_1^2},$$

then $\varphi(t)$ is a strictly increasing function. Indeed,

$$\varphi'(t)=\rho t^{\rho-1}\left[a_0+\left(a_1+\frac{\lambda}{\rho}b_1\right)\cos(\lambda\ln t)+\left(b_1-\frac{\lambda}{\rho}a_1\right)\sin(\lambda\ln t)\right]\,,$$

and by the elementary inequality

$$C_1\sin\alpha+C_2\cos\alpha\geqslant-\sqrt{C_1^2+C_2^2}$$

we obtain

$$\varphi'(t)\geqslant\rho t^{\rho-1}\left[a_0-\sqrt{\left(a_1+\frac{\lambda}{\rho}b_1\right)^2+\left(b_1-\frac{\lambda}{\rho}a_1\right)^2}\,\right]=\rho t^{\rho-1}\left(a_0-\sqrt{1+\frac{\lambda^2}{\rho^2}}\sqrt{a_1^2+b_1^2}\right)\geqslant0\,.$$

In the definition of the function f we let $n(t)=\lfloor\varphi(t)\rfloor$; as in Theorem 4, here we employ a standard notation for the integer part of a number. We observe that in fact, we deal not with a particular entire function f, but with a family of entire functions depending on five parameters ρ, a_0, λ, a_1 and b_1.

The Azarin limiting set $\operatorname{Fr}f$ for an entire function f is defined as the Azarin limiting set of a subharmonic function $\ln|f(z)|$. Theorem 6 shows that the Azaring limiting set $\operatorname{Fr}f$ and the indicator $h_f(\theta)$ of the function f in the mentioned family are determined by the identities

$$\operatorname{Fr}f=\left\{\left(a_0\frac{\cos\rho(\pi-\theta)}{\sin\rho\pi}+(a_1C_\rho(\lambda,\theta)+b_1D_\rho(\lambda,\theta))\cos\varphi+\right.\right.$$

$$+\ (-a_1 D_\rho(\lambda,\theta)+b_1 C_\rho(\lambda,\theta))\sin\varphi\Big)r^\rho:\ \varphi\in[0,2\pi]\Big\},$$

$$h_f(\theta)=a_0\frac{\cos\rho(\pi-\theta)}{\sin\rho\pi}+\sqrt{a_1^2+b_1^2}\sqrt{C_\rho^2(\lambda,\theta)+D_\rho^2(\lambda,\theta)}\,.$$

These relations hold also without the assumption

$$a_0\geqslant\sqrt{1+\frac{\lambda^2}{\rho^2}}\sqrt{a_1^2+b_1^2}\,,$$

but in the general case, the function f is meromorphic.

If we take an auxiliary function of the form:

$$\varphi(t)=t^\rho\left(a_0+\sum_{k=1}^{n}(a_k\cos(\lambda_k\ln t)+b_k\sin(\lambda_k\ln t))\right),\qquad t>0,$$

and construct then the function $f(z)$ by the proposed scheme, then by means of Theorem 6 we can obtain an asymptotic formula for $\ln|f(z)|$. In the case, when $\varphi(t)$ is an increasing function, the function f is entire. In this way one can obtain asymptotic formulae for a wide class of irregularly growing entire functions. It would be interesting to compare such formulae with general results of work [7], but this issue requires an independent study.

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