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ASYMPTOTIC EXPANSION OF SOLUTION TO SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM WITH CONVEX INTEGRAL QUALITY FUNCTIONAL WITH TERMINAL PART DEPENDING ON SLOW AND FAST VARIABLES

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Abstract. We consider an optimal control problem with a convex integral quality functional for a linear system with fast and slow variables in the class of piecewise continuous controls with smooth constraints on the control

$$\begin{cases} \dot{x}_\varepsilon = A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1u, \qquad t\in[0,T], \qquad \|u\|\leqslant 1,\\ \varepsilon\dot{y}_\varepsilon = A_{22}y_\varepsilon + B_2u, \quad x_\varepsilon(0)=x^0, \qquad y_\varepsilon(0)=y^0, \qquad \nabla\varphi_2(0)=0,\\ J(u):=\varphi_1\left(x_\varepsilon(T)\right)+\varphi_2\left(y_\varepsilon(T)\right)+\int\limits_0^T\|u(t)\|^2\,dt\to\min, \end{cases}$$

where $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, $u\in\mathbb{R}^r$; A_{ij} and B_i, $i,j=1,2$, are constant matrices of corresponding dimension, and the functions $\varphi_1(\cdot),\varphi_2(\cdot)$ are continuously differentiable in $\mathbb{R}^n,\mathbb{R}^m$, strictly convex, and cofinite in the sense of the convex analysis. In the general case, for such problem, the Pontryagin maximum principle is a necessary and sufficient optimality condition and there exist unique vectors l_ε and ρ_ε determining an optimal control by the formula

$$u_\varepsilon(T-t):=\frac{C^*_{1,\varepsilon}(t)l_\varepsilon+C^*_{2,\varepsilon}(t)\rho_\varepsilon}{S\left(\|C^*_{1,\varepsilon}(t)l_\varepsilon+C^*_{2,\varepsilon}(t)\rho_\varepsilon\|\right)},$$

where

$$C^*_{1,\varepsilon}(t):=B_1^*e^{A_{11}^*t}+\varepsilon^{-1}B_2^*\mathcal{W}^*{}_\varepsilon(t), \quad C^*_{2,\varepsilon}(t):=\varepsilon^{-1}B_2^*e^{A_{22}^*t/\varepsilon},$$

$$\mathcal{W}_\varepsilon(t):=e^{A_{11}t}\int\limits_0^t e^{-A_{11}\tau}A_{12}e^{A_{22}\tau/\varepsilon}\,d\tau, \quad S(\xi):=\begin{cases}2, & 0\leqslant\xi\leqslant 2,\\ \xi, & \xi>2.\end{cases}$$

The main difference of our problem from the previous papers is that the terminal part of quality functional depends on the slow and fast variables and the controlled system is a more general form. We prove that in the case of a finite number of control change points, a power asymptotic expansion can be constructed for the initial vector of dual state $\lambda_\varepsilon=(l^*_\varepsilon\ \rho^*_\varepsilon)^*$, which determines the type of the optimal control.

Keywords: optimal control, singularly perturbed problems, asymptotic expansion, small parameter.

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1. Introduction

The paper is devoted to studying the asymptotics of the vector of the dual state in the problem of optimal control [1, 2, 3] of linear system with fast and slow variables, see survey [4], with an convex integral quality functional [3, Ch. 3] and smooth geometric constraints on a control.

In [5, 6], there were considered problems related with a limiting problem for problems of optimal control by a linear system with fast and slow variables. For other formulation, the asymptotics of solutions of perturbed control problem were considered in [7]–[9]. We note that a controlled system of our form but with a terminal quality functional depending on slow variables only was considered in [8].

In the present work we obtain a complete asymptotic expansion of the vector of dual system determining the optimal control. The main difference of our problem in comparison with that considered in [10] is the dependence of the terminal part of the control functional not only on slow variables but also on fast ones.

2. Formulation of problem and main relations

In the class of piece-wise continuous controls we consider the following optimal control problem:

$$\begin{cases} \dot{x}_\varepsilon = A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1 u, \quad t \in [0,T], \qquad \|u\| \leqslant 1, \\ \varepsilon \dot{y}_\varepsilon = A_{22}y_\varepsilon + B_2 u, \quad x_\varepsilon(0) = x^0, \qquad y_\varepsilon(0) = y^0, \qquad \nabla\varphi_2(0) = 0, \\ J(u) := \varphi_1\left(x_\varepsilon(T)\right) + \varphi_2\left(y_\varepsilon(T)\right) + \int\limits_0^T \|u(t)\|^2\,dt \to \min, \end{cases} \tag{1}$$

where $x_\varepsilon \in \mathbb{R}^n$, $y_\varepsilon \in \mathbb{R}^m$, $u \in \mathbb{R}^r$; A_{ij}, B_i, i, $j = 1, 2$, are constant matrices of an appropriate dimension and $\varphi_1(\cdot)$, $\varphi_2(\cdot)$ are continuously differentiable on $\mathbb{R}^n$ and $\mathbb{R}^m$ functions strictly convex and cofinite in the sense of the convex analysis [11, Sect. 13]. All spaces $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^r$ are equipped with the Euclidean norm, which is everywhere denoted by the same symbol $\|\cdot\|$. We note that the terminal part of the quality functional depends on slow and fast variables.

For each fixed $\varepsilon > 0$, the controlled system and the quality functional in problem (1) are of the form:

$$\begin{cases} \dot{z}_\varepsilon = \mathcal{A}_\varepsilon z_\varepsilon + \mathcal{B}_\varepsilon u, \qquad t \in [0,T], \\ z_\varepsilon(0) = z^0, \qquad \|u\| \leqslant 1, \\ J(u) := \varphi\left(z_\varepsilon(T)\right) + \int\limits_0^T \|u(t)\|^2\,dt \to \min, \end{cases}$$

where

$$z_\varepsilon(t) = \begin{pmatrix} x_\varepsilon(t) \\ y_\varepsilon(t) \end{pmatrix}, \qquad z_\varepsilon(0) := z^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \qquad \varphi\left(z_\varepsilon(T)\right) := \varphi_1\left(x_\varepsilon(T)\right) + \varphi_2\left(y_\varepsilon(T)\right),$$

$$\mathcal{A}_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \varepsilon^{-1}A_{22} \end{pmatrix}, \qquad \mathcal{B}_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix}.$$

We observe that in the considered convex integral quality functional J, the terminal part can be interpreted as a penalty for the error of the control at the final moment of time T, while the second part reflect the energy spent for the realization of the control.

We shall say that a pair of matrices (A, B) is completely controllable if the system $\dot{x} = Ax + Bu$ is controllable.

Assumption 1. *For all sufficiently small* $\varepsilon > 0$*, the pair* $(\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon)$ *is completely controllable, that is,* $\operatorname{rank}\big(\mathcal{B}_\varepsilon, \mathcal{A}_\varepsilon\mathcal{B}_\varepsilon, \ldots, \mathcal{A}_\varepsilon^{n+m-1}\mathcal{B}_\varepsilon\big) = n+m$.

Assumption 2. *All eigenvalues of the matrix* A_{22} *have negative real parts.*

Under Assumption 1, the Pontryagin maximum principle is necessary and sufficient condition of the optimality giving the unique solution of problem (1) [3, Sect. 3.5, Thm. 14].

It was shown in [10, Prop. 1, Eq. (1.6)] that the function $u_\varepsilon(t)$ is the only optimal control in problem (1), it is of the form

$$u_\varepsilon(T-t) := \frac{\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon^* t}\lambda_\varepsilon}{S\left(\|\mathcal{B}_\varepsilon{}^* e^{\mathcal{A}_\varepsilon^* t}\lambda_\varepsilon\|\right)}, \qquad S(\xi) := \begin{cases} 2, & 0 \leqslant \xi \leqslant 2, \\ \xi, & \xi > 2, \end{cases} \tag{2}$$

and the vector λ_ε is the unique solution (in view of the cofiniteness of the function φ; [11, Thm. 26.6]) of the equation

$$\nabla\varphi^*(-\lambda) = e^{\mathcal{A}_\varepsilon T} z^0 + \int_0^T e^{\mathcal{A}_\varepsilon \tau}\mathcal{B}_\varepsilon \frac{\mathcal{B}_\varepsilon^* e^{\mathcal{A}_\varepsilon^* \tau}\lambda}{S\big(\|\mathcal{B}_\varepsilon^* e^{\mathcal{A}_\varepsilon^* \tau}\lambda\|\big)}\, d\tau. \tag{3}$$

Here $\nabla\varphi^*$ is the gradient of the function φ^* dual to the function φ in the sense of the convex analysis, see [11, Sect. 12].

We note that in the considered case

$$\varphi^*(\lambda) = \varphi_1^*(l) + \varphi_2^*(\rho) \qquad \text{and} \qquad \nabla\varphi_2^*(0) = 0. \tag{4}$$

We shall consider the vector λ_ε determining the optimal control in problem (1) as $\lambda_\varepsilon = \begin{pmatrix} l_\varepsilon \\ \rho_\varepsilon \end{pmatrix}$, where $l_\varepsilon \in \mathbb{R}^n$, $\rho_\varepsilon \in \mathbb{R}^m$.

Straightforward calculation of the matrix exponent of the controlled system in problem (1) gives:

$$e^{\mathcal{A}_\varepsilon t} := \begin{pmatrix} e^{A_{11}t} & \mathcal{W}_\varepsilon(t) \\ 0 & e^{\frac{A_{22}t}{\varepsilon}} \end{pmatrix}, \tag{5}$$

where $\mathcal{W}_\varepsilon'(t) = A_{11}\mathcal{W}_\varepsilon(t) + A_{12}e^{\frac{A_{22}t}{\varepsilon}}$ and $\mathcal{W}_\varepsilon(0) = 0$. This is why

$$\mathcal{W}_\varepsilon(t) := e^{A_{11}t}\int_0^t e^{-A_{11}\tau}A_{12}e^{\frac{A_{22}\tau}{\varepsilon}}\, d\tau. \tag{6}$$

Integrating by parts in the right hand side in identity (6), we obtain

$$\mathcal{W}_\varepsilon(t) = \varepsilon\Big(A_{12}e^{\frac{A_{22}t}{\varepsilon}} - e^{A_{11}t}A_{12}\Big)A_{22}^{-1} + \varepsilon A_{11}\mathcal{W}_\varepsilon(t)A_{22}^{-1},$$

and by the boundedness of $A_{12}e^{\frac{A_{22}t}{\varepsilon}} - e^{A_{11}t}A_{12}$ on $[0,T]$,

$$\mathcal{W}_\varepsilon(t) = \varepsilon\sum_{k=0}^{\infty}\varepsilon^k A_{11}^k\Big(A_{12}e^{\frac{A_{22}t}{\varepsilon}} - e^{A_{11}t}A_{12}\Big)A_{22}^{-(k+1)}. \tag{7}$$

We shall make use of the following notation:

$$C_\varepsilon(t) = \begin{pmatrix} C_{1,\varepsilon}(t) \\ C_{2,\varepsilon}(t) \end{pmatrix} := e^{\mathcal{A}_\varepsilon t}\mathcal{B}_\varepsilon = \begin{pmatrix} e^{A_{11}t}B_1 + \varepsilon^{-1}\mathcal{W}_\varepsilon(t)B_2 \\ \varepsilon^{-1}e^{\frac{A_{22}t}{\varepsilon}}B_2 \end{pmatrix}. \tag{8}$$

According identity (4) and notation (8), equation (3) is transformed into the system of equations

$$\begin{cases} \nabla\varphi_1^*(-l_\varepsilon) = e^{A_{11}T}x^0 + \mathcal{W}_\varepsilon(T)y^0 + \int\limits_0^T C_{1,\varepsilon}(t)u_\varepsilon(T-t)\,dt, \\ \nabla\varphi_2^*(-\rho_\varepsilon) = e^{A_{22}T/\varepsilon}y^0 + \int\limits_0^T C_{2,\varepsilon}(t)u_\varepsilon(T-t)\,dt, \end{cases} \tag{9}$$

where

$$u_\varepsilon(T-t) := \frac{C_{1,\varepsilon}^*(t)l_\varepsilon + C_{2,\varepsilon}^*(t)\rho_\varepsilon}{S\left(\|C_{1,\varepsilon}^*(t)l_\varepsilon + C_{2,\varepsilon}^*(t)\rho_\varepsilon\|\right)}. \tag{10}$$

Definition 1. *A limiting problem for problem (1) is*

$$\begin{cases} \dot{x}_0 = A_0x_0 + B_0u, \qquad t\in[0,T], \qquad \|u\|\leqslant 1, \\ A_0 := A_{11}, \qquad B_0 := B_1 - A_{12}A_{22}^{-1}B_2, \qquad x_0(0) = x^0, \\ J_0(u) := \varphi_1(x_0(T)) + \int\limits_0^T \|u(t)\|^2\,dt \to \min. \end{cases}$$

Assumption 3. *The pairs of matrices* (A_0, B_0), (A_{22}, B_2) *are completely controllable.*

By [5], Assumptions 2 and 3 ensure Assumption 1 for all sufficiently small ε.

Formulae (5), (7) and (8) imply

$$C_{1,\varepsilon}(t) = C_{1,0}(t) + A_{12}A_{22}^{-1}e^{\frac{A_{22}t}{\varepsilon}}B_2 + O(\varepsilon), \quad \varepsilon\to 0, \quad C_{1,0}(t) := e^{A_0t}B_0, \tag{11}$$

$$\frac{\partial}{\partial t}C_{1,\varepsilon}(t) = \frac{d}{dt}C_{1,0}(t) + \varepsilon^{-1}A_{12}e^{\frac{A_{22}t}{\varepsilon}}B_2 + A_{11}A_{12}e^{\frac{A_{22}t}{\varepsilon}}A_{22}^{-1}B_2 + O(\varepsilon), \quad \varepsilon\to 0, \tag{12}$$

uniformly on the segment $[0,T]$.

We mention the known fact that under Assumption 2 there exist $\gamma>0$ and $K>0$ such that

$$\|e^{\frac{A_{22}t}{\varepsilon}}\| \leqslant Ke^{-\frac{\gamma t}{\varepsilon}}. \tag{13}$$

If a vector function $f_\varepsilon(t)$ is such that $f_\varepsilon(t) = O(\varepsilon^\alpha)$ as $\varepsilon\to 0$ for each $\alpha>0$ uniformly in $t\in[a,b]$, we shall write $\mathbb{O}$ instead of $f_\varepsilon(t)$. In particular,

$$\|e^{A_{22}t/\varepsilon}\| = \mathbb{O}, \qquad e^{-\gamma t/\varepsilon} = \mathbb{O} \quad \text{as} \quad t\in[\varepsilon^p, T], \quad p\in(0,1), \tag{14}$$

where $\gamma>0$.

It follows from formulae (11), (12) and estimate (13) that there exist $K_1>0$ and $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ and $t\in[\sqrt{\varepsilon}, T]$, the inequalities hold

$$\|C_{1,\varepsilon}^*(t) - C_{1,0}^*(t)\| \leqslant K_1\varepsilon, \qquad \left\|\frac{\partial}{\partial t}C_{1,\varepsilon}^*(t) - \frac{d}{dt}C_{1,0}^*(t)\right\| \leqslant K_1\varepsilon. \tag{15}$$

3. Auxiliary statements on cofinite functions

According [11, Thm. 26.6], if f is a differentiable strictly convex cofinite function on $\mathbb{R}^n$, then $\nabla f:\mathbb{R}^n\to\mathbb{R}^n$ is a one-to-one correspondence on $\mathbb{R}^n$ and f^* is a differentiable strictly convex cofinite function on $\mathbb{R}^n$.

Lemma 1. *Let f be a differentiable strictly convex cofinite function on $\mathbb{R}^n$, $\mathbb{L}$ be a non-negative linear operator in $\mathbb{R}^n$, that is,*

$$\langle\mathbb{L}l, l\rangle \geqslant 0 \qquad \textit{for all} \qquad l\in\mathbb{R}^n.$$

Then the $g(l) = f(l) + \frac{1}{2}\langle \mathbb{L}l, l\rangle$ *is a differentiable strictly convex cofinite function on* $\mathbb{R}^n$ *and* $\nabla g(l) = \nabla f(l) + \mathbb{L}l$.

Proof. We begin with proving that $g(l)$ is a differentiable strictly convex cofinite function on $\mathbb{R}^n$. We calculate the derivative of the scalar product $\frac{1}{2}\langle \mathbb{L}l, l\rangle$ along the direction of Δl:

$$D\left(\frac{1}{2}\langle \mathbb{L}l, l\rangle\right)(\Delta l) = \frac{\partial}{\partial l}\bigg|_{t=0} \frac{\langle \mathbb{L}(l + t\Delta l), l + t\Delta l\rangle}{2} = \langle \mathbb{L}l, \Delta l\rangle,$$

and we obtain that $\nabla\left(\frac{1}{2}\langle \mathbb{L}l, l\rangle\right) = \mathbb{L}l$. According the definition [11], a convex function f is cofinite if the following relation holds:

$$\lim_{\lambda\to+\infty} \frac{f(\lambda l)}{\lambda} = +\infty \qquad \text{for all} \qquad l \neq 0. \tag{16}$$

Let us show that the function $g(l)$ obeys this condition.

For each $\lambda > 0$ we have:

$$\frac{g(\lambda l)}{\lambda} = \frac{f(\lambda l)}{\lambda} + \frac{1}{2}\cdot\frac{\langle \mathbb{L}(\lambda l), \lambda l\rangle}{\lambda} = \frac{f(\lambda l)}{\lambda} + \frac{\lambda}{2}\cdot\langle \mathbb{L}l, l\rangle \geqslant \frac{f(\lambda l)}{\lambda} \to +\infty \quad \text{as} \quad \lambda \to +\infty.$$

□

Corollary 1. *Let a function* f *satisfies the assumptions of Lemma 1, and* f^* *is a dual function for* f *in the sense of the convex analysis. Then the equation* $\nabla f^*(l) + \mathbb{L}l = d$ *has the unique solution for each vector* d.

This corollary follows Lemma 1 and Theorem 26.6 in [11].

4. Limiting values of vectors l_ε and ρ_ε

Theorem 1. *Let Assumptions 1 and 2 hold and the vector* $\lambda_\varepsilon^* = (l_\varepsilon^* \quad \rho_\varepsilon^*)$ *is the unique solution of system (9). Then the vectors* l_ε, ρ_ε *are bounded and*

$$l_\varepsilon \to l_0 \qquad \text{as} \qquad \varepsilon \to +0, \tag{17}$$

where l_0 *is the unique solution of the equation*

$$0 = -\nabla\varphi_1^*(-l) + e^{A_{11}T}x^0 + \int_0^T C_{1,0}(t)\frac{C_{1,0}^*(t)l}{S\big(\|C_{1,0}^*(t)l\|\big)}dt. \tag{18}$$

Proof. It is known that at the final time T, the set of attainability of the controlled system in problem (1) is bounded uniformly in $\varepsilon \in (0, \varepsilon_0]$, see, for instance, [6, Thm. 3.1]. Hence, the left hand side of equation (3) is bounded. This is why, as $\varepsilon \to 0$, the quantity $\nabla\varphi^*(-\lambda_\varepsilon)$ is bounded as well. Since the function φ^* is cofinite, according [11, Lm. 26.7], the vector λ_ε is bounded. Therefore, the vectors l_ε, ρ_ε are bounded.

We partition the interval of integration in the first identity (9) into two pieces: $[0, \sqrt{\varepsilon}]$ and $[\sqrt{\varepsilon}, T]$. Taking into consideration identity (6) and the notation (8) being representations of matrices $\mathcal{W}_\varepsilon(t)$ and $C_\varepsilon(t)$ in system (9)–(10), we can write the first identity (9) as

$$\nabla\varphi_1^*(-l_\varepsilon) = e^{A_{11}T}x^0 + O(\sqrt{\varepsilon}) + \int_{\sqrt{\varepsilon}}^T C_{1,\varepsilon}(t)\frac{C_{1,\varepsilon}^*(t)l_\varepsilon}{S\|C_{1,\varepsilon}^*(t)l_\varepsilon\|}dt \quad \text{as} \quad \varepsilon \to 0. \tag{19}$$

Let l_0 be an arbitrary limiting point of the function l_ε as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ in identity (19), by inequalities (15) we obtain the identity

$$\nabla\varphi_1^*(-l_0) = e^{A_{11}T}x^0 + \int_0^T C_{1,0}(t)\frac{C_{1,0}^*(t)l_0}{S\|C_{1,0}^*(t)l_0\|}dt,$$

that is, l_0 satisfies equation (18). This equation reads as

$$\nabla\varphi_1^*(-l_0) + \mathbb{L}(-l_0) = e^{A_{11}T}x^0$$

and $\mathbb{L} \geqslant 0$. This is why by Corollary 1 of Lemma 1, this equation possesses the unique solution. Thus, l_0 is the unique limiting point for l_ε and $l_\varepsilon \to l_0$ as $\varepsilon \to 0$. □

Theorem 2. *Let the assumptions of Theorem 1 hold, and B_2 is a mapping of $\mathbb{R}^r$ onto $\mathbb{R}^m$; in particular, $r \geqslant m$. Then $\rho_\varepsilon \to 0$, the quantity $\{r_\varepsilon\}$ ($r_\varepsilon := \varepsilon^{-1}\rho_\varepsilon$) is bounded as $\varepsilon \to +0$ and all its limiting points r_0 satisfy the equation*

$$0 = \int_0^{+\infty} e^{A_{22}\tau}B_2\frac{B_0^*l_0 + B_2^*e^{A_{22}^*\tau}(r_0 + (A_{22}^*)^{-1}A_{12}^*l_0)}{S\left(\|B_0^*l_0 + B_2^*e^{A_{22}^*\tau}(r_0 + (A_{22}^*)^{-1}A_{12}^*l_0)\|\right)}\,d\tau. \tag{20}$$

Proof. We change the variable $\tau := t/\varepsilon$ in the integral in the second identity in system (9). We choose arbitrary $\delta > 0$ and taking into consideration estimate (13), we rewrite this identity as

$$\nabla\varphi_2^*(-\rho_\varepsilon) = \mathbb{O} + \int_0^\delta e^{A_{22}\tau}B_2\frac{\tilde{B}(\tau,\varepsilon)l_\varepsilon + B_2^*e^{A_{22}^*\tau}r_\varepsilon}{S\left(\|\tilde{B}(\tau,\varepsilon)l_\varepsilon + B_2^*e^{A_{22}^*\tau}r_\varepsilon\|\right)}\,d\tau + O(e^{-\gamma\delta}), \tag{21}$$

where $r_\varepsilon := \rho_\varepsilon/\varepsilon$, and

$$\tilde{B}(\tau,\varepsilon) := B_0^*e^{A_{11}^*\varepsilon\tau} + B_2^*e^{A_{22}^*\tau}(A_{22}^*)^{-1}A_{12}^*. \tag{22}$$

We note that $\tilde{B}(\tau,\varepsilon)l_\varepsilon \to \tilde{B}(\tau,0)l_0$ as $\varepsilon \to 0$ uniformly on $[0,\delta]$ and $\tilde{B}(\tau,0)$ is bounded on $[0,+\infty)$.

Let ρ_0 be an arbitrary limiting point of ρ_ε as $\varepsilon \to 0$, that is, there exists $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ and $\rho_k := \rho_{\varepsilon_k} \to \rho_0$.

We assume that $r_k := r_{\varepsilon_k}$ is unbounded. Without loss of generality we suppose that

$$r_k \to \infty, \quad \frac{r_k}{\|r_k\|} \to \overline{r}, \quad \|\overline{r}\| = 1, \quad \rho_0 = \|\rho_0\|\overline{r}. \tag{23}$$

Since the function $B_2^*e^{A_{22}^*\tau}r$ is jointly continuous in the variable τ and vector r, and as $r \neq 0$, by the injectivity of B_2^*, we have $B_2^*e^{A_{22}^*\tau}r \neq 0$, there exists $K_0(\delta) > 0$ such that

$$\|B_2^*e^{A_{22}^*\tau}r\| \geqslant K_0(\delta)\|r\|$$

for all r and all $\tau \in [0,\delta]$. This is why, by relations (23), for all sufficiently large k, the inequality holds:

$$\left\|C_{1,\varepsilon_k}^*(\varepsilon\tau)l_{\varepsilon_k} + B_2^*e^{A_{22}^*\tau}r_k\right\| > 2,$$

and identity (21) becomes

$$\nabla\varphi_2^*(-\rho_k) = \int_0^\delta e^{A_{22}\tau}B_2\frac{\frac{1}{\|r_k\|}\tilde{B}(\tau,\varepsilon_k)l_k + B_2^*e^{A_{22}^*\tau}\frac{r_k}{\|r_k\|}}{\left\|\frac{1}{\|r_k\|}\tilde{B}(\tau,\varepsilon_k)l_k + B_2^*e^{A_{22}^*\tau}\frac{r_k}{\|r_k\|}\right\|}\,d\tau + \mathbb{O} + O(e^{-\gamma\delta}). \tag{24}$$

We pass to the limit in k and then as $\delta \to +\infty$ in identity (24). Then in view of relations (23) we obtain the identity:

$$\nabla\varphi_2^*(-\|\rho_0\|\overline{r}) = \int\limits_0^{+\infty} e^{A_{22}\tau} B_2 \frac{B_2^* e^{A_{22}^*\tau}\overline{r}}{\|B_2^* e^{A_{22}^*\tau}\overline{r}\|}\, d\tau.$$

We calculate the scalar product of the latter equation with $\overline{r}$ and we obtain:

$$\langle \nabla\varphi_2^*(-\|\rho_0\|\overline{r}), \overline{r}\rangle = \int\limits_0^{+\infty} \left\|B_2^* e^{A_{22}^*\tau}\overline{r}\right\|\, d\tau. \tag{25}$$

By Assumption 3, the right hand side of the above identity is positive, while the left hand side is non-positive due to the monotonicity of $\nabla\varphi_2^*$ and the identity $\nabla\varphi_2^*(0) = 0$; this is a contradiction. Thus, $\rho_\varepsilon \to 0$. If r_ε is unbounded, reproducing the above arguing, we arrive at a contradicting inequality similar to (25):

$$0 = \int\limits_0^{+\infty} \left\|B_2^* e^{A_{22}^*\tau}\overline{r}\right\|\, d\tau.$$

Finally, if r_0 is a limiting point of r_ε, then we pass to the limit as $\varepsilon \to 0$ in (21) and then we pass to the limit as $\delta \to +\infty$. In view of notation (22) we obtain identity (20). □

Theorem 3. *Let the assumptions of Theorem 2 holds. Then equation (20) has the unique solution r_0 and $r_\varepsilon \to r_0$.*

Proof. We introduce the notations: $l := B_0^* l_0$, $r := r_0 + (A_{22}^*)^{-1} A_{12}^* l_0$. Then equation (20) casts into the form:

$$F(r) := \int\limits_0^{+\infty} e^{A_{22}\tau} B_2 \frac{l + B_2^* e^{A_{22}^*\tau} r}{S\left(\|l + B_2^* e^{A_{22}^*\tau} r\|\right)}\, d\tau = 0. \tag{26}$$

If $l = 0$, we multiply identity (26) by r and we obtain:

$$\int\limits_0^{+\infty} \frac{\|B_2^* e^{A_{22}^*\tau} r\|^2}{S\left(\|B_2^* e^{A_{22}^*\tau} r\|\right)}\, d\tau = 0.$$

Since the integrand is continuous and non-negative, we have $\|B_2^* e^{A_{22}^*\tau} r\| \equiv 0$ and by Assumption 3 this implies $r = 0$.

Let $l \neq 0$. Assume that there exist two different solutions $r_1 \neq r_2$ to equation (26): $F(r_1) = F(r_2) = 0$. By the Lagrange formula,

$$0 = \langle F(r_1) - F(r_2), r_1 - r_2\rangle = \left\langle \frac{\partial}{\partial r} F(r)\Big|_{r=r'} (r_1 - r_2), r_1 - r_2 \right\rangle, \tag{27}$$

where $r' \in [r_1, r_2]$. Let us show that as $r_1 \neq r_2$, identity (27) is impossible.

We rewrite the integral in (26) as a sum of two integrals over two sets:

$$E_1(r) := \{\tau \in [0, +\infty) : \|l + B_2^* e^{A_{22}^*\tau} r\| \leqslant 2\}, \qquad E_2(r) := \{\tau \in [0, +\infty) : \|l + B_2^* e^{A_{22}^*\tau} r\| \geqslant 2\}.$$

Then the integral in the right hand side in equation (26) is split into two integrals:

$$F(r) = \int\limits_{E_1(r)} e^{A_{22}\tau} B_2 \frac{l + B_2^* e^{A_{22}^*\tau} r}{2}\, d\tau + \int\limits_{E_2(r)} e^{A_{22}\tau} B_2 \frac{l + B_2^* e^{A_{22}^*\tau} r}{\|l + B_2^* e^{A_{22}^*\tau} r\|}\, d\tau. \tag{28}$$

Since $B_2^* e^{A_{22}^*\tau} r \to 0$ as $\tau \to +\infty$, the sets $E_1(r)$ and $E_2(r)$ consist of finitely many segments.

Let us find the derivative $DF(r')(\Delta r)$ of the function F at the point r' along the direction Δr. We employ representation (28) and the known formula:

$$D\left(\int\limits_{\alpha(r)}^{\beta(r)} f(t,r)dt\right)(\Delta r) = \int\limits_{\alpha(r)}^{\beta(r)} \frac{\partial f(t,r)}{\partial r}(\Delta r)dt + +f(\beta(r),r)\frac{\partial \beta}{\partial r}(\Delta r) - f(\alpha(r),r)\frac{\partial \alpha}{\partial r}(\Delta r).$$

Since the integrands coincide at the common points of $E_1(r)$ and $E_2(r)$, the final formula for DF involves no non-integral terms.

Since

$$\frac{\partial}{\partial r}\left(e^{A_{22}\tau}B_2\frac{l+B_2^* e^{A_{22}^*\tau}r}{2}\right)(\Delta r) = C(\tau)\frac{C^*(\tau)\Delta r}{2}, \quad C(\tau) := e^{A_{22}\tau}B_2,$$

and

$$\frac{\partial}{\partial r}\left(e^{A_{22}\tau}B_2\frac{l+B_2^* e^{A_{22}^*\tau}r}{\|l+B_2^* e^{A_{22}^*\tau}r\|}\right)$$
$$= C(\tau)\frac{C^*(\tau)\Delta r\|l+C^*(\tau)r\|^2 - \langle C^*(\tau)\Delta r, l+C^*(\tau)r\rangle(l+C^*(\tau)r)}{\|l+C^*(\tau)r\|^3},$$

then

$$DF(r')(\Delta r) = DF_1(r')(\Delta r) + DF_2(r')(\Delta r),$$

$$DF_1(r')(\Delta r) = \frac{1}{2}\int\limits_{E_1(r')} e^{A_{22}\tau}B_2B_2^* e^{A_{22}^*\tau}\Delta r\, d\tau, \tag{29}$$

$$DF_2(r')(\Delta r) = \int\limits_{E_2(r')} C(\tau)\frac{C^*(\tau)\Delta r\|l+C^*(\tau)r\|^2 - \langle C^*(\tau)\Delta r, l+C^*(\tau)r\rangle(l+C^*(\tau)r)}{\|l+C^*(\tau)r\|^3}d\tau.$$

If $E_1(r') \neq \emptyset$, the latter identity in (29) implies $DF_1(r') > 0$. It follows from the Cauchy-Schwarz inequality and relations (29) that $DF_2(r') \geqslant 0$. This is why, if $E_1(r') \neq \emptyset$, then $DF(r') > 0$ and identity (27) is possible only as $\Delta r = r_1 - r_2 = 0$.

Since $\Delta r \neq 0$, it follows from identity (27) that

$$E_1(r') = \emptyset$$

and by the Cauchy-Schwarz inequality, the vector $l + B_2^* e^{A_{22}^*\tau}r'$ is parallel to the vector $B_2^* e^{A_{22}^*\tau}\Delta r$ for all τ. The identity $E_1(r') = \emptyset$ means that

$$\|l_1 + e^{A_{22}^*\tau}r'\| \geqslant 2 \quad \text{for all} \quad \tau. \tag{30}$$

By the assumptions of the theorem, $B_2^* e^{A_{22}^*\tau}\Delta r \neq 0$. Hence, there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$l + B_2^* e^{A_{22}^*\tau}r' = \beta(\tau)B_2^* e^{A_{22}^*\tau}\Delta r \quad \text{for all} \quad \tau.$$

Hence, l reads as $B_2^* l_1$. Thus, if $l \notin \operatorname{Im}(B_2^*)$, identity (27) is impossible.

By the injectivity of the operator B_2^* we obtain that

$$\forall\, \tau \quad l_1 + e^{A_{22}^*\tau}r' = \beta(\tau)e^{A_{22}^*\tau}\Delta r. \tag{31}$$

We multiply identity (31) by $e^{-A_{22}^*\tau}$ and we get:

$$e^{-A_{22}^*\tau}l_1 + r' = \beta(\tau)\Delta r. \tag{32}$$

Hence, the function $\beta(\tau)$ is infinitely differentiable. We differentiate identity (32) twice in τ and we obtain:

$$-A_{22}^* e^{-A_{22}^*\tau}l_1 = \beta'(\tau)\Delta r, \quad (A_{22}^*)^2 e^{-A_{22}^*\tau}l_1 = \beta''(\tau)\Delta r.$$

As $\tau = 0$, this gives the identities:

$$-A_{22}^* l_1 = \beta'(0)\Delta r, \quad (A_{22}^*)^2 l_1 = \beta''(0)\Delta r. \tag{33}$$

If $\beta'(0) = 0$ or $\beta''(0) = 0$, then $l_1 = 0$ that contradicts the assumptions of the theorem.

It follows from identity (33) that

$$\beta''(0)\Delta r = (A_{22}^*)^2 l_1 = -A_{22}^* \beta'(0)\Delta r,$$

that is, the vector Δr is an eigenvector of the matrix A_{22}^*. Hence,

$$A_{22}^*\Delta r = -\alpha\Delta r, \quad \alpha > 0, \tag{34}$$

where $\alpha = \beta''(0)/\beta'(0)$ is an eigenvalue of the matrix A_{22}^*. If the matrix A_{22}^* has no real eigenvalues, identity (27) is impossible.

It follows from identities (33) and (34) that the vector l_1 is parallel to the vector Δr. This is why by identity (32) and r' is parallel to the vector l_1. Since $r' = r_1 - \beta_0\Delta r$ for some β_0, it follows that the vectors r_1, r_2 are parallel to the vector l_1. Thus, in this case,

$$r_1 = \beta_1 l_1, \quad r_2 = \beta_1 l_2, \quad r' = \beta_3 l_1.$$

and identity (26) being valid for r_i, $i = 1, 2$ after calculating its scalar product with l_1, casts into the form:

$$\int\limits_0^{+\infty} \frac{\left(1 + \beta_i e^{-\alpha\tau}\right) e^{-\alpha\tau} \|B_2^* l_1\|^2}{S\Big(\left|1 + \beta_i e^{-\alpha\tau}\right| \cdot \|B_2^* l_1\|\Big)}\, d\tau = 0, \quad i = 1, 2. \tag{35}$$

The above identity (35) is impossible if $1 + \beta_i e^{-\alpha\tau}$ is sign-definite on $[0, +\infty)$. Since $e^{-\alpha\tau}$ is strictly decreasing and $e^{-\alpha\tau} \to 0$ as $\tau \to +\infty$, we obtain that $\beta_i < -1$, $i = 1, 2$. By the relation $r' \in [r_1, r_2]$ this implies that $\beta_3 < -1$. But then there exists $\tau_0 > 0$ such that $\left|1 + \beta_3 e^{-\alpha\tau_0}\right| \cdot \|B_2^* l_1\| = 0$ and this contradicts inequality (30). □

In what follows we suppose that

$$r = m, \quad A_{22} = -I, \quad B_2 = I. \tag{36}$$

Here I stands for the identity mapping of $\mathbb{R}^m$ onto $\mathbb{R}^m$.

Lemma 2. *Let conditions (36) and the assumptions of Theorem 1 are satisfied. Then*

$$r_\varepsilon \to r_0 = A_{12}^* l_0 - 2B_0^* l_0 \quad as \quad \varepsilon \to 0.$$

Proof. Under (36), equation (20) becomes

$$\int\limits_0^{+\infty} e^{-\tau} \frac{l + e^{-\tau} r}{S\left(\|l + e^{-\tau} r\|\right)}\, d\tau = 0, \tag{37}$$

where $l := B_0^* l_0$, $r := r_0 + (A_{22}^*)^{-1} A_{12}^* l_0$. Thanks to Theorem 3, it is sufficient to confirm that the vector $(-2l)$ is its solution. We substitute $r = -2l$ into the left hand side of equation (37), we obtain:

$$\int\limits_0^{+\infty} e^{-\tau} \frac{\left(1 - 2e^{-\tau}\right) l}{S\Big(\left|1 - 2e^{-\tau}\right| \cdot \|l\|\Big)}\, d\tau = \Big[\xi = e^{-\tau}\Big] = \int\limits_0^1 \frac{(1 - 2\xi)}{S\Big(|1 - 2\xi| \cdot \|l\|\Big)}\, d\xi\, l = \Big[\eta = 1 - 2\xi\Big]$$

$$= \frac{1}{2} \int\limits_{-1}^1 \frac{\eta}{S\big(|\eta| \cdot \|l\|\big)}\, d\eta\, l = 0$$

since the integrand is odd. □

5. Asymptotic expansion of vector λ_ε under conditions (36)

We observe that by conditions (36) we have:

$$B_0 = B_1 + A_{12}, \qquad r_0 = \big(A_{12}^* - 2B_0^*\big)l_0, \tag{38}$$

$$C_{1,\varepsilon}^*(t) = B_1^* e^{A_{11}^* t} + A_{12}^*\Big(e^{A_{11}^* t} - e^{-\frac{t}{\varepsilon}} I\Big)\sum_{k=0}^{\infty}(-1)^k\varepsilon^k (A_{11}^*)^k. \tag{39}$$

It follows from identities (38) and (39) that

$$\begin{aligned} C_\varepsilon^*(t)\lambda_\varepsilon =& C_{1,0}^*(t)l_0 + C_{1,0}^*\Delta l - \varepsilon A_{12}^* e^{A_{11}^* t} A_{11}^* l_0 - \\ &- 2e^{-\frac{t}{\varepsilon}} B_0^* l_0 - A_{12}^* e^{-\frac{t}{\varepsilon}}\Delta l + \varepsilon A_{12}^* e^{-\frac{t}{\varepsilon}} A_{11}^* l_0 + e^{-\frac{t}{\varepsilon}}\Delta r + \mathcal{F}_2(\varepsilon, \Delta l, \Delta r). \end{aligned} \tag{40}$$

Here $\Delta l := l_\varepsilon - l_0$, $\Delta r := r_\varepsilon - r_0$, and $\mathcal{F}_2(\varepsilon, \Delta l, \Delta r)$ is a function of a second order of smallness in $\{\varepsilon, \Delta l, \Delta r\}$.

We begin with the case, when the limiting problem has a single point of the change of the type of optimal control. Suppose that for the limiting problem and the initial state of the system x^0 there exists the only moment of time $t = t_0 \in (0, T)$ such that

$$\begin{aligned} &\|C_{1,0}^*(t)l_0\| < 2, \qquad \|C_{1,0}^*(t_0)l_0\| = 2 \quad \text{for all} \quad t < t_0, \\ &\|C_{1,0}^*(t)l_0\| > 2 \quad \text{for all} \quad t > t_0, \\ &\frac{d}{dt}\|C_{1,0}^*(t)l_0\|^2\bigg|_{t=t_0} \neq 0. \end{aligned} \tag{41}$$

Lemma 3. *If the condition*

$$\|B_0^* l_0\| < 2 \tag{42}$$

holds, then

$$\forall\, l_\varepsilon \to l_0\, \forall\, r_\varepsilon \to \big(A_{12}^* - 2B_0^*\big)l_0\; \exists\, \varepsilon_0 > 0\, \forall\, \varepsilon \in (0, \varepsilon_0)\, \forall\, t \in [0, \sqrt{\varepsilon}]\; \|C_\varepsilon^*(t)\lambda_\varepsilon\| < 2. \tag{43}$$

Proof. We assume the opposite; then there exits sequences $\{t_k\} \subset [0, \sqrt{\varepsilon}]$ and $\{\varepsilon_k\}$ such that $\varepsilon_k \to +0$ and

$$\|C_{\varepsilon_k}^*(t_k)\lambda_{\varepsilon_k}\| \geqslant 2. \tag{44}$$

We let $\tau_k := t_k/\varepsilon_k$, $l_k := l_{\varepsilon_k}$, $r_k := r_{\varepsilon_k}$ and $\lambda_k := \lambda_{\varepsilon_k}$. Then by identity (40) we get:

$$\begin{aligned} &C_{\varepsilon_k}^*(t_k)\lambda_{\varepsilon_k} = C_{1,0}^*(\varepsilon_k\tau_k)l_0 - 2e^{-\tau_k}B_0^* l_0 + \mathcal{F}_1(\varepsilon_k, \Delta l_k, \Delta r_k), \\ &\Delta l_k := l_k - l_0, \qquad \Delta r_k := r_k - r_0, \qquad \mathcal{F}_1(\varepsilon_k, \Delta l_k, \Delta r_k) \to 0. \end{aligned} \tag{45}$$

Let τ_0 be a limiting point of the sequence $\{\tau_k\}$; to shorten the notation, we suppose that $\tau_k \to \tau_0$. If $\tau_0 = +\infty$, we pass to the limit as $k \to \infty$ in identity (45) and taking into consideration that $l_k \to l_0$, $r_k \to (A_{12}^* - 2B_0^*)l_0$, we obtain: $C_{\varepsilon_k}^*(\varepsilon_k\tau_k)\lambda_k \to B_0^* l_0$. But $\|B_0^* l_0\| < 2$ by assumption (41) and this contradicts condition (44).

Thus, all limiting points τ_0 are finite. Then $\varepsilon_k\tau_n \to 0$ and this is why $C_{\varepsilon_k}^*(\varepsilon_k\tau_k)\lambda_k \to \big(1 - 2e^{-\tau_0}\big)B_0^* l_0$. But

$$\Big\|\big(1 - 2e^{-\tau_0}\big)B_0^* l_0\Big\| = \big|1 - 2e^{-\tau_0}\big| \cdot \|B_0^* l_0\| \leqslant \|B_0^* l_0\| < 2,$$

and this contradicts condition (44). □

Theorem 4. *Under condition (42), there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a single point t_ε of the change of the type of optimal control in problem (1), that is,*

$$\|C_\varepsilon^*(t)\lambda_\varepsilon\| < 2, \qquad \|C_\varepsilon^*(t_\varepsilon)\lambda_\varepsilon\| = 2 \quad \textit{for all} \quad t < t_\varepsilon, \qquad \|C_\varepsilon^*(t)\lambda_\varepsilon\| > 2 \quad \textit{for all} \quad t > t_\varepsilon.$$

At that, $t_\varepsilon \to t_0$ as $\varepsilon \to 0$.

Proof. We note that by assumption (41) there exists $\delta_0 > 0$ such that

$$\frac{d}{dt}\|C^*_{1,0}(t)l_0\|^2\bigg|_{t=t_0} > 0 \quad \text{for all} \quad t \in [t_0-\delta_0, t_0+\delta_0].$$

By (17) and (15) and since $\|C^*_{1,0}(t_0-\delta_0)l_0\| < 2$ and $\|C^*_{1,0}(t_0+\delta_0)l_0\| > 2$, there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1)$ and $t \in [t_0-\delta_0, t_0+\delta_0]$ the inequalities hold:

$$\|C^*_\varepsilon(t_0-\delta_0)\lambda_\varepsilon\| < 2, \qquad \|C^*_\varepsilon(t_0+\delta_0)\lambda_\varepsilon\| > 2, \qquad \frac{\partial}{\partial t}\left(\|C^*_\varepsilon(t)\lambda_\varepsilon\|^2\right) > 0.$$

This implies the existence of a single point $t_\varepsilon \in [t_0-\delta_0, t_0+\delta_0]$ such that $\|C^*_\varepsilon(t_\varepsilon)\lambda_\varepsilon\| = 2$.

Let us show that for all sufficiently small $\varepsilon > 0$ $(0 < \varepsilon < \varepsilon_0 \leqslant \varepsilon_1)$ there are no other points t obeying identity $\|C^*_\varepsilon(t)\lambda_\varepsilon\| = 2$.

By condition (41) there exists $\gamma > 0$ such that as $|t-t_0| \geqslant \delta_0$, the estimate holds:

$$\big|\|C^*_{1,0}(t)l_0\| - 2\big| \geqslant \gamma > 0.$$

Then it follows from estimate (11) and condition (17) that for all sufficiently small $\varepsilon > 0$, $t \in [\sqrt{\varepsilon}, T]$ and $\|t-t_0\| \geqslant \delta_0$ the inequality holds:

$$\big|\|C^*_\varepsilon(t)\lambda_\varepsilon\| - 2\big| \geqslant \frac{\gamma}{2} > 0.$$

Hence, $\|C^*_\varepsilon(t)\lambda_\varepsilon\| \neq 2$ for such ε and t. On the remaining segment $[0,\sqrt{\varepsilon}]$, the relation $\|C^*_\varepsilon(t)\lambda_\varepsilon\| \neq 2$ holds thanks to condition (43). □

Thus, in the considered case, the integral in (3) is also split into the sum of two integrals:

$$\int\limits_0^T \frac{C_\varepsilon(t)C^*_\varepsilon(t)\lambda}{S\left(\|C^*_\varepsilon(t)\lambda\|\right)}dt = \frac{1}{2}\int\limits_0^{t_\varepsilon} C_\varepsilon(t)C^*_\varepsilon(t)\lambda\,dt + \int\limits_{t_\varepsilon}^T C_\varepsilon(t)\frac{C^*_\varepsilon(t)\lambda}{\|C^*_\varepsilon(t)\lambda\|}dt. \tag{46}$$

Let $\Delta l_\varepsilon := l_\varepsilon - l_0$, $\Delta r_\varepsilon := r_\varepsilon - r_0$, $\Delta t_\varepsilon := t_\varepsilon - t_0$. Then

$$\lambda_\varepsilon = \begin{pmatrix} l_0 + \Delta l_\varepsilon \\ \varepsilon(r_0 + \Delta r_\varepsilon) \end{pmatrix}, \qquad \Delta l_\varepsilon = o(1), \qquad \Delta r_\varepsilon = o(1), \qquad \Delta t_\varepsilon = o(1)$$

as $\varepsilon \to 0$, and by identities (2), (3), (46) and Theorem 4, the triple $\{\Delta l_\varepsilon,\ \Delta r_\varepsilon\ \Delta t_\varepsilon\}$ solves the following system of equations depending on the parameter ε:

$$\begin{cases} 0 = F_1(\varepsilon,\Delta l,\Delta r,\Delta t) := -\nabla\varphi_1^*(-l_\varepsilon) + \nabla\varphi_1^*(-l_0) \\ \qquad + \mathcal{W}_\varepsilon(T)y_0 + \dfrac{1}{2}\displaystyle\int\limits_0^{t_\varepsilon} C_{1,\varepsilon}(t)C^*_\varepsilon(t)\lambda_\varepsilon\,dt + \int\limits_{t_\varepsilon}^T C_{1,\varepsilon}(t)\frac{C^*_\varepsilon(t)\lambda_\varepsilon}{\|C^*_\varepsilon(t)\lambda_\varepsilon\|}\,dt, \\ 0 = F_2(\varepsilon,\Delta l,\Delta r,\Delta t) := -\nabla\varphi_2^*(-\varepsilon r_\varepsilon) + \nabla\varphi_2^*(0) \\ \qquad + \dfrac{1}{2}\displaystyle\int\limits_0^{t_\varepsilon} \varepsilon^{-1}C_{2,\varepsilon}(t)C^*_\varepsilon(t)\lambda_\varepsilon\,dt + \int\limits_{t_\varepsilon}^T \varepsilon^{-1}C_{2,\varepsilon}(t)\frac{C^*_\varepsilon(t)\lambda_\varepsilon}{\|C^*_\varepsilon(t)\lambda_\varepsilon\|}\,dt, \\ 0 = G(\varepsilon,\Delta l,\Delta r,\Delta t) := \|C^*_\varepsilon(t+\triangle t)\lambda_\varepsilon\|^2 - \|C^*_{1,0}(t_0)l_0\|^2. \end{cases} \tag{47}$$

We note that the functions F_1, F_2 and G are continuous, and G is infinitely differentiable. Let us study their asymptotic expansions with respect to infinitesimals Δl, Δr and Δt.

By the infinite differentiability of the functions φ_1^* and φ_2^* and in view of identity $\varphi_2^*(0) = 0$ we obtain:

$$\begin{aligned} -\nabla\varphi_1^*(-l_0 - \Delta l) + \nabla\varphi_1^*(-l_0) &\sim D^2\varphi_1^*(-l_0)\Delta l + \sum_{k=2}^{\infty} \Phi_{1,k}(\Delta l), \\ -\nabla\varphi_2^*(-\varepsilon r_\varepsilon) + \nabla\varphi_2^*(0) &\sim D^2\varphi_2^*(0)r_0\varepsilon + \sum_{k=2}^{\infty} \Phi_{2,k}(\varepsilon, \Delta r), \end{aligned} \tag{48}$$

where $D^2\varphi_1^*(-l_0)$ and$D^2\varphi_2^*(0)$ are second order differentials of φ_1^* and φ_2^* at the points $(-l_0)$ and 0, respectively, and $\Phi_{1,k}(\Delta l)$ and $\Phi_{2,k}(\varepsilon, \Delta l)$ are homogeneous functions of order k, namely, polynomials of the components of the vector Δl and ε.

By identity (7),

$$\mathcal{W}_\varepsilon(T)y_0 \sim \varepsilon e^{A_{11}T} A_{12}y_0 + \sum_{k=2}^{\infty} \varepsilon^k y_k, \tag{49}$$

where y_k are known vectors.

We split each integral in the first and second identity in system of equations (47) into two parts

$$\int_0^{t_0+\Delta t} = \int_0^{t_0} + \int_{t_0}^{t_0+\Delta t}, \qquad \int_{t_0+\Delta t}^{T} = \int_{t_0+\Delta t}^{t_0} + \int_{t_0}^{T}$$

and we denote the integrals by $I_1(\varepsilon, \Delta\lambda)$, $I_2(\varepsilon, \Delta\lambda)$, $I_3(\varepsilon, \Delta\lambda)$ and $I_4(\varepsilon, \Delta\lambda)$, respectively.

We note that by identity (7), the asymptotics of integrands in $I_2 - I_4$ is power in ε and the components of the vector $\Delta\lambda$ with coefficients smoothly depending on t.

To expand the integrals I_2 and I_3 in Δt, we should additionally expand the coefficients depending on t into the Taylor series at the point t_0 and to integrate the obtained expansions over the mentioned segments.

We observe that in I_2 and I_3, the terms of the first order of smallness in Δt are of the form:

$$\frac{C_{1,0}(t_0)C_{1,0}^*(t_0)l_0}{2}\Delta t, \qquad -\frac{C_{1,0}(t_0)C_{1,0}^*(t_0)l_0}{\|C_{1,0}^*(t_0)l_0\|}\Delta t,$$

respectively. Since

$$\|C_{1,0}^*(t_0)l_0\| = 2, \qquad I_2(\varepsilon, \Delta\lambda) = O(\Delta t), \qquad I_2(\varepsilon, \Delta\lambda) = O(\Delta t),$$

the expansions of the $I_2 + I_3$ contains no terms of the first order of smallness in Δl, Δr, Δt and ε.

By estimate (14) and identity (39), on $[t_0, T]$ we have asymptotic identities:

$$C_{1,\varepsilon}^*(t) = B_1^*(t)e^{A_{11}^*t} + A_{12}^*e^{A_{11}^*t}\sum_{k=0}^{\infty}(-1)^k\varepsilon^k(A_{11}^*)^k, \qquad C_{2,\varepsilon}^*(t) = \mathbb{O} \quad \text{as} \qquad \varepsilon \to 0. \tag{50}$$

Hence,

$$\frac{1}{2}\int_0^{t_\varepsilon} \varepsilon^{-1}C_{2,\varepsilon}(t)C_\varepsilon^*(t)\lambda_\varepsilon\,dt + \int_{t_\varepsilon}^{T} \varepsilon^{-1}C_{2,\varepsilon}(t)\frac{C_\varepsilon^*(t)\lambda_\varepsilon}{\|C_\varepsilon^*(t)\lambda_\varepsilon\|}\,dt = \frac{1}{2}\int_0^{t_0} \varepsilon^{-1}C_{2,\varepsilon}(t)C_\varepsilon^*(t)\lambda_\varepsilon\,dt + \mathbb{O}$$
$$=: I_5(\varepsilon, \Delta\lambda) + \mathbb{O},$$

while the power asymptotics of the integrals I_i, $i = 2, 3, 4$ contains no Δr.

We introduce the notation: $\big(I_i(\varepsilon, \Delta\lambda)\big)_1$ is a linear in Δl, Δr, Δt and ε part of the integral $I_i(\varepsilon, \Delta\lambda)$.

By Theorem 4, identities (50) and

$$\int\limits_0^{t_0} e^{-\frac{t}{\varepsilon}} f(t, l_\varepsilon, r_\varepsilon)\,dt = O(\varepsilon),$$

if $f(t, l_\varepsilon, r_\varepsilon)$ is uniformly bounded on $[0, t_0]$, by simple calculations we get:

$$\big(I_1(\varepsilon, \Delta\lambda)\big)_1 = \frac{1}{2}\int\limits_0^{t_0} C_{1,0}(t)C^*_{1,0}(t)\,dt\,\Delta l + \varepsilon f_1 \ =:\ D_{11}\Delta l + \varepsilon f_1, \tag{51}$$

$$\begin{aligned}\big(I_3(\varepsilon, \Delta\lambda)\big)_1 = &\int\limits_{t_0}^{T} C_{1,0}(t)\frac{C^*_{1,0}(t)\triangle l\|C^*_{1,0}(t)l_0\|^2 - \langle C^*_{1,0}(t)\triangle l, C^*_{1,0}(t)l_0\rangle C^*_{1,0}(t)l_0}{\|C^*_{1,0}(t)l_0\|^3}dt\\ &+\varepsilon f_3 =:\ D_{12}\Delta l + \varepsilon f_3,\end{aligned} \tag{52}$$

$$\big(I_5(\varepsilon, \Delta\lambda)\big)_1 = \frac{1}{4}\Delta r + \frac{1}{4}\big(2B^*_0 - A^*_{12}\big)\Delta l + \varepsilon f_5, \tag{53}$$

where f_1, f_3 and f_5 are uniquely calculated by l_0. At that, by assumption (36) and Cauchy-Schwarz inequality we have:

$$D_{11} > 0, \qquad D_{12} \geqslant 0. \tag{54}$$

By identity (50) we can find the asymptotics for the function $G(\varepsilon, \Delta l, \Delta t)$ as Δl, Δt and ε tend to zero:

$$\begin{aligned}G(\varepsilon, \Delta l, \Delta t) \sim &2\langle C^*_{1,0}(t_0)l_0, C^*_{1,0}(t_0)\Delta l + (C^*_{1,0})'(t_0)l_0\Delta t + \varepsilon A^*_{11}e^{A^*_{11}t_0}l_0\rangle\\ &+\sum_{k=2}^{\infty} G_k(\varepsilon, \Delta l, \Delta t), \qquad (C^*_{1,0})'(t_0) := \frac{d}{dt}C^*_{1,0}(t)\Big|_{t=t_0},\end{aligned} \tag{55}$$

where $G_k(\varepsilon, \Delta l, \Delta t)$ are some homogeneous functions of order k in ε and the components of the vectors Δl and Δr.

Thus, by identities (48), (49), (51)–(53) and (55), the system for the first corrector of (47) reads as

$$\begin{cases}\varepsilon g_1 = D^2\varphi^*_1(-l_0)\Delta l_1 + D_{11}\Delta l_1 + D_{12}\Delta l_1\\ \varepsilon g_2 = \dfrac{1}{4}\Delta r_1 + \dfrac{1}{4}\big(2B^*_0 - A^*_{12}\big)\Delta l_1\\ \varepsilon g_3 = 2\langle C^*_{1,0}(t_0)l_0, C^*_{1,0}(t_0)\Delta l_1\rangle + \langle C^*_{1,0}(t_0)l_0, (C^*_{1,0})'(t_0)l_0\rangle\Delta t_1.\end{cases} \tag{56}$$

By the convexity of φ_1 and inequalities (54), we have

$$D^2\varphi^*_1(-l_0) + D_{11} + D_{12} > 0,$$

and this is why the first equation in system (56) determines uniquely $\Delta l_1 = \varepsilon l_1$. After that by the second equation in system (56) we uniquely find $\Delta r_1 = \varepsilon r_1$. Finally, by conditions (41), the coefficient at Δt_1 is non-zero and hence, by the third equation in system (56) we uniquely determine $\Delta t_1 = \varepsilon t_1$. Thus, the linear operator of the first corrector for system (56), that is, the operator

$$\mathcal{D}\begin{pmatrix}\Delta l_1\\ \Delta r_1\\ \Delta t_1\end{pmatrix} = \begin{pmatrix}D^2\varphi^*_1(-l_0)\Delta l_1 + D_{11}\Delta l_1 + D_{12}\Delta l_1\\ \frac{1}{4}\Delta r_1 + \frac{1}{4}\big(2B^*_0 - A^*_{12}\big)\Delta l_1\\ 2\langle C^*_{1,0}(t_0)l_0, C^*_{1,0}(t_0)\Delta l_1\rangle + \langle C^*_{1,0}(t_0)l_0, (C^*_{1,0})'(t_0)l_0\rangle\Delta t_1\end{pmatrix}$$

is continuously invertible.

The process of determining next terms in the expansions of Δl, Δr and Δt is continued in a standard way. Assume that we have approximations of Δl, Δr and Δt up to Nth order. Then the quantities

$$\Delta l_{N+1} := \Delta l - \sum_{k=1}^{N} \varepsilon^k l_k, \quad \Delta r_{N+1} := \Delta r - \sum_{k=1}^{N} \varepsilon^k r_k, \quad \Delta t_{N+1} := \Delta t - \sum_{k=1}^{N} \varepsilon^k t_k$$

satisfy the relations

$$\mathcal{D}\begin{pmatrix}\Delta l_{N+1}\\ \Delta r_{N+1}\\ \Delta t_{N+1}\end{pmatrix} = O\big(\varepsilon^{N+1}\big) + O\big(\varepsilon\|z_{N+1}\|\big) + O\big(\|z_{N+1}\|^2\big), \qquad z_{N+1} := \begin{pmatrix}\Delta l_{N+1}\\ \Delta r_{N+1}\\ \Delta t_{N+1}\end{pmatrix}. \tag{57}$$

By the continuous invertibility of the operator $\mathcal{D}$, by relations (57) we obtain:

$$z_{N+1} = O\big(\varepsilon^{N+1}\big) + O\big(\varepsilon\|z_{N+1}\|\big) + O\big(\|z_{N+1}\|^2\big). \tag{58}$$

As it was shown in [10, Stat. 2], it follows from (58) that $z_{N+1} = O\big(\varepsilon^{N+1}\big)$. Thus, we have proved the following theorem.

Theorem 5. *Let Assumptions 2 and 3 be satisfied as well as conditions (41) and (42). Then the vectors l_ε, r_ε and the moment of time t_ε are expanded into power asymptotic series*

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \quad r_\varepsilon \overset{as}{=} \big(A_{12}^* - 2B_0^*\big) l_0 + \sum_{k=1}^{\infty} \varepsilon^k r_k, \quad t_\varepsilon \overset{as}{=} t_0 + \sum_{k=1}^{\infty} \varepsilon^k t_k, \qquad \varepsilon \to 0,$$

whose coefficients can be found in a recurrent way.

Similar results are true in a more general case, when there exist finitely many points $\{t_1, t_2, \ldots, t_p\} \subset (0, T)$ such that

$$\|C_0^*(t) l_0\| \neq 2, \qquad \|C_0^*(t_i) l_0\|^2 = 4, \qquad \frac{d}{dt}\|C_0^*(t_i) l_0\|^2 \Big|_{t=t_i} \neq 0, \tag{59}$$

for all $t \in [0, T] \setminus \{t_i\}_{i=1}^p$ and condition (42) holds true.

In this case an analogue of Theorem 4 reads as follows.

Theorem 6. *Let Assumptions (36), (42) and (59) hold true. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exist the points $\{t_{1,\varepsilon}, t_{2,\varepsilon}, \ldots, t_{p,\varepsilon}\} \subset (0, T)$ of the change of the type of optimal control in problem (1). There are no other points of the change of the type of optimal control and $t_{i,\varepsilon} \to t_i$ as $\varepsilon \to 0$ for each $i = 1, \ldots, p$.*

The proof of this theorem is similar to that of Theorem 4.

We note that in this case the system of equations similar to system (47) contains a set of p equations $0 = G_p$ instead of one scalar equation $0 = G$; these equations correspond to the points $t_{i,\varepsilon}$ and the unknowns are Δl, Δr and Δt_i, $i = 1, \ldots, p$.

Similar to Theorem 5, we can prove the following final theorem.

Theorem 7. *Let Assumptions 2 and 3 are satisfied as well as conditions (36), (42) and (59). Then the vectors l_ε, r_ε and the moments of time $\{t_{1,\varepsilon}, t_{2,\varepsilon}, \ldots, t_{p,\varepsilon}\}$ are expanded into power asymptotic series*

$$l_\varepsilon \overset{as}{=} l_0 + \sum_{k=1}^{\infty} \varepsilon^k l_k, \qquad r_\varepsilon \overset{as}{=} \big(A_{12}^* - 2B_0^*\big) l_0 + \sum_{k=1}^{\infty} \varepsilon^k r_k,$$

$$t_{i,\varepsilon} \overset{as}{=} t_i + \sum_{k=1}^{\infty} \varepsilon^k t_{i,k}, \qquad i = 1, \ldots, p, \qquad \varepsilon \to 0,$$

whose coefficients can be found in a recurrent way.

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