BOUNDARY VALUE PROBLEMS FOR DEGENERATE AND DEGENERATE FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH NON-LOCAL LINEAR SOURCE AND DIFFERENCE METHODS FOR THEIR NUMERICAL IMPLEMENTATION

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Abstract. In the paper we study non-local boundary value problems for differential and partial differential equations of fractional order with a non-local linear source being mathematical models of the transfer of water and salts in soils with fractal organization. Apart of the Cartesian case, in the paper we consider one-dimensional cases with cylindrical and spherical symmetry. By the method of energy inequalities, we obtain apriori estimates of solutions to nonlocal boundary value problems in differential form. We construct difference schemes and for these schemes, we prove analogues of apriori estimates in the difference form and provide estimates for errors assuming a sufficient smoothness of solutions to the equations. By the obtained apriori estimates, we get the uniqueness and stability of the solution with respect to the the initial data and the right par, as well as the convergence of the solution of the difference problem to the solution of the corresponding differential problem with the rate of $O(h^2 + \tau^2)$.

Keywords: boundary value problem, apriori estimate, the equation of moisture transfer, the differential equation of fractional order, Gerasimov-Caputo fractional derivative.

Mathematics Subject Classification: 65N06; 65N12

Introduction

As non-local boundary value problems, one usually calls problem, in which on the boundary, instead of prescribing the values of a solution and its derivatives, a relation between these values and similar values on some other internal or boundary manifolds is given. The theory of non-local boundary value problems is important as a part of general theory of boundary value problems for partial differential equations due to numerous applications in mechanics, physics, biology and other natural sciences.

At present, differential equations involving fractional derivatives both in time and spatial variables attract attention by mathematicians and physicians since such equations serve as mathematical models of various processes [1]–[9]. A lot of works are devoted to studying various local and non-local initial-boundary value problems for Sobolev type differential equations and their subclass of pseudo-parabolic equations [10]–[19]. In works [20]–[25], the finite difference method was employed for studying various boundary value problems for Sobolev type differential equations with varying coefficients.

In the present work we consider nonlocal boundary value problems for Sobolev type differential equations with a Gerasimov-Caputo fractional derivative in time and with a nonlocal linear source. Apart of the Cartesian case, in the work we consider one-dimensional cases with a cylindrical and spherical symmetry.

1. Boundary value problem for a pseudo-parabolic equation with a nonlocal linear source

In a closed cylinder $\overline{Q}_T = \{(x, t) : 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$ we consider the following boundary value problem:

$$\partial_{0t}^{\alpha} u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + \partial_{0t}^{\alpha}\frac{\partial}{\partial x}\left(\eta(x)\frac{\partial u}{\partial x}\right) + r(x,t)\frac{\partial u}{\partial x}$$
$$- \int_0^x q(s,t)u(s,t)ds + f(x,t), \qquad 0 < x < l, \qquad 0 < t \leqslant T, \tag{1.1}$$

$$\Pi(0,t) = \beta_{11}(t)u(0,t) + \beta_{12}(t)\partial_{0t}^{\alpha}u(0,t) - \mu_1(t), \qquad 0 \leqslant t \leqslant T, \tag{1.2}$$

$$-\Pi(l,t) = \beta_{21}(t)u(l,t) + \beta_{22}(t)\partial_{0t}^{\alpha}u(l,t) - \mu_2(t), \qquad 0 \leqslant t \leqslant T, \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad 0 \leqslant x \leqslant l, \tag{1.4}$$

where

$$0 < c_0 \leqslant k(x,t), \eta(x), \qquad \beta_{12}(t), \qquad \beta_{22}(t) \leqslant c_1,$$
$$|\beta_{11}(t),\ \beta_{21}(t),\ r(x,t),\ q(x,t),\ \eta_x(x), k_x(x,t), r_x(x,t)| \leqslant c_2, \tag{1.5}$$

and

$$\partial_{0t}^{\alpha} u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}d\tau$$

is a Gerasimov-Caputo fractional derivative of order α, $0 < \alpha < 1$, [26], [27], $\Pi(x,t) = ku_x + \partial_{0t}^{\alpha}\big(\eta(x)u_x\big)$, c_i, $i = 0, 1, 2$ are positive constants,

$$\partial_{0t}^{\alpha} u = D_{0t}^{\alpha}u - \frac{u(0)}{\Gamma(1-\alpha)t^\alpha}, \qquad D_{0t}^{\alpha}u = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{ud\tau}{(t-\tau)^\alpha}$$

is the Riemann-Liouville fractional derivative of order α.

In what follows we assume that problem (1.1)–(1.4) possesses the unique solutions with all needed derivatives. We shall assume that the coefficients in the equation and boundary conditions satisfy all smoothness conditions arising in calculations and ensuring a needed approximation order of the difference scheme. By M_i, $i = 1, 2, \ldots$, we shall denote positive constants depending only on the data of the considered problem.

2. Apriori estimate in differential form

To obtain apriori estimate for the solution of problem (1.1)–(1.4) in a a differential form, we introduce a scalar product and a norm:

$$\left(a, b\right) = \int_0^l ab\,dx, \qquad \left(a, a\right) = \|a\|_0^2,$$

where a, b are some given on $[0, l]$ functions. We multiply equation (1.1) by $U = u + \partial_{0t}^{\alpha}u$ in the sense of the scalar product:

$$\left(\partial_{0t}^{\alpha}u, U\right) = \left((ku_x)_x, U\right) + \left(\partial_{0t}^{\alpha}(\eta u_x)_x, U\right) + \left(ru_x, U\right) - \left(\int_0^x quds, U\right) + \left(f, U\right). \tag{2.1}$$

The following lemma holds true [28].

Lemma 1. *For each absolutely continuous on $[0,T]$ function $v(t)$, the inequality holds:*

$$v(t)\partial_{0t}^{\alpha}v(t) \geqslant \frac{1}{2}\partial_{0t}^{\alpha}\big(v^2(t)\big), \qquad 0 < \alpha < 1.$$

We rewrite the terms in identity (2.1) by employing the Cauchy inequality with ε [29] and Lemma 1:

$$\Big(\partial_{0t}^{\alpha}u, U\Big) = \Big(\partial_{0t}^{\alpha}u, u + \partial_{0t}^{\alpha}u\Big) = \Big(1, u\partial_{0t}^{\alpha}u\Big) + \Big(1, (\partial_{0t}^{\alpha}u)^2\Big) \geqslant \frac{1}{2}\partial_{0t}^{\alpha}\|u\|_0^2 + \|\partial_{0t}^{\alpha}u\|_0^2, \qquad (2.2)$$

$$\begin{aligned}
\Big((ku_x)_x, U\Big) &= \Big((ku_x)_x, u + \partial_{0t}^{\alpha}u\Big) = Uku_x\Big|_0^l - \Big(ku_x, u_x + \partial_{0t}^{\alpha}u_x\Big) \\
&= Uku_x\Big|_0^l - \Big(k, u_x^2\Big) - \Big(k, u_x\partial_{0t}^{\alpha}u_x\Big) \\
&\leqslant Uku_x\Big|_0^l - c_0\|u_x\|_0^2 - \frac{1}{2}\int_0^l k\partial_{0t}^{\alpha}(u_x)^2 dx.
\end{aligned} \qquad (2.3)$$

$$\begin{aligned}
\Big(\partial_{0t}^{\alpha}(\eta u_x)_x, U\Big) &= \Big(\partial_{0t}^{\alpha}(\eta u_x)_x, u + \partial_{0t}^{\alpha}u\Big) = U\partial_{0t}^{\alpha}(\eta u_x)\Big|_0^l - \Big(\partial_{0t}^{\alpha}(\eta u_x), u_x + \partial_{0t}^{\alpha}u_x\Big) \\
&= -\Big(\eta, u_x\partial_{0t}^{\alpha}u_x\Big) - \Big(\eta, (\partial_{0t}^{\alpha}u_x)^2\Big) + U\partial_{0t}^{\alpha}(\eta u_x)\Big|_0^l \\
&= U\partial_{0t}^{\alpha}(\eta u_x)\Big|_0^l - \frac{1}{2}\int_0^l \eta(x)\partial_{0t}^{\alpha}(u_x)^2 dx - c_0\|\partial_{0t}^{\alpha}u_x\|_0^2.
\end{aligned} \qquad (2.4)$$

$$\begin{aligned}
\Big(ru_x, U\Big) &= \Big(ru_x, u + \partial_{0t}^{\alpha}u\Big) = \Big(ru_x, u\Big) + \Big(ru_x, \partial_{0t}^{\alpha}u\Big) \\
&\leqslant \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + M_1^{\varepsilon}\Big(\|u\|_0^2 + \|u_x\|_0^2\Big),
\end{aligned} \qquad (2.5)$$

$$\begin{aligned}
-\Big(\int_0^s quds, U\Big) &= -\Big(\int_0^s quds, u + \partial_{0t}^{\alpha}u\Big) = -\Big(\int_0^s quds, u\Big) - \Big(\int_0^s quds, \partial_{0t}^{\alpha}u\Big) \\
&\leqslant \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + \frac{1}{2}\|u\|_0^2 + M_2^{\varepsilon}\Big(1, \Big(\int_0^s quds\Big)^2\Big) \\
&\leqslant M_3^{\varepsilon}\int_0^l\int_0^s u^2 dsdx + \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + \frac{1}{2}\|u\|_0^2 \leqslant \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + M_4^{\varepsilon}\|u\|_0^2.
\end{aligned} \qquad (2.6)$$

$$\Big(f, U\Big) = \Big(f, u + \partial_{0t}^{\alpha}u\Big) = \Big(f, u\Big) + \Big(f, \partial_{0t}^{\alpha}u\Big) \leqslant \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + M_5^{\varepsilon}\|f\|_0^2 + \|u\|_0^2. \qquad (2.7)$$

Taking into considerations transformations (2.2)–(2.7), by (2.1) we find

$$\begin{aligned}
&\frac{1}{2}\partial_{0t}^{\alpha}\|u\|_0^2 + \|\partial_{0t}^{\alpha}u\|_0^2 + c_0\|u_x\|_0^2 + \frac{1}{2}\int_0^l (k + \eta(x))\partial_{0t}^{\alpha}(u_x)^2 dx + c_0\|\partial_{0t}^{\alpha}u_x\|_0^2 \\
&\qquad \leqslant U\Pi(x,t)\Big|_0^l + \varepsilon\|\partial_{0t}^{\alpha}u\|_0^2 + M_6^{\varepsilon}\Big(\|u\|_0^2 + \|u_x\|_0^2\Big) + M_7^{\varepsilon}\|f\|_0^2.
\end{aligned} \qquad (2.8)$$

Letting $\varepsilon = \frac{1}{2}$, by (2.8) we get:

$$\begin{aligned}
&\frac{1}{2}\partial_{0t}^{\alpha}\|u\|_0^2 + \frac{1}{2}\|\partial_{0t}^{\alpha}u\|_0^2 + c_0\|u_x\|_0^2 + \frac{1}{2}\int_0^l (k + \eta(x))\partial_{0t}^{\alpha}(u_x)^2 dx + c_0\|\partial_{0t}^{\alpha}u_x\|_0^2 \\
&\qquad \leqslant U\Pi(x,t)\Big|_0^l + M_8\Big(\|u\|_0^2 + \|u_x\|_0^2\Big) + M_9\|f\|_0^2.
\end{aligned} \qquad (2.9)$$

Estimating the first term in the right hand side of the above inequality, we obtain:

$$
\begin{aligned}
U(x,t)\Pi(x,t)\Big|_0^l ={}& \Big(u(l,t) + \partial_{0t}^\alpha u(l,t)\Big)\Big(\mu_2(t) - \beta_{21}(t)u(l,t) - \beta_{22}(t)\partial_{0t}^\alpha u(l,t)\Big)\\
& + \Big(u(0,t) + \partial_{0t}^\alpha u(0,t)\Big)\Big(\mu_1(t) - \beta_{11}u(0,t) - \beta_{12}(t)\partial_{0t}^\alpha u(0,t)\Big)\\
={}& \mu_2(t)u(l,t) + \mu_2(t)\partial_{0t}^\alpha u(l,t) - \beta_{21}(t)u^2(l,t) - \beta_{21}(t)u(l,t)\partial_{0t}^\alpha u(l,t)\\
& - \beta_{22}(t)u(l,t)\partial_{0t}^\alpha u(l,t) - \beta_{22}(t)\Big(\partial_{0t}^\alpha u(l,t)\Big)^2 + \mu_1(t)u(0,t)\\
& + \mu_1(t)\partial_{0t}^\alpha u(0,t) - \beta_{11}(t)u^2(0,t) - \beta_{11}u(0,t)\partial_{0t}^\alpha u(0,t)\\
& - \beta_{12}(t)u(0,t)\partial_{0t}^\alpha u(0,t) - \beta_{12}\Big(\partial_{0t}^\alpha u(0,t)\Big)^2\\
\leqslant{}& M_{10}^{\varepsilon_1,\varepsilon_2}(\mu_1^2 + \mu_2^2) + \varepsilon_1\Big(\partial_{0t}^\alpha u(l,t)\Big)^2 + \varepsilon_2\Big(\partial_{0t}^\alpha u(0,t)\Big)^2\\
& + M_{11}^\varepsilon\Big(\|u\|_0^2 + \|u_x\|_0^2\Big) - \beta_{22}(t)\Big(\partial_{0t}^\alpha u(l,t)\Big)^2 - \frac{1}{2}\beta_{22}(t)\partial_{0t}^\alpha u^2(l,t)\\
& - \beta_{12}(t)\Big(\partial_{0t}^\alpha u(0,t)\Big)^2 - \frac{1}{2}\beta_{12}(t)\partial_{0t}^\alpha u^2(0,t)\\
\leqslant{}& -\frac{\beta_{12}(t)}{2}\Big(\partial_{0t}^\alpha u(0,t)\Big)^2 - \frac{\beta_{22}(t)}{2}\Big(\partial_{0t}^\alpha u(l,t)\Big)^2 - \frac{\beta_{12}(t)}{2}\partial_{0t}^\alpha u^2(0,t)\\
& - \frac{\beta_{12}(t)}{2}\partial_{0t}^\alpha u^2(l,t) + M_{12}\Big(\|u\|_0^2 + \|u_x\|_0^2\Big) + M_{13}\Big(\mu_1^2 + \mu_2^2\Big).
\end{aligned}
\tag{2.10}
$$

In view of (2.10), by (2.9) we find:

$$
\begin{aligned}
\partial_{0t}^\alpha \|u\|_0^2 &+ \int_0^l \big(k + \eta(x)\big)\partial_{0t}^\alpha(u_x)^2 dx + \|u_x\|_0^2 + \|\partial_{0t}^\alpha u\|_0^2 + \|\partial_{0t}^\alpha u_x\|_0^2\\
&\leqslant M_{14}\|u\|_{W_2^1(0,l)}^2 + M_{15}\Big(\|f\|_0^2 + \mu_1^2(t) + \mu_2^2(t)\Big),
\end{aligned}
\tag{2.11}
$$

where $\|u\|_{W_2^1(0,l)}^2 = \|u\|_0^2 + \|u_x\|_0^2$.

We apply the operator of fractional integration $D_{0t}^{-\alpha}$ to both sides of inequality (2.11) and we obtain:

$$
\begin{aligned}
\|u\|_{W_2^1(0,l)}^2 &+ D_{0t}^{-\alpha}\Big(\|u_x\|_0^2 + \|\partial_{0t}^\alpha u\|_0^2 + \|\partial_{0t}^\alpha u_x\|_0^2\Big) \leqslant M_{14}D_{0t}^{-\alpha}\|u\|_{W_2^1(0,l)}^2\\
&+ M_{16}\left(D_{0t}^{-\alpha}\Big(\|f\|_0^2 + \mu_1^2(t) + \mu_2^2(t)\Big) + \|u_0(x)\|_{W_2^1(0,l)}^2\right).
\end{aligned}
\tag{2.12}
$$

To estimate the first term in the right hand side, we employ Lemma [28].

Lemma 2. *Let a non-negative absolutely continuous function $y(t)$ for almost all t in $[0,T]$ satisfy the inequality*

$$
\partial_{0t}^\alpha y(t) \leqslant c_1 y(t) + c_2(t), \qquad 0 \leqslant \alpha \leqslant 1,
$$

where $c_1 > 0$, $c_2(t)$ is a summable on $[0,T]$ non-negative function. Then

$$
y(t) \leqslant y(0)E_\alpha(c_1 t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(c_1 t^\alpha)D_{0t}^{-\alpha}c_2(t),
$$

where

$$
E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad E_{\alpha,\mu}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \mu)}
$$

and Mittag-Leffler functions.

By Lemma 2 we can estimate the first term in the right hand side in (2.12).

Let $y(t) = D_{0t}^{-\alpha} \|u\|_{W_2^1(0,l)}^2$, $\partial_{0t}^\alpha y(t) = \|u\|_{W_2^1(0,l)}^2$, then

$$D_{0t}^{-\alpha} \|u\|_{W_2^1(0,l)}^2 \leqslant M_{17}\left(D_{0t}^{-2\alpha}\left(\|f\|_0^2 + \mu_1^2(t) + \mu_2^2(t)\right) + \|u_0(x)\|_{W_2^1(0,l)}^2\right). \tag{2.13}$$

The following lemma holds.

Lemma 3. *Each non-negative integrable on $[0,T]$ function $g(t)$ satisfies the inequality*

$$D_{0t}^{-2\alpha} g(t) = \frac{t^\alpha \Gamma(\alpha)}{\alpha \Gamma^2(\alpha) - \Gamma(2\alpha)} D_{0t}^{-\alpha} g(t). \tag{2.14}$$

Proof. We transform a fractional integral in the left hand side:

$$D_{0t}^{-2\alpha} g(t) = \frac{1}{\Gamma(2\alpha)} \int_0^t (t-\tau)^{2\alpha-1} g(\tau) d\tau = \frac{1}{\Gamma(2\alpha)} \int_0^t (t-\tau)^\alpha (t-\tau)^{\alpha-1} g(\tau) d\tau. \tag{2.15}$$

Integrating by parts and employing the formula $B(\alpha, \alpha) = \dfrac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}$, by simple transformations we arrive at (2.14).

Let us show that $\alpha \Gamma^2(\alpha) > \Gamma(2\alpha)$ for all $\alpha \in (0,1)$ in (2.14), or

$$(2\alpha)! < 2(\alpha!)^2, \qquad \text{for all} \qquad \alpha \in (0,1). \tag{2.16}$$

In order to do this, we consider the inequality $2^{\alpha^2} < 2$, which is true for all $\alpha \in (-1,1)$. Then by (2.16) we find

$$(2\alpha)! \leqslant 2^{\alpha^2}(\alpha!)^2 < 2(\alpha!)^2 \qquad \text{for all} \qquad \alpha \in (0,1). \tag{2.17}$$

We are going to prove

$$(2\alpha)! \leqslant 2^{\alpha^2}(\alpha!)^2 \qquad \text{for all} \qquad \alpha \in \mathbb{R} \tag{2.18}$$

by induction. Indeed, (2.18) is true as $\alpha = 0$. Assume that it holds for all $\alpha = n$. Let us prove that (2.18) holds for all $\alpha = n + 1$, then we obtain

$$(2n+2)! \leqslant 2^{(n+1)^2}((n+1)!)^2. \tag{2.19}$$

We transform the right hand side of (2.19)

$$(2n+2)! = 2n!(2n+1)(2n+2) \leqslant 2^{n^2}(n!)^2(2n+1)(2n+2) \leqslant 2^{n^2} 2^{2n}(n!)^2 2(n+1)^2.$$

This implies:

$$2n+1 \leqslant 2^{2n}(n+1) \qquad \text{for all} \qquad n \in \mathbb{R}. \tag{2.20}$$

The latter identity can be proved by induction. $\qquad \square$

By means of Lemma 3, (2.12), (2.13), (2.14) we find a needed apriori estimate:

$$\|u\|_{W_2^1(0,l)}^2 + D_{0t}^{-\alpha}\left(\|u_x\|_0^2 + \|\partial_{0t}^\alpha u\|_0^2 + \|\partial_{0t}^\alpha u_x\|_0^2\right)$$

$$\leqslant M\left(D_{0t}^{-\alpha}\left(\|f\|_0^2 + \mu_1^2(t) + \mu_2^2(t)\right) + \|u_0(x)\|_{W_2^1(0,l)}^2\right), \tag{2.21}$$

where M is a positive constant depending only on data (1.1)–(1.4),

$$D_{0t}^{-\alpha} u = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u \, d\tau}{(t-\tau)^{1-\alpha}}$$

is a fractional Riemann-Liouville integral of order α, $0 < \alpha < 1$.

Theorem 1. *If $k(x,t) \in C^{1,0}(\overline{Q}_T)$, $\eta(x) \in C^1[0,l]$, $r(x,t), q(x,t), f(x,t) \in C(\overline{Q}_T)$, $u(x,t) \in C^{2,0}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, $\partial_{0t}^\alpha u(x,t) \in C(\overline{Q}_T)$ and conditions (1.5) are satisfied, then the solution of problem (1.1)–(1.4) obeys apriori estimate (2.21).*

Apriori estimate (2.21) implies the uniqueness of the solution and the stability in the initial data and the right hand side in the sense of the norm

$$\|u\|_1^2 = \|u\|_{W_2^1(0,l)}^2 + D_{0t}^{-\alpha}\left(\|u_x\|_0^2 + \|\partial_{0t}^{\alpha}u\|_0^2 + \|\partial_{0t}^{\alpha}u_x\|_0^2\right).$$

3.   STABILITY AND CONVERGENCE OF DIFFERENCE SCHEME

To solve problem (1.1)–(1.4), we apply the finite element method. We construct a monotone scheme of second order approximation involving the derivatives taking into consideration the sign of $r(x,t)$. In order to do this, instead of equation (1.1), we consider the following equation with perturbed coefficients [28]:

$$\partial_{0t}^{\alpha}u = \varkappa(ku_x)_x + \partial_{0t}^{\alpha}(\eta u_x)_x + ru_x - \int_0^x q(s,t)u(s,t)ds + f(x,t), \qquad (3.1)$$

where $\varkappa = \frac{1}{1+R}$, $R = \frac{0.5h|r|}{k}$ is the difference Reynolds number.

On the uniform grid $\overline{\omega}_{h\tau}$ we introduce a difference scheme of approximation order $O(h^2 + \tau^2)$ corresponding to differential problem (1.1)–(1.4):

$$\Delta_{0t_{j+\sigma}}^{\alpha}y = \varkappa_i^j\left(a_i^j y_{\bar{x}}^{(\sigma)}\right)_x + \Delta_{0t_{j+\sigma}}^{\alpha}\left(\gamma_i y_{\bar{x}}\right)_x + b_i^{-j}a_i^j y_{\bar{x},i}^{(\sigma)} + b_i^{+j}a_{i+1}^j y_{x,i}^{(\sigma)} - \sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar + \varphi_i^j, \quad (3.2)$$

$$\varkappa_0 a_1 y_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_1 y_{x,0}) = \beta_{11}y_0^{(\sigma)} + 0.25h^2 d_0^j y_0^{(\sigma)} + \tilde{\beta}_{12}\Delta_{0t_{j+\sigma}}^{\alpha}y_0 - \tilde{\mu}_1, \; t \in \overline{\omega}_\tau, \qquad (3.3)$$

$$-\left(\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_N y_{\bar{x},N})\right) = \beta_{21}y_N^{(\sigma)} + 0.5h\sum_{s=0}^{N} d_s^j y_s^{(\sigma)}\hbar + \tilde{\beta}_{22}\Delta_{0t_{j+\sigma}}^{\alpha}y_N - \tilde{\mu}_2, \qquad (3.4)$$

$$y(x,0) = u_0(x), \qquad x \in \overline{\omega}_h, \qquad (3.5)$$

where

$$\tilde{\beta}_{12} = \beta_{12} + 0.5h, \qquad \tilde{\mu}_1(t_{j+\sigma}) = \mu_1(t_{j+\sigma}) + 0.5h\varphi_0^j, \qquad \tilde{\beta}_{22} = \beta_{22}(t_{j+\sigma}) + 0.5h,$$

$$\tilde{\mu}_2(t_{j+\sigma}) = \mu_2(t_{j+\sigma}) + 0.5h\varphi_N^j, \qquad a_i^j = k(x_{i-0.5}, t^{j+\sigma}), \qquad \gamma_i = \eta(x_{i-0.5}),$$

$$b_i^j = \frac{r(x, t_{j+\sigma})}{k(x, t_{j+\sigma})}, \qquad \varphi = f(x_i, t_{j+\sigma}) \qquad y^{(\sigma)} = \sigma y^{j+1} + (1-\sigma)y^j, \qquad d_i^j = d(x_i, t_{j+\sigma}),$$

$$a_0^{(\alpha,\sigma)} = \sigma^{1-\alpha}, \qquad a_l^{(\alpha,\sigma)} = (l+\sigma)^{1-\alpha} - (l-1+\sigma)^{1-\alpha}, \quad l \geqslant 1, \quad \sigma = 1 - \frac{\alpha}{2},$$

$$b_l^{(\alpha,\sigma)} = \frac{1}{2-\alpha}\left[(l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha}\right] - \frac{1}{2}\left[(l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\alpha}\right], \; l \geqslant 1,$$

$$c_0^{(\alpha,\sigma)} = a_0^{(\alpha,\sigma)} \qquad \text{as} \qquad j = 0,$$

$$c_s^{(\alpha,\sigma)} = \begin{cases} a_0^{(\alpha,\sigma)} + b_1^{(\alpha,\sigma)}, & s = 0, \\ a_s^{(\alpha,\sigma)} + b_{s+1}^{(\alpha,\sigma)} - b_s^{(\alpha,\sigma)}, & 1 \leqslant s \leqslant j-1, \qquad \text{as} \qquad j > 0, \\ a_j^{(\alpha,\sigma)} - b_j^{(\alpha,\sigma)}, & s = j, \end{cases}$$

$$c_s^{(\alpha,\sigma)} > \frac{1-\alpha}{2}(s+\sigma)^{-\alpha} > 0,$$

and

$$\Delta_{0t_{j+\sigma}}^{\alpha}y = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{j} c_{j-s}^{(\alpha,\sigma)} y_t^s$$

is a discrete analogue of Gerasimov-Caputo fractional derivative of order α, $0 < \alpha < 1$ [30]. We introduce scalar products and a norm:

$$[u, v] = \sum_{i=0}^{N} u_i v_i \hbar, \quad \hbar = \begin{cases} 0.5h, & i = 0, N, \\ h, & i \neq 0, N, \end{cases} \quad (u, v) = \sum_{i=1}^{N} u_i v_i h, \quad [u, u] = [1, u^2] = |[u]|_0^2,$$

We rewrite (3.2)–(3.5) in an operator form:

$$\Delta_{0t_{j+\sigma}}^{\alpha} y = \overline{\Lambda}(t_{j+\sigma}) y^{(\sigma)} + \overline{\delta} y + \overline{\Phi}, \tag{3.6}$$

$$y(x, 0) = u_0(x), \qquad x \in \overline{\omega}_h, \tag{3.7}$$

where

$$\overline{\Lambda}(t_{j+\sigma}) y^{(\sigma)} = \begin{cases} \tilde{\Lambda} y_i^{(\sigma)} = \varkappa(a y_{\bar{x}}^{(\sigma)})_x + b^- a y_{\bar{x}}^{(\sigma)} + b^+ a^{(+1)} y_x^{(\sigma)} - \sum_{s=0}^{i} d_s^j y_s^{(\sigma)} \hbar, \\[2mm] \Lambda^- y_0^{(\sigma)} = \dfrac{2}{h} \left(\varkappa_0 a_1 y_{x,0}^{(\sigma)} - \beta_{11} y_0^{(\sigma)} - 0.25 h^2 d_0^j y_0^{(\sigma)} \right), \qquad i = 0, \\[2mm] \Lambda^+ y_N^{(\sigma)} = \dfrac{2}{h} \left(-\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \beta_{21} y_N^{(\sigma)} - 0.5 h \sum_{s=0}^{N} d_s^j y_s^{(\sigma)} \hbar \right), \qquad i = N, \end{cases}$$

$$\overline{\delta} y = \begin{cases} \delta y_i = \Delta_{0t_{j+\sigma}}^{\alpha} (\gamma_i y_{\bar{x}})_x, \qquad i = \overline{1, N-1}, \\[2mm] \delta^- y_0 = \dfrac{2}{h} \left(\Delta_{0t_{j+\sigma}}^{\alpha} (\gamma_1 y_{x,0}) - \beta_{12} \Delta_{0t_{j+\sigma}}^{\alpha} y_0 \right), \qquad i = 0, \\[2mm] \delta^+ y_N = \dfrac{2}{h} \left(-\Delta_{0t_{j+\sigma}}^{\alpha} (\gamma_N y_{\bar{x},N}) - \beta_{22} \Delta_{0t_{j+\sigma}}^{\alpha} y_N \right), \qquad i = N, \end{cases}$$

$$\overline{\Phi} = \begin{cases} \varphi = \varphi_i, \qquad i = \overline{1, N-1}, \\[2mm] \varphi^- = \dfrac{2}{h} \left(\mu_1(t_{j+\sigma}) + 0.5 h \varphi_0^j \right), \qquad i = 0, \\[2mm] \varphi^+ = \dfrac{2}{h} \left(\mu_2(t_{j+\sigma}) + 0.5 h \varphi_N^j \right), \qquad i = N, \end{cases} \qquad \begin{aligned} \varkappa &= \dfrac{1}{1 + \frac{0.5h|r|}{k}} \\[3mm] \varkappa_0 &= \dfrac{1}{1 + \frac{0.5h|r_0|}{k_{0.5}}}, \qquad r_0 \leqslant 0 \\[3mm] \varkappa_N &= \dfrac{1}{1 + \frac{0.5h|r_N|}{k_{N-0.5}}}, \qquad r_N \geqslant 0. \end{aligned}$$

We calculate the scalar product of (3.6) with $\bar{y} = y^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha} y$:

$$\left[\Delta_{0t_{j+\sigma}}^{\alpha} y, \bar{y} \right] = \left[\overline{\Lambda}(t_{j+\sigma}) y^{(\sigma)}, \bar{y} \right] + \left[\overline{\delta} y, \bar{y} \right] + \left[\overline{\Phi}, \bar{y} \right]. \tag{3.8}$$

The following lemma holds [30].

Lemma 4. *Each function $y(t)$ defined on the grid $\bar{\omega}_\tau$ satisfies the inequality:*

$$y^{(\sigma)} \Delta_{0t_{j+\sigma}}^{\alpha} y \geqslant \frac{1}{2} \Delta_{0t_{j+\sigma}}^{\alpha} (y^2).$$

By means of Lemma 4, we estimate the sums in (3.8):

$$\begin{aligned} \left[\Delta_{0t_{j+\sigma}}^{\alpha} y, \bar{y} \right] &= \left[\Delta_{0t_{j+\sigma}}^{\alpha} y, y^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha} y \right] = \left[\Delta_{0t_{j+\sigma}}^{\alpha} y, y^{(\sigma)} \right] + \left[1, (\Delta_{0t_{j+\sigma}}^{\alpha} y)^2 \right] \\ &\geqslant \frac{1}{2} \Delta_{0t_{j+\sigma}}^{\alpha} |[y]|_0^2 + |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2, \end{aligned} \tag{3.9}$$

$$\left[\overline{\Lambda}(t_{j+\sigma})y^{(\sigma)}, \bar{y}\right] = \left(\tilde{\Lambda}y^{(\sigma)}, \bar{y}\right) + 0.5h\bar{y}_0\Lambda^- y_0^{(\sigma)} + 0.5h\bar{y}_N\Lambda^+ y_N^{(\sigma)}$$

$$= \left(\varkappa(ay_{\bar{x}}^{(\sigma)})_x, \bar{y}\right) + \left(b^- ay_{\bar{x}}^{(\sigma)}, \bar{y}\right) + \left(b^+ a^{(+1)}y_x^{(\sigma)}, \bar{y}\right)$$

$$- \left(\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar, \bar{y}\right) + \bar{y}_0\varkappa_0 a_1 y_{x,0}^{(\sigma)} - \beta_{11}y_0^{(\sigma)}\bar{y}_0 \tag{3.10}$$

$$- \varkappa_N a_N y_{\bar{x},N}^{(\sigma)}\bar{y}_N - 0.25h^2 d_0^j y_0^{(\sigma)}\bar{y}_0$$

$$- \beta_{21}y_N^{(\sigma)}\bar{y}_N - 0.5hd_N\bar{y}_N \sum_{s=0}^{N} d_s^j y_s^{(\sigma)}\hbar.$$

We transform the terms in the right hand side in (3.10):

$$\left(\varkappa(ay_{\bar{x}}^{(\sigma)})_x, \bar{y}\right) = \bar{y}\varkappa ay_{\bar{x}}^{(\sigma)}\Big|_0^N - \left(ay_{\bar{x}}^{(\sigma)}, (\varkappa\bar{y})_{\bar{x}}\right]$$

$$= \bar{y}_N\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \bar{y}_0\varkappa_0 a_1 y_{x,0}^{(\sigma)} - \left(ay_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}}\bar{y} + \varkappa^{(-1)}\bar{y}_{\bar{x}}\right]$$

$$= \bar{y}_N\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \bar{y}_0\varkappa_0 a_1 y_{x,0}^{(\sigma)} - \left(ay_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}}y^{(\sigma)}\right] - \left(ay_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}}\Delta_{0t_{j+\sigma}}^{\alpha}y\right]$$

$$- \left(ay_{\bar{x}}^{(\sigma)}, \varkappa^{(-1)}y_{\bar{x}}^{(\sigma)}\right) - \left(ay_{\bar{x}}^{(\sigma)}, \varkappa^{(-1)}\Delta_{0t_{j+\sigma}}^{\alpha}y_{\bar{x}}\right] \tag{3.11}$$

$$\leqslant \bar{y}_N\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \bar{y}_0\varkappa_0 a_1 y_{x,0}^{(\sigma)} + \varepsilon|[\Delta_{0t_{j+\sigma}}^{\alpha}y]|_0^2 + M_1^\varepsilon\left(|[y^{(\sigma)}]|_0^2 + |[y_{\bar{x}}^{(\sigma)}]|_0^2\right)$$

$$- \frac{1}{1+hM_2}\left(a\varkappa, (y_{\bar{x}}^{(\sigma)})^2\right] - \frac{c_0}{2(1+hM_2)}\left(\varkappa, \Delta_{0t_{j+\sigma}}^{\alpha}y_{\bar{x}}^2\right]$$

$$\leqslant \bar{y}_N\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \bar{y}_0\varkappa_0 a_1 y_{x,0}^{(\sigma)} + \varepsilon|[\Delta_{0t_{j+\sigma}}^{\alpha}y]|_0^2 + M_1^\varepsilon\left(|[y^{(\sigma)}]|_0^2 + \|y_{\bar{x}}^{(\sigma)}\|_0^2\right)$$

$$- M_3\|y_{\bar{x}}^{(\sigma)}]|_0^2 - M_4\Delta_{0t_{j+\sigma}}^{\alpha}\|y_{\bar{x}}]|_0^2,$$

$$\left(b^- ay_{\bar{x}}^{(\sigma)}, \bar{y}\right) + \left(b^+ a^{(+1)}y_x^{(\sigma)}, \bar{y}\right) = \left(b^- ay_{\bar{x}}^{(\sigma)}, y^{(\sigma)}\right) + \left(b^- ay_{\bar{x}}^{(\sigma)}, \Delta_{0t_{j+\sigma}}^{\alpha}y\right)$$

$$+ \left(b^+ a^{(+1)}y_x^{(\sigma)}, y^{(\sigma)}\right) + \left(b^+ a^{(+1)}y_x^{(\sigma)}, \Delta_{0t_{j+\sigma}}^{\alpha}y\right) \tag{3.12}$$

$$\leqslant \varepsilon|[\Delta_{0t_{j+\sigma}}^{\alpha}y]|_0^2 + M_5^\varepsilon\left(|[y^{(\sigma)}]|_0^2 + \|y_{\bar{x}}^{(\sigma)}]|_0^2\right),$$

$$-\left(\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar, \bar{y}\right) - 0.25h^2 d_0^j y_0^{(\sigma)}\bar{y}_0 - 0.5hd_N\bar{y}_N \sum_{s=0}^{N} d_s^j y_s^{(\sigma)}\hbar - \beta_{11}y_0^{(\sigma)}\bar{y}_0 - \beta_{21}y_N^{(\sigma)}\bar{y}_N$$

$$= -\left[\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar, y^{(\sigma)}\right] - \left[\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar, \Delta_{0t_{j+\sigma}}^{\alpha}y\right]$$

$$- \beta_{11}(y_0^{(\sigma)})^2 - \beta_{11}y_0^{(\sigma)}\Delta_{0t_{j+\sigma}}^{\alpha}y_0$$

$$- \beta_{21}(y_N^{(\sigma)})^2 - \beta_{21}y_0^{(\sigma)}\Delta_{0t_{j+\sigma}}^{\alpha}y_N \tag{3.13}$$

$$\leqslant \varepsilon_1|[\Delta_{0t_{j+\sigma}}^{\alpha}y]|_0^2 + M_6^{\varepsilon_1}\left[1, \left(\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar\right)^2\right] + M_7^{\varepsilon_2,\varepsilon_3}|[y^{(\sigma)}]|_0^2$$

$$+ \varepsilon_2\left(\Delta_{0t_{j+\sigma}}^{\alpha}y_0\right)^2 + \varepsilon_3\left(\Delta_{0t_{j+\sigma}}^{\alpha}y_N\right)^2$$

$$\leqslant \varepsilon_1|[\Delta_{0t_{j+\sigma}}^{\alpha}y]|_0^2 + M_8^{\varepsilon_1,\varepsilon_2,\varepsilon_3}|[y^{(\sigma)}]|_0^2 + \varepsilon_2\left(\Delta_{0t_{j+\sigma}}^{\alpha}y_0\right)^2 + \varepsilon_3\left(\Delta_{0t_{j+\sigma}}^{\alpha}y_N\right)^2.$$

In view of (3.11)–(3.13), by (3.10) we obtain

$$
\begin{aligned}
\left[\overline{\Lambda}(t_{j+\sigma})y^{(\sigma)}, \bar{y}\right] \leqslant & \varepsilon_1 |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2 + \varepsilon_2 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 + \varepsilon_3 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_N\right)^2 \\
& + M_7^{\varepsilon_1, \varepsilon_2, \varepsilon_3} \left(|[y^{(\sigma)}]|_0^2 + \|y_{\bar{x}}^{(\sigma)}\|_0^2\right) - M_3 \|y_{\bar{x}}^{(\sigma)}\|_0^2 - M_4 \Delta_{0t_{j+\sigma}}^{\alpha} \|y_{\bar{x}}\|_0^2,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\left[\bar{\delta}y, \bar{y}\right] = & \left(\delta y, \bar{y}\right) + 0.5h\delta^- y_0 \bar{y}_0 + 0.5h\delta^+ y_N \bar{y}_N \\
= & \left(\Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{\bar{x}})_x, \bar{y}\right) + \bar{y}_0 \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{x,0}) - \bar{y}_0 \beta_{12} \Delta_{0t_{j+\sigma}}^{\alpha} y_0 \\
& - \bar{y}_N \Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_N y_{\bar{x},N}) - \bar{y}_N \beta_{22} \Delta_{0t_{j+\sigma}}^{\alpha} y_N \\
= & -\left(\Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{\bar{x}}), \bar{y}_{\bar{x}}\right] - \beta_{12}\bar{y}_0 \Delta_{0t_{j+\sigma}}^{\alpha} y_0 - \bar{y}_N \beta_{22} \Delta_{0t_{j+\sigma}}^{\alpha} y_N \\
= & -\left(\Delta_{0t_{j+\sigma}}^{\alpha}(\gamma_i y_{\bar{x}}^{(\sigma)}), y_{\bar{x}}^{(\sigma)}\right] - \left(\gamma_i, (\Delta_{0t_{j+\sigma}}^{\alpha} y_{\bar{x}})^2\right] - \beta_{12} y_0^{(\sigma)} \Delta_{0t_{j+\sigma}}^{\alpha} y_0 \\
& - \beta_{12}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 - -\beta_{22} y_N^{(\sigma)} \Delta_{0t_{j+\sigma}}^{\alpha} y_N - \beta_{22}(\Delta_{0t_{j+\sigma}}^{\alpha} y_N)^2 \\
\leqslant & -\frac{c_0}{2} \Delta_{0t_{j+\sigma}}^{\alpha} \|y_{\bar{x}}\|_0^2 - c_0 \|\Delta_{0t_{j+\sigma}}^{\alpha} y_{\bar{x}}\|_0^2 - \frac{\beta_{12}}{2} \Delta_{0t_{j+\sigma}}^{\alpha} y_0^2 \\
& - \frac{\beta_{22}}{2} \Delta_{0t_{j+\sigma}}^{\alpha} y_N^2 - \beta_{12}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 - \beta_{22}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_N\right)^2,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
\left[\overline{\Phi}, \bar{y}\right] = & (\varphi, \bar{y}) + 0.5h\varphi^- \bar{y}_0 + 0.5h\varphi^+ \bar{y}_N \\
= & (\varphi, \bar{y}) + \bar{y}_0 \left(\mu_1 + 0.5h\varphi_0\right) + \bar{y}_N \left(\mu_2 + 0.5h\varphi_N\right) \\
= & (\varphi, \bar{y}) + \bar{y}_0 \mu_1 + 0.5h\varphi_0 \bar{y}_0 + \bar{y}_N \mu_2 + 0.5h\varphi_N \bar{y}_N \\
= & \left[\varphi, \bar{y}\right] + \mu_1 \bar{y}_0 + \mu_2 \bar{y}_N \\
= & \left[\varphi, y^{(\sigma)}\right] + \left[\varphi, \Delta_{0t_{j+\sigma}}^{\alpha} y\right] + \mu_1 y_0^{(\sigma)} + \mu_1 \Delta_{0t_{j+\sigma}}^{\alpha} y_0 + \mu_2 y_N^{(\sigma)} + \mu_2 \Delta_{0t_{j+\sigma}}^{\alpha} y_N \\
\leqslant & \varepsilon_1 |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2 + \varepsilon_2 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 + \varepsilon_3 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_N\right)^2 + M_9^{\varepsilon_2, \varepsilon_3} \left(\mu_1^2 + \mu_2^2\right) \\
& + M_{10}\left(|[y^{(\sigma)}]|_0^2 + \|y_{\bar{x}}^{(\sigma)}\|_0^2\right) + M_{11}^{\varepsilon_1} |[\varphi]|_0^2.
\end{aligned}
\tag{3.16}
$$

Due to (3.9)–(3.16), by (3.8) we find:

$$
\begin{aligned}
& \frac{1}{2}\Delta_{0t_{j+\sigma}}^{\alpha} |[y]|_0^2 + |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2 + M_{12}\Delta_{0t_{j+\sigma}}^{\alpha} \|y_{\bar{x}}\|_0^2 + c_0 \|\Delta_{0t_{j+\sigma}}^{\alpha} y_{\bar{x}}\|_0^2 + M_3 \|y_{\bar{x}}^{(\sigma)}\|_0^2 \\
& \qquad + \frac{\beta_{12}}{2}\Delta_{0t_{j+\sigma}}^{\alpha} y_0^2 + \frac{\beta_{22}}{2}\Delta_{0t_{j+\sigma}}^{\alpha} y_N^2 + \beta_{12}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 + \beta_{22}\left(\Delta_{0t_{j+\sigma}}^{\alpha} y_N\right)^2 \\
& \leqslant \varepsilon_1 |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2 + \varepsilon_2 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_0\right)^2 + \varepsilon_3 \left(\Delta_{0t_{j+\sigma}}^{\alpha} y_N\right)^2 \\
& \qquad + M_{13}^{\varepsilon_1, \varepsilon_2, \varepsilon_3} \left(|[y^{(\sigma)}]|_0^2 + \|y_{\bar{x}}^{(\sigma)}\|_0^2\right) + M_{14}^{\varepsilon_2, \varepsilon_3} \left(\mu_1^2 + \mu_2^2\right) + M_{11}^{\varepsilon_1} |[\varphi]|_0^2.
\end{aligned}
\tag{3.17}
$$

Choosing $\varepsilon_1 = \dfrac{1}{2}$, $\varepsilon_2 = \dfrac{\beta_{12}}{2}$, $\varepsilon_3 = \dfrac{\beta_{22}}{2}$, by (3.17) we get:

$$
\begin{aligned}
& \Delta_{0t_{j+\sigma}}^{\alpha} |[y]|_{W_2^1(0,l)}^2 + \|y_{\bar{x}}^{(\sigma)}\|_0^2 + |[\Delta_{0t_{j+\sigma}}^{\alpha} y]|_0^2 + \|\Delta_{0t_{j+\sigma}}^{\alpha} y_{\bar{x}}\|_0^2 \\
& \qquad \leqslant M_{15} |[y^{(\sigma)}]|_{W_2^1(0,l)}^2 + M_{16}\left(|[\varphi]|_0^2 + \mu_1^2 + \mu_2^2\right),
\end{aligned}
\tag{3.18}
$$

where $|[y]|^2_{W^1_2(0,l)} = |[y]|^2_0 + \|y_{\bar{x}}\|^2_0$. We rewrite (3.18) as

$$\Delta^\alpha_{0t_{j+\sigma}}|[y]|^2_{W^1_2(0,l)} \leqslant M^\sigma_{17}|[y^{j+1}]|^2_{W^1_2(0,l)} + M^\sigma_{18}|[y^j]|^2_{W^1_2(0,l)} + M_{16}\Big(|[\varphi]|^2_0 + \mu^2_1 + \mu^2_2\Big). \qquad (3.19)$$

The following lemma holds true.

Lemma 5. *Let* $\{p_j\}$ *be a sequence obeying the following conditions:*

$$p_0 = 1, \qquad \overline{\sigma}^{1-\alpha} p_j = \sum_{s=1}^{j}(c^{\alpha,\sigma}_{s-1} - c^{\alpha,\sigma}_s)p_{j-s}, \qquad j \geqslant 1,$$

then

$$0 < p_j < 1, \qquad \sum_{s=k}^{j} p_{j-s}c^{\alpha,\sigma}_{s-k} = \overline{\sigma}^{1-\alpha}, \qquad 1 \leqslant k \leqslant j, \qquad (3.20)$$

where

$$\overline{\sigma}^{1-\alpha} = \frac{1}{2-\alpha}\Big((1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}\Big) - \frac{1}{2}\Big((1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}\Big).$$

Proof. We follow [31] to prove identity (3.20). Since $c_s < c_{s-1}$ as $s \geqslant 1$, we obtain:

$$\sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_s < \sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_{s-1}, \qquad (3.21)$$

where

$$\sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_{s-1} = \sum_{s=0}^{j} p_{j-s}c^{\alpha,\sigma}_s. \qquad (3.22)$$

By (3.21), (3.22) we find

$$\sum_{s=j}^{j} p_{j-s}c^{\alpha,\sigma}_{s-j} = p_0 c_0 = \overline{\sigma}^{1-\alpha}, \qquad (3.23)$$

where

$$\overline{\sigma}^{1-\alpha} = \begin{cases} \sigma^{1-\alpha}, & j = 0, \\ \dfrac{1}{2-\alpha}\Big((1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}\Big) - \dfrac{1}{2}\Big((1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}\Big), & j \geqslant 1, \end{cases}$$

$$\sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_s < \sum_{s=0}^{j} p_{j-s}c^{\alpha,\sigma}_s = p_0 c_0 = \overline{\sigma}^{1-\alpha}. \qquad (3.24)$$

Thanks to (3.23), (3.24), we obtain

$$\sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_{s-1} = \overline{\sigma}^{1-\alpha}, \quad \sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_s < \overline{\sigma}^{1-\alpha}, \quad \sum_{s=1}^{j} p_{j-s}(c^{\alpha,\sigma}_{s-1} - c^{\alpha,\sigma}_s) < \overline{\sigma}^{1-\alpha},$$

$$\sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_s < \sum_{s=1}^{j} p_{j-s}c^{\alpha,\sigma}_s + p_j c_0, \quad p_j c_0 > 0, \quad c_0 = \overline{\sigma}^{1-\alpha}. \qquad (3.25)$$

By (3.22) we find

$$c_0 p_j = \sum_{s=1}^{j}(c^{\alpha,\sigma}_{s-1} - c^{\alpha,\sigma}_s)p_{j-s}.$$

It follows from (3.24), (3.25) that

$$0 < p_j c_0 < \overline{\sigma}^{1-\alpha}, \qquad 0 < p_j < 1.$$

Let $s = l + k - 1$, then by (3.23) we obtain:

$$\sum_{s=k}^{j} p_{j-s} c_{s-k}^{\alpha,\sigma} = \sum_{l=1}^{j-k+1} p_{j-k+1-l} c_{l-1}^{\alpha,\sigma} = \overline{\sigma}^{1-\alpha}, \qquad 1 \leqslant k \leqslant j.$$

The proof is complete. □

Lemma 6. *Assume that (3.20) holds, then*

$$\frac{\Gamma(2-\alpha)}{\Gamma(1+(m-1)\alpha)} \sum_{s=1}^{j} p_{j-s} s^{(m-1)\alpha} \leqslant \frac{\overline{\sigma}^{1-\alpha} j^{m\alpha}}{\Gamma(1+m\alpha)}, \qquad m \in \mathbb{N}. \qquad (3.26)$$

The proof of this lemma is similar to that of Lemma 3.2 in [31].

Lemma 7. *Let* $\overrightarrow{e} = (1, 1, \ldots, 1)^T \in R^j$,

$$J = 2\overline{\sigma}^{\alpha-1}\Gamma(2-\alpha)\lambda\tau^\alpha \begin{bmatrix} 0 & p_1 & \ldots & p_{j-2} & p_{j-1} \\ 0 & 0 & \ldots & p_{j-3} & p_{j-2} \\ & & \ldots & & \\ 0 & 0 & \ldots & 0 & p_1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}_{j \times j}$$

and (3.26) holds. Then

$$J^i = 0, \qquad i \geqslant j,$$

$$J^m \overrightarrow{e} \leqslant \frac{1}{\Gamma(1+m\alpha)} \left((2\lambda t_j^\alpha)^m, (2\lambda t_{j-1}^\alpha)^m, \ldots, (2\lambda t_1^\alpha)^m \right)^T, \quad m = 0, 1, 2, \ldots$$

$$\sum_{s=0}^{i} J^s \overrightarrow{e} = \sum_{s=0}^{j-1} J^s \overrightarrow{e} \leqslant \left(E_\alpha(2\lambda t_j^\alpha), E_\alpha(2\lambda t_{j-1}^\alpha), \ldots, E_\alpha(2\lambda t_1^\alpha) \right)^T, \quad i \geqslant j.$$

The proof of the above lemma is similar to that of Lemma 3.3 in [31].

Lemma 8. *Assume that non-negative sequences* y^j, φ^j, $j = 0, 1, 2, \ldots$, *satisfy the inequality*

$$\Delta_{0t_{j+\sigma}}^\alpha y^j \leqslant \lambda_1 y^{j+1} + \lambda_2 y^j + \varphi^j, \quad j \geqslant 1$$

where $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ *are constants. Then there exist* τ_0 *such that as* $\tau \leqslant \tau_0$, *we have*

$$y^{j+1} \leqslant 2\left(y^0 + \frac{t_j^\alpha}{\Gamma(1+\alpha)} \max_{0 \leqslant j' \leqslant j} \varphi^{j'} \right) E_\alpha(2\lambda t_j^\alpha), \quad 1 \leqslant j \leqslant j_0,$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$$

is a Mittag-Leffler function and $\lambda = \lambda_1 + \dfrac{\lambda_2}{2 + 2^{1-\alpha}}$.

This lemma can be proved by employing Lemmata 4-6 similar to Lemma 3.1 in [31].
By Lemma 8 and (3.18) we obtain

$$||[y^{j+1}]||_{W_2^1(0,l)}^2 \leqslant M\left(||[y^0]||_{W_2^1(0,l)}^2 + \frac{t_j^\alpha}{\Gamma(1+\alpha)} \max_{0 \leqslant j' \leqslant j} \left(||[\varphi^{j'}]||_0^2 + \mu_1^2 + \mu_2^2 \right) \right), \qquad (3.27)$$

where M is a positive constant independent of h and τ.

Theorem 2. *Assume that conditions (1.5) are satisfied, then there exists* τ_0 *such that as* $\tau \leqslant \tau_0$, *the solution of difference problem (3.2)–(3.5) obeys apriori estimate (3.27).*

Apriori estimate (3.27) implies the uniqueness of solution to problem (3.2)–(3.5) and its stability in initial data and the right hand side.

Let $u(x,t)$ be the solution of problem (1.1)–(1.4), $y(x_i, t_j) = y_i^j$ be the solution of difference problem (3.2)–(3.5). To estimate the exactness of difference scheme (3.2)–(3.5), we consider the difference $z_i^j = y_i^j - u_i^j$, where $u_i^j = u(x_i, t_j)$. Substituting $y = z + u$ into relations (3.2)–(3.5), we obtain a problem for the function z:

$$\Delta_{0t_{j+\sigma}}^\alpha z = \varkappa_i^j \left(a_i^j z_{\bar{x}}^{(\sigma)} \right)_x + \Delta_{0t_{j+\sigma}}^\alpha \left(\gamma_i z_{\bar{x}} \right)_x + b_i^{-j} a_i^j z_{\bar{x},i}^{(\sigma)} + b_i^{+j} a_{i+1}^j z_{x,i}^{(\sigma)} - \sum_{s=0}^i d_s^j z_s^{(\sigma)} \hbar + \Psi_i^j, \quad (3.28)$$

$$\varkappa_0 a_1 z_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^\alpha (\gamma_1 z_{x,0}) = \beta_{11} z_0^{(\sigma)} + 0.25 h^2 d_0^j z_0^{(\sigma)} + \tilde{\beta}_{12} \Delta_{0t_{j+\sigma}}^\alpha z_0 - \tilde{\nu}_1, \ t \in \overline{\omega}_\tau, \quad (3.29)$$

$$- \left(\varkappa_N a_N z_{\bar{x},N}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^\alpha (\gamma_N z_{\bar{x},N}) \right) = \beta_{21} z_N^{(\sigma)} + 0.5 h \sum_{s=0}^N d_s^j z_s^{(\sigma)} \hbar + \tilde{\beta}_{22} \Delta_{0t_{j+\sigma}}^\alpha z_N - \tilde{\nu}_2, \quad (3.30)$$

$$z(x,0) = 0, \qquad x \in \overline{\omega}_h, \quad (3.31)$$

where $\Psi = O(h^2 + \tau^2)$, $\tilde{\nu}_1 = O(h^2 + \tau^2)$, $\tilde{\nu}_2 = O(h^2 + \tau^2)$ are the errors of approximation of differential problem (1.1)–(1.4) by difference scheme (3.2)–(3.5) in the class of solutions $u = u(x,t)$ of problem (1.1)–(1.4).

Applying apriori estimate (3.27) to the solution of problem (3.28)–(3.31), we obtain the inequality

$$|[z^{j+1}]|_{W_2^1(0,l)}^2 \leqslant M \max_{0 \leqslant j' \leqslant j} \left(|[\Psi^{j'}]|_0^2 + \nu_1^{j'2} + \nu_2^{j'2} \right), \quad (3.32)$$

where M is a positive constant independent of h and τ.

Apriori estimate (3.32) implies the convergence of the solution of difference problem (3.2)–(3.5) to the solution of differential problem (1.1)–(1.4) in the sense of the norm $|[z^{j+1}]|_{W_2^1(0,l)}^2$ on each fiver so that there exists τ_0 such that as $\tau \leqslant \tau_0$, the estimate holds:

$$|[y^{j+1} - u^{j+1}]|_{W_2^1(0,l)}^2 \leqslant M \left(h^2 + \tau^2 \right).$$

Corollary 1. *The results obtained in the present work holds also for the case, when equation (1.1) reads as*

$$\partial_{0t}^\alpha u = \frac{\partial}{\partial x} \left(k(x,t) \frac{\partial u}{\partial x} \right) + \partial_{0t}^\alpha \frac{\partial}{\partial x} \left(\eta(x) \frac{\partial u}{\partial x} \right) + r(x,t) \frac{\partial u}{\partial x}$$

$$- \int_0^l q(x,t) u(x,t) dx + f(x,t), \qquad 0 < x < l, \qquad 0 < t \leqslant T,$$

if we assume that $|q| \leqslant c_2$.

4. Boundary value problem for a degenerating pseudo-parabolic equation with a nonlocal linear source

In a closed cylinder $\overline{Q}_T = \{(x,t) : 0 \leqslant x \leqslant l, \ 0 \leqslant t \leqslant T\}$ we consider the following nonlocal boundary value problem:

$$\partial_{0t}^\alpha u = \frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m k \frac{\partial u}{\partial x} \right) + \frac{1}{x^m} \partial_{0t}^\alpha \frac{\partial}{\partial x} \left(x^m \eta(x) \frac{\partial u}{\partial x} \right)$$
$$+ r \frac{\partial u}{\partial x} - \int_0^x q(s,t) u(s,t) ds + f(x,t), \qquad 0 < x < l, \qquad 0 < t \leqslant T, \quad (4.1)$$

$$\lim_{x \to 0} x^m \Pi(x,t) = 0, \qquad 0 \leqslant t \leqslant T, \quad (4.2)$$

$$-\Pi(l,t) = \beta_1(t) u(l,t) + \beta_2(t) \partial_{0t}^\alpha u(l,t) - \mu(t), \qquad 0 \leqslant t \leqslant T, \quad (4.3)$$

$$u(x,0) = u_0(x), \qquad 0 \leqslant x \leqslant l, \quad (4.4)$$

where $0 \leqslant m \leqslant 2$. At $x = 0$, we assume the boundedness of the solution $|u(0, t)| < \infty$ that is equivalent to condition (4.2), which is equivalent to the identity $\Pi(0, t) = 0$ [25] if the functions $r(0, t)$, $k(0, t)$, $q(0, t)$, $f(0, t)$ are finite.

5. Apriori estimate in differential form

We are going to obtain an apriori estimate by the method of energy inequalities. In order to do this, we calculate the scalar product of equation (4.1) with $x^m U = x^m(u + \partial_{0t}^\alpha u)$:

$$
\begin{aligned}
\left(\partial_{0t}^\alpha u, x^m U \right) = & \left(\left(x^m k u_x \right)_x, U \right) + \left(\partial_{0t}^\alpha \left(x^m \eta u_x \right)_x, U \right) \\
& + \left(r u_x, x^m U \right) - \left(\int_0^s q u \, ds, x^m U \right) + \left(f, x^m U \right).
\end{aligned}
\tag{5.1}
$$

Taking into consideration (2.2)-(2.7), after some simple transformations by (5.1) we find:

$$
\begin{aligned}
\frac{1}{2} \partial_{0t}^\alpha \| x^{\frac{m}{2}} u \|_0^2 + \frac{1}{2} \int_0^l \left(k + \eta(x) \right) \partial_{0t}^\alpha (x^{\frac{m}{2}} u_x)^2 dx + c_0 \| x^{\frac{m}{2}} u_x \|_0^2 + \frac{1}{2} \| \partial_{0t}^\alpha x^{\frac{m}{2}} u \|_0^2 \\
c_0 \| \partial_{0t}^\alpha x^{\frac{m}{2}} u_x \|_0^2 \leqslant x^m U \Pi(x, t) \big|_0^l + M_7 \left(\| x^{\frac{m}{2}} u \|_0^2 + \| x^{\frac{m}{2}} u_x \|_0^2 \right) + M_8 \| x^{\frac{m}{2}} f \|_0^2.
\end{aligned}
\tag{5.2}
$$

We estimate the first term in the right hand side in (5.2):

$$
\begin{aligned}
x^m U \Pi(x, t) \big|_0^l = & l^m \left(u(l, t) + \partial_{0t}^\alpha u(l, t) \right) \Pi(l, t) \\
= & l^m \left(u(l, t) + \partial_{0t}^\alpha u(l, t) \right) \left(\mu(t) - \beta_1(t) u(l, t) - \beta_2(t) \partial_{0t}^\alpha u(l, t) \right) \\
= & l^m u(l, t) \mu(t) + l^m \mu(t) \partial_{0t}^\alpha u(l, t) - l^m u^2(l, t) \beta_1(t) - l^m \beta_1(t) u(l, t) \partial_{0t}^\alpha u(l, t) \\
& - l^m \beta_2(t) u(l, t) \partial_{0t}^\alpha u(l, t) - l^m \beta_2(t) (\partial_{0t}^\alpha u(l, t))^2 \\
\leqslant & - l^m \beta_2(t) (\partial_{0t}^\alpha u(l, t))^2 - \frac{l^m \beta_2(t)}{2} \partial_{0t}^\alpha u^2(l, t) \\
& + \varepsilon \left(\partial_{0t}^\alpha u(l, t) \right)^2 + M_9 \left(\| x^{\frac{m}{2}} u \|_0^2 + \| x^{\frac{m}{2}} u_x \|_0^2 \right) + M_{10} \mu^2(t) \\
\leqslant & - \frac{l^m \beta_2(t)}{2} (\partial_{0t}^\alpha u(l, t))^2 - \frac{l^m \beta_2(t)}{2} \partial_{0t}^\alpha u^2(l, t) \\
& + M_9 \left(\| x^{\frac{m}{2}} u \|_0^2 + \| x^{\frac{m}{2}} u_x \|_0^2 \right) + M_{10} \mu^2(t).
\end{aligned}
\tag{5.3}
$$

In view of (5.3), by (5.2) we find:

$$
\begin{aligned}
\partial_{0t}^\alpha \| x^{\frac{m}{2}} u \|_0^2 + \int_0^l \left(k + \eta(x) \right) \partial_{0t}^\alpha (x^{\frac{m}{2}} u_x)^2 dx + \| x^{\frac{m}{2}} u_x \|_0^2 + \| \partial_{0t}^\alpha x^{\frac{m}{2}} u \|_0^2 + \| \partial_{0t}^\alpha x^{\frac{m}{2}} u_x \|_0^2 \\
\leqslant M_{11} \| x^{\frac{m}{2}} u \|_{W_2^1(0,l)}^2 + M_{12} \left(\| x^{\frac{m}{2}} f \|_0^2 + \mu_2(t) \right),
\end{aligned}
\tag{5.4}
$$

where $\| x^{\frac{m}{2}} u \|_{W_2^1(0,l)}^2 = \| x^{\frac{m}{2}} u \|_0^2 + \| x^{\frac{m}{2}} u_x \|_0^2$. Applying the operator of the fractional integration $D_{0t}^{-\alpha}$ to the both sides of inequality (5.4), we find:

$$
\begin{aligned}
\| x^{\frac{m}{2}} u \|_{W_2^1(0,l)}^2 + D_{0t}^{-\alpha} \left(\| x^{\frac{m}{2}} u_x \|_0^2 + \| \partial_{0t}^\alpha x^{\frac{m}{2}} u \|_0^2 + \| \partial_{0t}^\alpha x^{\frac{m}{2}} u_x \|_0^2 \right) \\
\leqslant M_{13} D_{0t}^{-\alpha} \| x^{\frac{m}{2}} u \|_{W_2^1(0,l)}^2 + M_{15} \left(D_{0t}^{-\alpha} \left(\| x^{\frac{m}{2}} f \|_0^2 + \mu_2^2(t) \right) + \| x^{\frac{m}{2}} u_0 \|_{W_2^1(0,l)}^2 \right).
\end{aligned}
\tag{5.5}
$$

By Lemma 2 and (5.5) we obtain the sought apriori estimate:

$$\|x^{\frac{m}{2}}u\|_{W_2^1(0,l)}^2 + D_{0t}^{-\alpha}\left(\|x^{\frac{m}{2}}u_x\|_0^2 + \|\partial_{0t}^\alpha x^{\frac{m}{2}}u\|_0^2 + \|\partial_{0t}^\alpha x^{\frac{m}{2}}u_x\|_0^2\right)$$

$$\leqslant M\left(D_{0t}^{-\alpha}(\|x^{\frac{m}{2}}f\|_0^2 + \mu_2^2(t)) + \|x^{\frac{m}{2}}u_0(x)\|_{W_2^1(0,l)}^2\right), \tag{5.6}$$

where M is a positive constant depending only on the data of problem (4.1)–(4.4), and

$$D_{0t}^{-\alpha}u = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u d\tau}{(t-\tau)^{1-\alpha}}$$

is the fractional Riemann-Liouville integral of order α, $0 < \alpha < 1$.

Theorem 3. *If $k(x,t) \in C^{1,0}(\overline{Q}_T)$, $\eta(x) \in C^1[0,l]$, $r(x,t), q(x,t), f(x,t) \in C(\overline{Q}_T)$, $u(x,t) \in C^{2,0}(Q_T) \cap C^{1,0}(\overline{Q}_T)$, $\partial_{0t}^\alpha u(x,t) \in C(\overline{Q}_T)$ and conditions (1.5) are satisfied, then the solution of problem (4.1)–(4.4) obeys apriori estimate (5.6).*

Apriori estimate (5.6) implies the uniqueness of the solution and the stability with respect to the initial data and the right hand side in the sense of the norm:

$$\|x^{\frac{m}{2}}u\|_1^2 = \|x^{\frac{m}{2}}u\|_{W_2^1(0,l)}^2 + D_{0t}^{-\alpha}\left(\|x^{\frac{m}{2}}u_x\|_0^2 + \|\partial_{0t}^\alpha x^{\frac{m}{2}}u\|_0^2 + \|\partial_{0t}^\alpha x^{\frac{m}{2}}u_x\|_0^2\right).$$

6. Stability and convergence of difference scheme

On the uniform grid $\overline{\omega}_{h\tau}$, we consider a difference scheme for differential problem (4.1)–(4.4) of approximation order $O(h^2 + \tau^2)$:

$$\bar{\varkappa}\Delta_{0t_{j+\sigma}}^\alpha y = \frac{\varkappa}{x_i^m}\left(x_{i-0.5}^m a^j y_{\bar{x}}^{(\sigma)}\right)_x + \frac{1}{x_i^m}\Delta_{0t_{j+\sigma}}^\alpha (x_{i-0.5}^m \gamma_i y_{\bar{x},i})_x + \frac{b^{-j}}{x_i^m}\left(x_{i-0.5}^m a_i^j y_{\bar{x},i}^{(\sigma)}\right)$$

$$+ \frac{b^{+j}}{x_i^m}\left(x_{i+0.5}^m a_{i+1}^j y_{x,i}^{(\sigma)}\right) - \sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar + \varphi_i^j, \qquad (x,t) \in \omega_{h,\tau}, \tag{6.1}$$

$$\varkappa_0 a_1 y_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^\alpha\left(\gamma_1 y_{\bar{x},0}\right) = \frac{0.5h}{m+1}\left(\Delta_{0t_{j+\sigma}}^\alpha y_0 + 0.5h d_0^j y_0^{(\sigma)}\right) - \tilde{\mu}_1, \tag{6.2}$$

$$-\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - \Delta_{0t_{j+\sigma}}^\alpha\left(\gamma_N y_{\bar{x},N}\right) = \tilde{\beta}_1 y_N^{(\sigma)} + 0.5h\sum_{s=0}^N d_s^j y_s^{(\sigma)}\hbar + \tilde{\beta}_2 \Delta_{0t_{j+\sigma}}^\alpha y_N - \tilde{\mu}_2, \tag{6.3}$$

$$y(x,0) = u_0(x), \qquad x \in \overline{\omega}_h, \tag{6.4}$$

where

$$\tilde{\beta}_1 = \tilde{\varkappa}\beta_1(t_{j+\sigma}), \qquad \tilde{\beta}_2 = \tilde{\varkappa}\beta_2 + 0.5h, \qquad \tilde{\mu}_1 = \frac{0.5h}{m+1}\varphi_0^j, \qquad \tilde{\mu}_2 = \tilde{\varkappa}\mu(t_{j+\sigma}) + 0.5h\varphi_N^j,$$

$$\varkappa_0 = \frac{1}{1 + \frac{0.5h|r_0|}{(m+1)k_{0.5}^{j+\sigma}}} \quad \text{if} \quad r_0^{j+\sigma} \leqslant 0, \qquad \varkappa_N = \frac{1}{1 + \frac{0.5h|r_N^{j+\sigma}|}{k_{N-0.5}}} \quad \text{if} \quad r_N^{j+\sigma} \geqslant 0,$$

$$r = r^+ + r^-, \qquad |r| = r^+ - r^-, \qquad r^+ = 0.5\left(r + |r|\right) \geqslant 0, \qquad r^- = 0.5\left(r - |r|\right) \leqslant 0,$$

$$a_i^j = k(x_{i-0.5}, t^{j+\sigma}), \qquad \gamma_i = \eta(x_{i-0.5}), \qquad b_i^{\pm j} = \frac{\bar{\varkappa}_i r_i^{\pm j+\sigma}}{k_i^{j+\sigma}}, \qquad \bar{\varkappa}_i = 1 + \frac{m(m-1)h^2}{24x_i^2},$$

$$d_i^j = \begin{cases} \bar{\varkappa}_i q^{j+\sigma}, & i = \overline{1, N-1}, \\ q^{j+\sigma}, & i = 0, N, \end{cases} \qquad \varphi_i^j = \begin{cases} \bar{\varkappa}_i f_i^{j+\sigma}, & i = 1, N-1, \\ f_i^{j+\sigma}, & i = 0, N, \end{cases}$$

$$\hbar = \begin{cases} 0.5h, & i = 0, \\ h, & i = \overline{1, N-1}, \end{cases} \qquad \tilde{\varkappa} = 1 + \frac{0.5hm}{l} = \frac{1}{1 - \frac{0.5hm}{l}},$$

$$\varkappa_i = \frac{1}{1 + R_i}, \qquad R_i = \frac{0.5h|r_i|\bar{\varkappa}_i}{k_{i-0.5}}.$$

We are going to obtain an apriori estimate by the method of energy inequalities. In order to do this, we rewrite (6.1)–(6.4) in the operator form:

$$\bar{\bar{\varkappa}}\Delta^\alpha_{t_{j+\sigma}}y = \bar{\Lambda}(t_{j+\sigma})y^{(\sigma)} + \bar{\delta}y + \bar{\Phi}, \qquad (6.5)$$

$$y(x, 0) = u_0(x), \qquad (6.6)$$

where

$$\bar{\bar{\varkappa}} = \begin{cases} \bar{\varkappa}_i, & x \in \omega_h, \\ 1, & x = 0, l, \end{cases} \qquad \bar{\varkappa}_i = 1 + \frac{m(m-1)h^2}{24x_i^2},$$

$$\bar{\delta}y = \begin{cases} \delta y_i = \dfrac{1}{x_i^m}\Delta^\alpha_{0t_{j+\sigma}}(x^m_{i-0.5}\gamma_i y_{\bar{x},i})_x, & (x, t) \in \omega_{h,\tau} \\[2mm] \delta^- y_0 = \dfrac{m+1}{0.5h}\Delta^\alpha_{0t_{j+\sigma}}(\gamma_1 y_{x,0}), & x = 0, \\[2mm] \delta^+ y_N = -\dfrac{2}{h}\Big(\Delta^\alpha_{0t_{j+\sigma}}(\gamma_N y_{\bar{x},N}) + \tilde{\varkappa}\beta_2 \Delta^\alpha_{0t_{j+\sigma}} y_N\Big), & x = l. \end{cases}$$

$$\bar{\Lambda}(t^{j+\sigma})y^\sigma = \begin{cases} \tilde{\Lambda}(t^{j+\sigma})y_i^{(\sigma)} = \dfrac{\varkappa_i}{x_i^m}\Big(x^m_{i-0.5}a_i^j y_{\bar{x},i}^{(\sigma)}\Big)_x + \dfrac{b^{-j}}{x_i^m}\Big(x^m_{i-0.5}a_i^j y_{\bar{x}}^{(\sigma)}\Big) \\[2mm] \qquad\qquad + \dfrac{b^{+j}}{x_i^m}\Big(x^m_{i+0.5}a_{i+1}^j y_x^{(\sigma)}\Big) - \displaystyle\sum_{s=0}^{i} d_s^j y_s^{(\sigma)}\hbar, \\[4mm] \Lambda^- y_0^{(\sigma)} = \dfrac{m+1}{0.5h}\Big(\varkappa_0 a_1 y_{x,0}^{(\sigma)} - 0.5h d_0^j y_0^{(\sigma)}\Big), & x = 0, \\[3mm] \Lambda^+ y_N^{(\sigma)} = -\dfrac{2}{h}\Big(\varkappa_N a_N y_{x,N}^{(\sigma)} + \tilde{\beta}_1 y_N^{(\sigma)} + 0.5h\displaystyle\sum_{s=0}^{N} d_s^j y_s^{(\sigma)}\hbar\Big), & x = l, \end{cases}$$

$$\bar{\Phi} = \begin{cases} \varphi = \varphi_i, & (x, t) \in \omega_{h\tau}, \\[2mm] \varphi^- = \dfrac{m+1}{0.5h}\tilde{\mu}_1, & x = 0, \\[2mm] \varphi^+ = \dfrac{1}{0.5h}\tilde{\mu}_2, & x = l. \end{cases}$$

We calculate the scalar product of (6.5) with $x^m\bar{y} = x^m y^{(\sigma)} + x^m\Delta^\alpha_{0t_{j+\sigma}}y$:

$$\Big(\bar{\bar{\varkappa}}\Delta^\alpha_{0t_{j+\sigma}}y, x^m\bar{y}\Big) = \Big(\bar{\Lambda}(t_{j+\sigma})y^{(\sigma)}, x^m\bar{y}\Big] + \Big(\bar{\delta}y, x^m\bar{y}\Big] + \Big(\bar{\Phi}, x^m\bar{y}\Big], \qquad (6.7)$$

where

$$(u, v] = \sum_{i=1}^{N} u_i v_i \hbar, \qquad \|u\|_0^2 = \sum_{i=1}^{N} u_i^2 \hbar, \qquad \hbar = \begin{cases} 0.5h, & i = 0, N \\ h, & i \neq 0, N. \end{cases}$$

We transform the sums in identity (6.7) employing the Cauchy inequality with ε:

$$\begin{aligned} \Big(\bar{\bar{\varkappa}}\Delta^\alpha_{0t_{j+\sigma}}y, x^m\bar{y}\Big) &= \Big(\bar{\bar{\varkappa}}\Delta^\alpha_{0t_{j+\sigma}}y, x^m y^{(\sigma)}\Big] + \Big(\bar{\bar{\varkappa}}\Delta^\alpha_{0t_{j+\sigma}}y, x^m\Delta^\alpha_{0t_{j+\sigma}}y\Big] \\ &\geqslant \Big(\frac{\bar{\bar{\varkappa}}}{2}, \Delta^\alpha_{0t_{j+\sigma}}(x^{\frac{m}{2}}y)^2\Big] + \Big(\bar{\bar{\varkappa}}, \Big(\Delta^\alpha_{0t_{j+\sigma}}(x^{\frac{m}{2}}y)\Big)^2\Big], \end{aligned} \qquad (6.8)$$

$$\left(\bar{\Lambda}(t^{j+\sigma})y^{(\sigma)}, x^m\bar{y}\right] = \left(\tilde{\Lambda}y^{(\sigma)}, x^m\bar{y}\right) + 0.5h\Lambda^+ y_N^{(\sigma)} x_N^m\bar{y}_N$$

$$= \left(\varkappa(x_{i-0.5}^m a_i y_{\bar{x}}^{(\sigma)})_x, \bar{y}\right) + \left(b^-(x_{i-0.5}^m a_i^j y_{\bar{x}}^{(\sigma)}), \bar{y}\right) + \left(b^+(x_{i+0.5}^m a_{i+1}^j y_x^{(\sigma)}), \bar{y}\right)$$

$$- \left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x_i^m\bar{y}\right) - x_N^m\bar{y}_N\left(\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} + \tilde{\beta}_1 y_N^{(\sigma)} + 0.5h\sum_{s=0}^N d_s^j y_s^{(\sigma)}\hbar\right)$$

$$= - \left(x_{i-0.5}^m a_i y_{\bar{x}}^{(\sigma)}, (\varkappa\bar{y})_{\bar{x}}\right] + \left(b^-(\bar{x}^m a_i y_{\bar{x}}^{(\sigma)}), \bar{y}\right) + \left(b^+ x_{i+0.5}^m a^{(+1)}, y_x^{(\sigma)}\bar{y}\right)$$

$$- \left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x^m\bar{y}\right) - \tilde{\beta}_1 x_N^m y_N^{(\sigma)}\bar{y}_N - x_N^m 0.5h\bar{y}_N\sum_{s=0}^N d_s^j y_s^{(\sigma)}\hbar$$

$$+ \left(\bar{x}_N^m - x_N^m\right)\bar{y}_N\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} - x_{0.5}^m\varkappa_0 a_1 y_{x,0}^{(\sigma)}\bar{y}_0. \tag{6.9}$$

Let us transform the terms in the right hand of the above identity:

$$-\left(x_{i-0.5}^m a_i y_{\bar{x}}^{(\sigma)}, (\varkappa\bar{y})_{\bar{x}}\right] = -\left(x_{i-0.5}^m a_i y_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}}\bar{y} + \varkappa^{(-1)}\bar{y}_{\bar{x}}\right]$$

$$= -\left(\bar{x}^m a y_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}} y^{(\sigma)}\right] - \left(\bar{x}^m a y_{\bar{x}}^{(\sigma)}, \varkappa_{\bar{x}}\Delta_{0t_{j+\sigma}}^\alpha y\right]$$

$$- \left(\bar{x}^m a y_{\bar{x}}^{(\sigma)}, \varkappa^{(-1)} y_{\bar{x}}^{(\sigma)}\right] - \left(\bar{x}^m a y_{\bar{x}}^{(\sigma)}, \varkappa^{(-1)}\Delta_{0t_{j+\sigma}}^\alpha y_{\bar{x}}\right]$$

$$\leqslant \varepsilon\|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}}y]\|_0^2 + M_1^\varepsilon\left(\|x^{\frac{m}{2}}y^{(\sigma)}]\|_0^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}]\|_0^2\right) \tag{6.10}$$

$$- \frac{1}{1+hM_2}\left(\bar{x}^m a\varkappa, (y_{\bar{x}}^{(\sigma)})^2\right] - \frac{1}{2(1+hM_2)}\left(\bar{x}^m a\varkappa, \Delta_{0t_{j+\sigma}}^\alpha y_{\bar{x}}^2\right]$$

$$\leqslant \varepsilon\|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}}y]\|_0^2 + M_1^\varepsilon\left(\|x^{\frac{m}{2}}y^{(\sigma)}]\|_0^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}]\|_0^2\right)$$

$$- M_3\|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}]\|_0^2 - M_4\Delta_{0t_{j+\sigma}}^\alpha\|\bar{x}^{\frac{m}{2}}y_{\bar{x}}]\|_0^2,$$

$$\left(b^-(\bar{x}^m a_i^j y_{\bar{x}}^{(\sigma)}), \bar{y}\right) + \left(b^+ x_{i+0.5}^m a^{(+1)}, y_x^{(\sigma)}\bar{y}\right)$$

$$= \left(b^-\bar{x}^m a y_{\bar{x}}^{(\sigma)}, y^{(\sigma)}\right) + \left(b^-\bar{x}^m a y_{\bar{x}}^{(\sigma)}, \Delta_{0t_{j+\sigma}}^\alpha y\right)$$

$$+ \left(b^+ x_{i+0.5}^m a^{(+1)} y_x^{(\sigma)}, y^{(\sigma)}\right) + \left(b^+ x_{i+0.5}^m a^{(+1)} y_x^{(\sigma)}, \Delta_{0t_{j+\sigma}}^\alpha y\right) \tag{6.11}$$

$$\leqslant \varepsilon\|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}}y]\|_0^2 + M_5^\varepsilon\left(\|x^{\frac{m}{2}}y^{(\sigma)}]\|_0^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}]\|_0^2\right),$$

$$-\left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x^m\bar{y}\right) - \tilde{\beta}_1 x_N^m y_N^{(\sigma)}\bar{y}_N - x_N^m 0.5h\bar{y}_N\sum_{s=0}^N d_s^j y_s^{(\sigma)}\hbar$$

$$= -\left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x^m\bar{y}\right] - \tilde{\beta}_1 x_N^m y_N^{(\sigma)}\bar{y}_N$$

$$\leqslant -\left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x^m y^{(\sigma)}\right] - \left(\sum_{s=0}^i d_s^j y_s^{(\sigma)}\hbar, x^m\Delta_{0t_{j+\sigma}}^\alpha y\right] \tag{6.12}$$

$$- \tilde{\beta}_1 x_N^m (y_N^{(\sigma)})^2 - \tilde{\beta}_1 x_N^m y_N^{(\sigma)}\Delta_{0t_{j+\sigma}}^\alpha y_N$$

$$\leqslant \varepsilon_1\|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}}y]\|_0^2 + \varepsilon_2\left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$+ M_6^{\varepsilon_1,\varepsilon_2}\left(\|x^{\frac{m}{2}}y^{(\sigma)}]\|_0^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}]\|_0^2\right).$$

Taking into consideration (6.9)-(6.12), we obtain:

$$
\begin{aligned}
\Big(\bar\Lambda(t^{j+\sigma})y^{(\sigma)}, x^m\bar y\Big] \leqslant & \varepsilon_1\|\Delta^\alpha_{0t_{j+\sigma}}x^{\frac{m}{2}}y]|^2_0 + \varepsilon_2\Big(\Delta^\alpha_{0t_{j+\sigma}}y_N\Big)^2 \\
& + M_7^{\varepsilon_1,\varepsilon_2}\Big(\|x^{\frac{m}{2}}y^{(\sigma)}]|^2_0 + \|\bar x^{\frac{m}{2}}y^{(\sigma)}_{\bar x}]|^2_0\Big) \\
& - M_3\|\bar x^{\frac{m}{2}}y^{(\sigma)}_{\bar x}]|^2_0 - M_4\Delta^\alpha_{0t_{j+\sigma}}\|\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0 \\
& + \Big(\bar x^m_N - x^m_N\Big)\bar y_N\varkappa_N a_N y^{(\sigma)}_{\bar x,N} - x^m_{0.5}\varkappa_0 a_1 y^{(\sigma)}_{x,0}\bar y_0,
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
\Big(\bar\delta y, x^m\bar y\Big] = & \Big(\delta y, \bar y x^m\Big) + 0.5hx^m_N\bar y_N\delta^+ y_N \\
= & \Big(\Delta^\alpha_{0t_{j+\sigma}}(\bar x^m\gamma y_{\bar x})_x, \bar y\Big) + 0.5hx^m_N\bar y_N\delta^+ y_N \\
= & -\Big(\Delta^\alpha_{0t_{j+\sigma}}\bar x^m\gamma y_{\bar x}, \bar y_{\bar x}\Big] + \bar x^m_N\gamma_N\bar y_N\Delta^\alpha_{0t_{j+\sigma}}y_{\bar x,N} \\
& - x^m_N\bar y_N\gamma_N\Delta^\alpha_{0t_{j+\sigma}}y_{\bar x,N} - \tilde\varkappa\beta_2 x^m_N\bar y_N\Delta^\alpha_{0t_{j+\sigma}}y_N - x^m_{0.5}\bar y_0\Delta^\alpha_{0t_{j+\sigma}}(\gamma_1 y_{x,0}).
\end{aligned}
\tag{6.14}
$$

We transform the terms in the right hand side in (6.14):

$$
\begin{aligned}
-\Big(\Delta^\alpha_{0t_{j+\sigma}}\bar x^m\gamma y_{\bar x}, \bar y_{\bar x}\Big] = & -\Big(\Delta^\alpha_{0t_{j+\sigma}}\bar x^m\gamma y_{\bar x}, y^{(\sigma)}_{\bar x}\Big] - \Big(\Delta^\alpha_{0t_{j+\sigma}}\bar x^m\gamma y_{\bar x}, \Delta^\alpha_{0t_{j+\sigma}}y_{\bar x}\Big] \\
& \leqslant -\frac{c_0}{2}\Delta^\alpha_{0t_{j+\sigma}}\|\bar x^{\frac{m}{2}}y_x]|^2_0 - c_0\|\Delta^\alpha_{0t_{j+\sigma}}\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0,
\end{aligned}
\tag{6.15}
$$

$$
-\tilde\varkappa\beta_2 x^m_N\bar y_N\Delta^\alpha_{0t_{j+\sigma}}y_N = -\tilde\varkappa\beta_2 x^m_N y^{(\sigma)}_N\Delta^\alpha_{0t_{j+\sigma}}y_N - \tilde\varkappa\beta_2 x^m_N\Big(\Delta^\alpha_{0t_{j+\sigma}}y_N\Big)^2.
\tag{6.16}
$$

By (6.15) and (6.16) we obtain:

$$
\begin{aligned}
\Big(\bar\delta y, x^m\bar y\Big] \leqslant & -\frac{c_0}{2}\Delta^\alpha_{0t_{j+\sigma}}\|\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0 - c_0\|\Delta^\alpha_{0t_{j+\sigma}}\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0 + \Big(\bar x^m_N - x^m_N\Big)\bar y_N\Delta^\alpha_{0t_{j+\sigma}}(\gamma_N y_{\bar x,N}) \\
& - x^m_{0.5}\bar y_0\Delta^\alpha_{0t_{j+\sigma}}(\gamma_1 y_{x,0}) - \tilde\varkappa\frac{\beta_2}{2}x^m_N\Delta^\alpha_{0t_{j+\sigma}}(y_N)^2 - \tilde\varkappa x^m_N\beta_2\Big(\Delta^\alpha_{0t_{j+\sigma}}y_N\Big)^2,
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
\Big(\bar\Phi, x^m\bar y\Big] = & \Big(\varphi, x^m\bar y\Big) + 0.5hx^m_N\bar y_N\varphi^+ = \Big(\varphi, x^m\bar y\Big) + x^m_N\tilde\mu^2\bar y_N \\
= & \Big(\varphi, x^m y^{(\sigma)}\Big) + \Big(\varphi, x^m\Delta^\alpha_{0t_{j+\sigma}}y\Big) + x^m_N\tilde\mu_2\bar y_N \\
\leqslant & \varepsilon_1\|\Delta^\alpha_{0t_{j+\sigma}}x^{\frac{m}{2}}y]|^2_0 + M_8^{\varepsilon_1}\Big(\|x^{\frac{m}{2}}y^{(\sigma)}]|^2_0 \\
& + \|\bar x^{\frac{m}{2}}y^{(\sigma)}_{\bar x}]|^2_0\Big) + M_9^{\varepsilon_1}\|x^{\frac{m}{2}}\varphi\|^2_0 + x^m_N\tilde\mu_2\bar y_N.
\end{aligned}
\tag{6.18}
$$

Bearing in mind (6.8)–(6.18), by (6.7) we obtain:

$$
\begin{aligned}
\Big(\frac{\bar{\bar\varkappa}}{2}, \Delta^\alpha_{0t_{j+\sigma}}(x^{\frac{m}{2}}y)^2\Big] & + M_{10}\Delta^\alpha_{0t_{j+\sigma}}\|\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0 + M_3\|\bar x^{\frac{m}{2}}y^{(\sigma)}_{\bar x}]|^2_0 + \Big(\bar{\bar\varkappa}, \Big(\Delta^\alpha_{0t_{j+\sigma}}(x^{\frac{m}{2}}y)\Big)^2\Big] \\
& + c_0\|\Delta^\alpha_{0t_{j+\sigma}}\bar x^{\frac{m}{2}}y_{\bar x}]|^2_0 + \frac{\beta_2}{2}x^m_N\Delta^\alpha_{0t_{j+\sigma}}(y_N)^2 + \beta_2 x^m_N\Big(\Delta^\alpha_{0t_{j+\sigma}}y_N\Big)^2 \\
& \leqslant \varepsilon_1\|\Delta^\alpha_{0t_{j+\sigma}}x^{\frac{m}{2}}y]|^2_0 + \varepsilon_2\Big(\Delta^\alpha_{0t_{j+\sigma}}y_N\Big)^2 \\
& + \Big(\bar x^m_N - x^m_N\Big)\Big(x_N a_N y^{(\sigma)}_{\bar x,N} + \Delta^\alpha_{0t_{j+\sigma}}(\gamma_N y_{\bar x,N})\Big)\bar y_N \\
& - x^m_{0.5}\bar y_0\Big(\varkappa_0 a_1 y^{(\sigma)}_{x,0} + \Delta^\alpha_{0t_{j+\sigma}}(\gamma_1 y_{x,0})\Big) + M_8(\varepsilon_1)\|x^{\frac{m}{2}}\varphi\|^2_0 + x^m_N\tilde\mu_2\bar y_N \\
& + M_{11}(\varepsilon_1, \varepsilon_2)\Big(\|x^{\frac{m}{2}}y^{(\sigma)}]|^2_0 + \|\bar x^{\frac{m}{2}}y^{(\sigma)}_{\bar x}]|^2_0\Big).
\end{aligned}
\tag{6.19}
$$

The third, fourth and sixth term in the right hand in the above identity we get:

$$
\left(\bar{x}_N^m - x_N^m\right)\left(\varkappa_N a_N y_{\bar{x},N}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^\alpha(\gamma_N y_{\bar{x},N})\right)\bar{y}_N - x_{0.5}^m\bar{y}_0\left(\varkappa_0 a_1 y_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^\alpha(\gamma_1 y_{x,0})\right)
$$

$$
+ x_N^m\tilde{\mu}_2\bar{y}_N = x_{0.5}^m\bar{y}_0\left(\tilde{\mu}_1 - \frac{0.5h}{m+1}\left(\Delta_{0t_{j+\sigma}}^\alpha y_0 + 0.5hd_0^j y_0^{(\sigma)}\right)\right)
$$

$$
+ \left(\bar{x}_N^m - x_N^m\right)\bar{y}_N\left(\tilde{\mu}_2 - \tilde{\beta}_1 y_N^{(\sigma)} - \tilde{\beta}_2\Delta_{0t_{j+\sigma}}^\alpha y_N\right) + x_N^m\tilde{\mu}_2\bar{y}_N
$$

$$
= x_{0.5}^m y_0^{(\sigma)}\tilde{\mu}_1 + x_{0.5}^m\tilde{\mu}_1\Delta_{0t_{j+\sigma}}^\alpha y_0 - \frac{0.5h}{m+1}x_{0.5}^m y_0^{(\sigma)}\Delta_{0t_{j+\sigma}}^\alpha y_0
$$

$$
- \frac{0.5h}{m+1}x_{0.5}^m\left(\Delta_{0t_{j+\sigma}}^\alpha y_0\right)^2 - \frac{0.25h^2}{m+1}x_{0.5}^m y_0^{(\sigma)}d_0^j y_0^{(\sigma)}
$$

$$
- \frac{0.25h^2}{m+1}x_{0.5}^m d_0^j y_0^{(\sigma)}\Delta_{0t_{j+\sigma}}^\alpha y_0 + \bar{x}_N^m y_N^{(\sigma)}\tilde{\mu}_2 + \bar{x}_N^m\tilde{\mu}_2\Delta_{0t_{j+\sigma}}^\alpha y_N
$$

$$
\tag{6.20}
$$

$$
- \left(\bar{x}_N^m - x_N^m\right)\tilde{\beta}_1(y_N^\sigma)^2 - \left(\bar{x}_N^m - x_N^m\right)\tilde{\beta}_1 y_N^{(\sigma)}\Delta_{0t_{j+\sigma}}^\alpha y_N
$$

$$
- \left(\bar{x}_N^m - x_N^m\right)y_N^{(\sigma)}\tilde{\beta}_2\Delta_{0t_{j+\sigma}}^\alpha y_N - \left(\bar{x}_N^m - x_N^m\right)\tilde{\beta}_2\left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2
$$

$$
\leqslant \varepsilon_3\left(\Delta_{0t_{j+\sigma}}^\alpha y_0\right)^2 + \varepsilon_4\left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2 + M_{12}^{\varepsilon_3,\varepsilon_4}\left(\tilde{\mu}_1^2 + \tilde{\mu}_2^2\right)
$$

$$
+ M_{13}^{\varepsilon_3,\varepsilon_4}\left(\|x^{\frac{m}{2}}y^{(\sigma)}\|_0^2 + (x_{0.5}^{\frac{m}{2}}y_0)^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}\|_0^2\right)
$$

$$
- \frac{h}{4(m+1)}x_{0.5}^m\Delta_{0t_{j+\sigma}}^\alpha y_0^2 - \left(\bar{x}_N^m - x_N^m\right)\frac{\tilde{\beta}_2}{2}\Delta_{0t_{j+\sigma}}^\alpha(y_N)^2
$$

$$
- \frac{0.5h}{m+1}x_{0.5}^m\left(\Delta_{0t_{j+\sigma}}^\alpha y_0\right)^2 - \left(\bar{x}_N^m - x_N^m\right)\tilde{\beta}_2\left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2.
$$

In view of above identities, for

$$
\varepsilon_1 = \frac{\bar{\bar{\varkappa}}}{2}, \qquad \varepsilon_2 = \frac{\beta_2 x_N^m}{2}, \qquad \varepsilon_3 = \frac{hx_{0.5}^m}{4(m+1)}, \qquad \varepsilon_4 = (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}
$$

by (6.19) we find:

$$
\left(\frac{\bar{\bar{\varkappa}}}{2}, \Delta_{0t_{j+\sigma}}^\alpha(x^{\frac{m}{2}}y)^2\right] + M_{10}\Delta_{0t_{j+\sigma}}^\alpha\|\bar{x}^{\frac{m}{2}}y_{\bar{x}}\|_0^2 + M_3\|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}\|_0^2 + \left(\frac{\bar{\bar{\varkappa}}}{2}, \left(\Delta_{0t_{j+\sigma}}^\alpha(x^{\frac{m}{2}}y)\right)^2\right]
$$

$$
+ c_0\|\Delta_{0t_{j+\sigma}}^\alpha\bar{x}^{\frac{m}{2}}y_{\bar{x}}\|_0^2 + \frac{h}{4(m+1)}x_{0.5}^m\Delta_{0t_{j+\sigma}}^\alpha y_0^2
$$

$$
+ \left(\frac{\tilde{\varkappa}\beta_2}{2}x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right)\Delta_{0t_{j+\sigma}}^\alpha(y_N)^2
$$

$$
+ \frac{0.5h}{2(m+1)}x_{0.5}^m\left(\Delta_{0t_{j+\sigma}}^\alpha y_0\right)^2
$$

$$
\tag{6.21}
$$

$$
+ \left(\frac{\tilde{\varkappa}\beta_2}{2}x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right)\left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2
$$

$$
\leqslant M_{14}\|x^{\frac{m}{2}}\varphi\|_0^2 + M_{15}\left(\tilde{\mu}_1^2 + \tilde{\mu}_2^2\right)
$$

$$
+ M_{16}\left(\|x^{\frac{m}{2}}y^\sigma\|_0^2 + \|\bar{x}^{\frac{m}{2}}y_{\bar{x}}^{(\sigma)}\|_0^2 + \left(x_{0.5}^{\frac{m}{2}}y_0\right)^2\right).
$$

Now we transform the first, fourth, seventh and ninth terms in the left hand side in (6.21) bearing in mind the inequality $x_{N-0.5}^m \geqslant \frac{1}{6} x_N^m$:

$$\left(\frac{\bar{\bar{\varkappa}}}{2}, \Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)^2\right] + \left(\frac{\tilde{\varkappa}\beta_2}{2} x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right) \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2 + \left(\frac{\bar{\bar{\varkappa}}}{2}, \left(\Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)\right)^2\right]$$

$$+ \left(\frac{\tilde{\varkappa}\beta_2}{2} x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right) \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$= \left(\frac{\bar{\varkappa}}{2}, \Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)^2\right) + \frac{0.5h}{2} x_N^m \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2 + \left(\frac{\bar{\varkappa}}{2}, \left(\Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)\right)^2\right)$$

$$+ \frac{0.5h}{2} x_N^m \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2 + \left(\frac{\tilde{\varkappa}\beta_2}{2} x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right) \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2$$

$$+ \left(\frac{\tilde{\varkappa}\beta_2}{2} x_N^m + (\bar{x}_N^m - x_N^m)\frac{\tilde{\beta}_2}{2}\right) \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$= \left(\frac{\bar{\varkappa}}{2}, \Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)^2\right) + \left(\frac{\bar{\varkappa}}{2}, \left(\Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)\right)^2\right) \tag{6.22}$$

$$+ \left(\frac{\tilde{\varkappa}\beta_2}{2} \bar{x}_N^m + \frac{0.5h}{2} \bar{x}_N^m\right) \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2$$

$$+ \left(\frac{\tilde{\varkappa}\beta_2}{2} \bar{x}_N^m + \frac{0.5h}{2} \bar{x}_N^m\right) \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$\geqslant \frac{M_{17}}{2} \left(1, \Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)^2\right) + \frac{h}{4} \bar{x}_N^m \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2$$

$$+ \frac{M_{17}}{2} \left(1, \left(\Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)\right)^2\right) + \frac{h}{4} \bar{x}_N^m \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$\geqslant \frac{1}{4} \left(1, \Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)^2\right) + \frac{0.5h}{12} x_N^m \Delta_{0t_{j+\sigma}}^\alpha (y_N)^2$$

$$+ \frac{1}{4} \left(1, \left(\Delta_{0t_{j+\sigma}}^\alpha (x^{\frac{m}{2}} y)\right)^2\right) + \frac{0.5h}{12} x_N^m \left(\Delta_{0t_{j+\sigma}}^\alpha y_N\right)^2$$

$$\geqslant \frac{1}{12} \Delta_{0t_{j+\sigma}}^\alpha \|x^{\frac{m}{2}} y]\|_0^2 + \frac{1}{12} \|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}} y]\|_0^2,$$

where

$$M_{17} = \begin{cases} 1 & \text{as} \quad m = 0 \quad \text{or} \quad m \geqslant 1, \\ \frac{1}{2} & \text{as} \quad m \in (0,1), \quad h \leqslant h_0 = \sqrt{\dfrac{12x^2}{m(1-m)}}. \end{cases}$$

By (6.22) and (6.21) we obtain:

$$\Delta_{0t_{j+\sigma}}^\alpha \|x^{\frac{m}{2}} y]\|_1^2 + \|\bar{x}^{\frac{m}{2}} y_{\bar{x}}^{(\sigma)}]\|_0^2 + \|\Delta_{0t_{j+\sigma}}^\alpha x^{\frac{m}{2}} y]\|_0^2 + \|\Delta_{0t_{j+\sigma}}^\alpha \bar{x}^{\frac{m}{2}} y_{\bar{x}}]\|_0^2$$
$$\leqslant M_{18} \|x^{\frac{m}{2}} y^{(\sigma)}]\|_1^2 + M_{19}\left(\|x^{\frac{m}{2}} \varphi\|_0^2 + \tilde{\mu}_1^2 + \tilde{\mu}_2^2\right), \tag{6.23}$$

where

$$\|x^{\frac{m}{2}} y]\|_1^2 = \|x^{\frac{m}{2}} y\|_0^2 + \|\bar{x}^{\frac{m}{2}} y_{\bar{x}}]\|_0^2 + \left(x_{0.5}^{\frac{m}{2}} y_0\right)^2.$$

Reproducing the arguing in (3.18)-(3.27), by (6.23) we find the sought apriori estimate:

$$\|x^{\frac{m}{2}} y^{j+1}]\|_1^2 \leqslant M\left(\|x^{\frac{m}{2}} y^0]\|_1^2 + \frac{t_j^\alpha}{\Gamma(1+\alpha)} \max_{0 \leqslant j' \leqslant j} \left(\|x^{\frac{m}{2}} \varphi\|_0^2 + \tilde{\mu}_1^2 + \tilde{\mu}_2^2\right)\right), \tag{6.24}$$

where M is a positive constant independent of h and τ.

Theorem 4. *Assume that conditions (1.4), (4.5) hold. Then there exits τ_0, h_0 such that as $\tau \leqslant \tau_0$, $h \leqslant h_0$, the solution of difference problem (6.1)–(6.4) satisfies apriori estimate (6.24).*

Apriori estimate (6.24) imply the uniqueness of the solution of problem (6.1)-(6.4) and its stability with respect to the initial data and the right hand side.

Let $u(x,t)$ be the solution of problem (4.1)–(4.4), $y(x_i, t_j) = y_i^j$ be the solution of difference problem (6.1)–(6.4). To estimate the exactness of difference scheme (6.1)–(6.4), we consider the difference $z_i^j = y_i^j - u_i^j$, where $u_i^j = u(x_i, t_j)$. Then substituting $y = z + u$ into (6.1)–(6.4), we obtain the problem for the function z:

$$
\bar{\varkappa}\Delta_{0t_{j+\sigma}}^{\alpha} z = \frac{\varkappa}{x_i^m}\left(x_{i-0.5}^m a^j z_{\bar{x}}^{(\sigma)}\right)_x + \frac{1}{x_i^m}\Delta_{0t_{j+\sigma}}^{\alpha}(x_{i-0.5}^m \gamma_i z_{\bar{x},i})_x + \frac{b^{-j}}{x_i^m}\left(x_{i-0.5}^m a_i^j z_{\bar{x},i}^{(\sigma)}\right)
$$

$$
+ \frac{b^{+j}}{x_i^m}\left(x_{i+0.5}^m a_{i+1}^j z_{x,i}^{(\sigma)}\right) - \sum_{s=0}^{i} d_s^j z_s^{(\sigma)}\hbar + \Psi_i^j, \quad (x,t) \in \omega_{h,\tau}, \tag{6.25}
$$

$$
\varkappa_0 a_1 z_{x,0}^{(\sigma)} + \Delta_{0t_{j+\sigma}}^{\alpha}\left(\gamma_1 z_{\bar{x},0}\right) = \frac{0.5h}{m+1}\left(\Delta_{0t_{j+\sigma}}^{\alpha} z_0 + 0.5hd_0^j z_0^{(\sigma)}\right) - \tilde{\nu}_1, \tag{6.26}
$$

$$
-\varkappa_N a_N z_{\bar{x},N}^{(\sigma)} - \Delta_{0t_{j+\sigma}}^{\alpha}\left(\gamma_N z_{\bar{x},N}\right) = \tilde{\beta}_1 z_N^{(\sigma)} + 0.5h\sum_{s=0}^{N} d_s^j z_s^{(\sigma)}\hbar + \tilde{\beta}_2 \Delta_{0t_{j+\sigma}}^{\alpha} z_N - \tilde{\nu}_2, \tag{6.27}
$$

$$
z(x,0) = 0, \qquad x \in \overline{\omega}_h, \tag{6.28}
$$

where $\|x\Psi\|_0^2 = O\left(h^2 + \tau^2\right)$, $\tilde{\nu}_1 = O\left(h^2 + \tau^2\right)$, $\tilde{\nu}_2 = O\left(h^2 + \tau^2\right)$ are the errors of approximation differential problem (4.1)–(4.4) by difference scheme (6.1)–(6.4) in the class of solutions $u = u(x,t)$ of problem (4.1)–(4.4), see [25].

Applying apriori estimate (6.24) for solution of problem (6.25)–(6.28), we obtain the inequality

$$
\|x^{\frac{m}{2}} z^{j+1}\|_1^2 \leqslant M \max_{0\leqslant j'\leqslant j}\left(\|x^{\frac{m}{2}}\Psi^{j'}\|_0^2 + \nu_1^2 + \nu_2^2\right), \tag{6.29}
$$

where M is a positive constant independent of h and τ. This implies the apriori estimate

$$
\|xz^{j+1}\|_1^2 \leqslant \overline{M} \max_{0\leqslant j'\leqslant j}\left(\|x\Psi^{j'}\|_0^2 + \nu_1^2 + \nu_2^2\right), \tag{6.30}
$$

where $\overline{M}$ is a positive constant independent of h and τ. Apriori estimate (6.30) implies the convergence of the solution of difference problem (5.1)–(5.4) to the solution of differential problem (1.1), (1.2), (4.1), (1.4) in the sense of the norm $\|xz^{j+1}\|_1^2$ on each fiber so that there exist τ_0, h_0 such that as $\tau \leqslant \tau_0$, $h \leqslant h_0$, the apriori estimate holds:

$$
\|x\left(y^{j+1} - u^{j+1}\right)\|_1 \leqslant \overline{M}\left(h^2 + \tau^2\right).
$$

Corollary 2. *The obtained results hold true also in the case, when equation (4.1) is of the form:*

$$
\partial_{0t}^{\alpha} u = \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m k(x,t)\frac{\partial u}{\partial x}\right) + \frac{1}{x^m}\partial_{0t}^{\alpha}\frac{\partial}{\partial x}\left(x^m \eta(x)\frac{\partial u}{\partial x}\right)
$$

$$
+ r(x,t)\frac{\partial u}{\partial x} - \int_0^l q(x,t)u(x,t)dx + f(x,t), \qquad 0 < x < l, \qquad 0 < t \leqslant T,
$$

if we assume the inequality $|q| \leqslant c_2$.

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