ISSN 2304-0122 Ufa Mathematical Journal. Vol. 11. No 1 (2019). P. 19-26.

doi:10.13108/2019-11-1-19 UDC 517.5

ON AN INTERPOLATION PROBLEM IN THE CLASS OF FUNCTIONS OF EXPONENTIAL TYPE IN A HALF-PLANE

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Abstract. Solvability conditions for interpolation problem $f(n) = d_n, \quad n \in \mathbb{N}$ in the class of entire functions satisfying the condition $|f(z)| \leqslant e^{\pi|\operatorname{Im} z|+o(|z|)}, z \to \infty$ are well known. In the presented paper we study the interpolation problem $f(\lambda_n) = d_n$ in the class of exponential type functions in the half-plane. We find sufficient solvability conditions for the considerate problem. In particular, a sufficient part of Carleson's interpolation theorem is generalized and an analogue of a classic interpolation condition is found in the form

$$\sum_{j=k}^{\infty} \operatorname{Re}\left(-\xi_j \frac{\lambda_k^2 - 1}{\lambda_k + \overline{\lambda_j}}\right) \leqslant c_3, \qquad \xi_j := \frac{\operatorname{Re}\lambda_j}{1 + |\lambda_j|^2}.$$

The necessity of sufficient conditions is also discussed. The results are applied to studying a problem on splitting and searching an analogue of the identity $2\cos z = \exp(-iz) + \exp(iz)$ for each function of exponential type in the half-plane. We prove that each holomorphic in the right-hand half-plane function f obeying the , estimate $|f(z)| \leqslant O(\exp(\sigma|\operatorname{Im} z|))$ can be represented in the form $f = f_1 + f_2$ and the functions f_1 and f_2 holomorphic in the right-hand half-plane satisfy conditions

$$|f_1(z)| \leqslant O(\exp(|z|h_-(\varphi))) \quad \text{and}\, |f_2(z)| \leqslant O(\exp(|z|h_+(\varphi))),$$

where $\sigma \in [0; +\infty)$, $z = re^{i\varphi}$,

$$h_+(\varphi) = \begin{cases} \sigma|\sin\varphi|, & \varphi \in \left[0; \frac{\pi}{2}\right], \\ 0, & \varphi \in \left[-\frac{\pi}{2}; 0\right], \end{cases} \qquad h_-(\varphi) = \begin{cases} 0, & \varphi \in \left[0; \frac{\pi}{2}\right], \\ \sigma|\sin\varphi|, & \varphi \in \left[-\frac{\pi}{2}; 0\right]. \end{cases}$$

The paper uses methods works by L. Carleson, P. Jones, K. Kazaryan, K. Malyutin and other mathematicians.

Keywords: holomorphic functions of exponential type in the half-plane, interpolation, splitting of holomorphic functions

Mathematics Subject Classifcation: 30E05, 30D15

1. Introduction

It is known that for each sequence $d = (d_n) \in l^\infty$ there exists an entire function f such that [1]

$$f(n) = d_n, \qquad n \in \mathbb{N}, \tag{1.1}$$

$$|f(z)| \leqslant e^{\pi|\operatorname{Im} z|+o(|z|)}, \qquad z \to \infty. \tag{1.2}$$

In (1.2), "$o(|z|)$" can not be omitted [1], [2]. Our aim is to prove the following statement.

B.V. Vynnyts'kyi, V.L. Sharan, I.B. Sheparovych, On an interpolation problem in the class of functions of exponential type in a half-plane.

Submitted June 1, 2017.

Theorem 1. *For each sequence $(d_n) \in l^\infty$ there exists a holomorphic in the half-plane $\mathbb{C}_+ = \{z : \operatorname{Re} z > 0\}$ function f such that (1.1) holds and*

$$|f(z)| \leqslant c_1 e^{\pi|\operatorname{Im} z|}, \quad z \in \mathbb{C}_+. \tag{1.3}$$

Hereinafter c_j stand for positive constants.

We let $h \in C[-\pi/2; \pi/2]$, $\sigma \in [0; +\infty)$, $h_0(\varphi) = \sigma|\sin\varphi|$,

$$h_+(\varphi) = \begin{cases} \sigma|\sin\varphi|, & \varphi \in \left[0; \frac{\pi}{2}\right], \\ 0, & \varphi \in \left[-\frac{\pi}{2}; 0\right], \end{cases} \qquad h_-(\varphi) = \begin{cases} 0, & \varphi \in \left[0; \frac{\pi}{2}\right], \\ \sigma|\sin\varphi|, & \varphi \in \left[-\frac{\pi}{2}; 0\right]. \end{cases}$$

and let $H^\infty(\mathbb{C}_+; h)$ be the space of functions f holomorphic in $\mathbb{C}_+$ obeying

$$\|f\| := \sup\left\{|f(z)|e^{-rh(\varphi)} : z = x + iy = re^{i\varphi} \in \mathbb{C}_+\right\} < +\infty.$$

We employ Theorem 1 and its modifications for proving the following statement.

Theorem 2. *Let $\sigma \in [0; +\infty)$. Then each function $f \in H^\infty(\mathbb{C}_+; h_0)$ is represented as*

$$f = f_1 + f_2, \qquad f_1 \in H^\infty(\mathbb{C}_+; h_-), \qquad f_2 \in H^\infty(\mathbb{C}_+; h_+). \tag{1.4}$$

The problem on splitting (1.4), which is an analogue of the identity $\cos\sigma z = \frac{1}{2}e^{i\sigma z} + \frac{1}{2}e^{-i\sigma z}$, arises in seeking analogue of Paley-Wiener theorem for some weighted spaces and studying some convolution type equations (see [3, 4]). It was studied in works by V.M. Dilnyi [5, 6]. However, positive resolving is known mostly for spaces defined by L_2-metric. For the space $H^\infty(\mathbb{C}_+; h_0)$, the issue remained open. Theorem 2 positively resolves this. A more complicated and important similar question for the space of exponential type in the half-plane defined by L_1-metric remains open.

Let $\lambda = (\lambda_n) = (|\lambda_n|e^{i\varphi_n})$ be an arbitrary sequence of different complex numbers in the complex half-plane $\mathbb{C}_+$, $l^\infty(h; \lambda)$ be the space of sequences d, for which

$$\|d\| := \sup\left\{|d_n|e^{-|\lambda_n|h(\varphi_n)} : n \in \mathbb{N}\right\} < +\infty.$$

Let

$$S(r) := \sum_{1<|\lambda_k|\leqslant r} \left(\frac{1}{|\lambda_k|^2} - \frac{1}{r^2}\right) \operatorname{Re}\lambda_k.$$

Various interpolation problems in the classes of functions holomorphic in the half-plane were considered in many works, see [7–9] and the references therein. However, the solvability criterions of the interpolation problem

$$f(\lambda_n) = d_n, \qquad n \in \mathbb{N}, \tag{1.5}$$

in the class $H^\infty(\mathbb{C}_+; h_0)$ is not known.

We employ some ideas from [7–9] and obtain the above formulated theorems on the base of the following statement, which in fact contains a sufficient part of the interpolation Carleson theorem; its elementary proof for the half-plane was provided, for instance, in [9].

Theorem 3. *Let (λ_k) be a sequence of different complex numbers in the half-plane $\mathbb{C}_+ = \{z : \operatorname{Re} z > 0\}$ such that*

$$\sum_{|\lambda_k|\leqslant 1} \operatorname{Re}\lambda_k < +\infty, \tag{1.6}$$

$$\sup\left\{S(r) - \frac{\sigma}{\pi}\ln r : r \in [1; +\infty)\right\} < +\infty, \tag{1.7}$$

$$\sum_{j=k}^{\infty} \operatorname{Re}\left(-\xi_j \frac{\lambda_k^2 - 1}{\lambda_k + \overline{\lambda_j}}\right) \leqslant c_3, \quad \xi_j := \frac{\operatorname{Re}\lambda_j}{1 + |\lambda_j|^2}. \tag{1.8}$$

Moreover, let the sequence (λ_k) is a subsequence of zeroes of a holomorphic in $\mathbb{C}_+$ function Ω such that

$$\left|\frac{\Omega(z)\left(z + \overline{\lambda_k}\right)}{(z - \lambda_k)\operatorname{Re}\lambda_k \Omega'(\lambda_k)}\right| \leqslant c_0 e^{r h_0(\varphi)} e^{-|\lambda_k| h_0(\varphi_n)}, \quad z = x + iy = re^{i\varphi} \in \mathbb{C}_+, \quad k \in \mathbb{N}. \tag{1.9}$$

Then for each sequence $d \in l^\infty(h_0; \lambda)$ there exists a function $f \in H^\infty(\mathbb{C}_+; h_0)$ satisfying condition (1.5).

Remark 1. *If $\sigma = 0$, then conditions (1.6) and (1.7) are equivalent to the condition*

$$\sum_{j=1}^{\infty} \frac{\operatorname{Re}\lambda_j}{1 + |\lambda_j|^2} < +\infty,$$

and if $\Omega(z) = B(z)$ is the Blaschke product for $\mathbb{C}_+$, then condition (1.9) is equivalent to the Carleson condition

$$\inf\left\{\left|\prod_{k=1, k\neq n}^{\infty} \frac{\lambda_n - \lambda_k}{\lambda_n + \overline{\lambda_k}}\right| : n \in \mathbb{N}\right\} \geqslant \delta > 0,$$

while the latter implies (1.8), see, for instance, [9]. The issue on necessity of conditions (1.8) and (1.9) remains for us open. Some comments on this issues are given in the end of the paper.

2. Proof of Theorem 3

Let $s_0(t) = \sum\limits_{1<|\lambda_k|\leqslant t} \operatorname{Re}\lambda_k$. Since

$$\frac{1}{t^2} \leqslant \frac{4}{3}\left(\frac{1}{t^2} - \frac{1}{4s^2}\right) \quad \text{as} \quad |t| \leqslant |s|,$$

then

$$\begin{aligned} s_0(r) \leqslant & r^2 \sum_{1<|\lambda_k|\leqslant r} \frac{\operatorname{Re}\lambda_k}{|\lambda_k|^2} \leqslant r^2 \frac{4}{3} \sum_{1<|\lambda_k|\leqslant r} \left(\frac{1}{|\lambda_k|^2} - \frac{1}{(2r)^2}\right) \operatorname{Re}\lambda_k \\ \leqslant & r^2 \frac{4}{3} \sum_{1<|\lambda_k|\leqslant 2r} \left(\frac{1}{|\lambda_k|^2} - \frac{1}{(2r)^2}\right) \operatorname{Re}\lambda_k = \frac{4}{3} r^2 S(2r). \end{aligned}$$

This is why conditions (1.6) and (1.7) implies the convergence of the series $\sum\limits_{j=1}^{\infty} \left(\frac{\operatorname{Re}\lambda_j}{1+|\lambda_j|^2}\right)^2$ and $\sum\limits_{j=1}^{\infty} \operatorname{Re}\lambda_j \left(1 + |\lambda_j|^2\right)^{-3/2}$. Therefore, $\xi_j \to 0$. This is why, as in [7–9], in the proof of Theorem 3 we can assume that the sequence (ξ_j) is non-increasing. Let

$$\Psi_j(z) = -\xi_j \frac{z^2-1}{z+\overline{\lambda_j}} \quad \text{and} \quad F_k(z) = \exp\left(-\sum_{j=k}^{\infty} \Psi_j(z)\right).$$

The latter series converges uniformly on compact sets in $\mathbb{C}_+$. Let us show that the sought function is

$$f(z) = \sum_{k=1}^{\infty} d_k \frac{\Omega(z)\left(z+\overline{\lambda_k}\right)}{(z-\lambda_k)\Omega'(\lambda_k) 2\mathrm{Re}\,\lambda_k} \left(\frac{1+z}{1+\lambda_k}\right)^2 \left(\frac{2\mathrm{Re}\,\lambda_k}{z+\bar{\lambda}_k}\right)^2 \frac{e^{\xi_k\lambda_k}}{e^{\xi_k z}} \frac{F_k(z)}{F_k(\lambda_k)}.$$

Indeed,

$$\frac{z^2-1}{z+\overline{\lambda_j}} = z - \frac{1+z\overline{\lambda_j}}{z+\overline{\lambda_j}}, \qquad \mathrm{Re}\,\frac{1+z\overline{\lambda_j}}{z+\overline{\lambda_j}} = \frac{\left(1+|z|^2\right)\mathrm{Re}\,\lambda_j + \left(1+|\lambda_j|^2\right)\mathrm{Re}\,z}{\left|z+\overline{\lambda_j}\right|^2}$$

and

$$\mathrm{Re}\,\Psi_j(z) = \frac{\left(1+|z|^2\right)\mathrm{Re}^2\lambda_j}{\left(1+|\lambda_j|^2\right)\left|z+\overline{\lambda_j}\right|^2} + \frac{\mathrm{Re}\,\lambda_j \mathrm{Re}\,z}{\left|z+\overline{\lambda_j}\right|^2} - \xi_j \mathrm{Re}\,z.$$

Hence,

$$|F_k(z)| \leqslant \exp\left(\sum_{j=k}^{\infty}\left(-\frac{\left(1+|z|^2\right)\mathrm{Re}^2\lambda_j}{\left(1+|\lambda_j|^2\right)\left|z+\overline{\lambda_j}\right|^2} + \xi_j \mathrm{Re}\,z\right)\right)$$
$$\leqslant \exp\left(\xi_k \mathrm{Re}\,z\right)\exp\left(\sum_{j=k}^{\infty}\left(-\frac{\left(1+|z|^2\right)\mathrm{Re}^2\lambda_j}{\left(1+|\lambda_j|^2\right)\left|z+\overline{\lambda_j}\right|^2}\right)\right).$$

Moreover, see [9],

$$\left|\frac{2\mathrm{Re}\,\lambda_j}{1+\lambda_j}\frac{z+1}{z+\overline{\lambda_j}}\right|^2 \leqslant 4\frac{\mathrm{Re}\,\lambda_j}{\left(1+|\lambda_j|^2\right)\left|z+\overline{\lambda_j}\right|^2}\left(\left(|z|^2+1\right)\mathrm{Re}\,\lambda_j + \left(1+|\lambda_j|^2\right)\mathrm{Re}\,z\right)$$
$$= 4\mathrm{Re}\,\frac{\mathrm{Re}\,\lambda_j}{1+|\lambda_j|^2}\frac{1+z\overline{\lambda_j}}{z+\overline{\lambda_j}}.$$

In addition, according condition (1.8),

$$|F_k(\lambda_k)| = \exp\left(-\sum_{j=k}^{\infty}\mathrm{Re}\,\left(-\xi_j\frac{\lambda_k^2}{\lambda_k+\overline{\lambda_j}} + \xi_j\frac{1}{\lambda_k+\overline{\lambda_j}}\right)\right) \geqslant c_2.$$

Therefore,

$$\left|d_k \frac{\Omega(z)\left(z+\overline{\lambda_k}\right)}{(z-\lambda_k)\Omega'(\lambda_k)2\mathrm{Re}\,\lambda_k}\left(\frac{1+z}{1+\lambda_k}\right)^2\left(\frac{2\mathrm{Re}\,\lambda_k}{z+\bar{\lambda}_k}\right)^2\frac{e^{\xi_k\lambda_k}}{e^{\xi_k z}}\frac{F_k(z)}{F_k(\lambda_k)}\right|$$
$$\leqslant c_3\left|\left(\frac{1+z}{1+\lambda_k}\right)^2\left(\frac{2\mathrm{Re}\,\lambda_k}{z+\bar{\lambda}_k}\right)^2\frac{e^{\xi_k\lambda_k}}{e^{\xi_k z}}\frac{F_k(z)}{F_k(\lambda_k)}\right|$$
$$\leqslant c_4\frac{\mathrm{Re}\,\lambda_k}{1+|\lambda_k|^2}\mathrm{Re}\,\frac{1+z\overline{\lambda_k}}{z+\overline{\lambda_k}}\exp\left(-\sum_{j\geqslant k}\left(\frac{\mathrm{Re}\,\lambda_k}{1+|\lambda_k|^2}\mathrm{Re}\,\frac{1+z\overline{\lambda_k}}{z+\overline{\lambda_k}}\right)\right).$$

Since

$$\sum_{k=1}^{\infty}|a_k|\exp\left(-\sum_{j=k}^{\infty}|a_j|\right) < 1,$$

we arrive at the statement of Theorem 3.

3. Proof of Theorem 1

Lemma 3.1. *Let* $\sigma \in [0;+\infty)$, *the function* $\Omega \in H^\infty(\mathbb{C}_+;h_0)$ *has the zeroes at the points* $\lambda_k \in \mathbb{C}_+$,

$$\tilde{\Omega}_k(z) = \frac{\Omega(z)(z+\overline{\lambda_k})}{z-\lambda_k}, \qquad \tau_k = \frac{\delta_k}{1+\sqrt{1+\delta_k^2}},$$

where $\delta_k = 1$ *if* $\operatorname{Re}\lambda_k < 1$ *or if* $\sigma = 0$, *and* $\delta_k = (\operatorname{Re}\lambda_k)^{-1}$ *if* $\sigma > 0$ *and* $\operatorname{Re}\lambda_k \geqslant 1$. *Then*

$$\left|\tilde{\Omega}_k(z)\right| \leqslant \frac{c_2}{\tau_k}\exp(\sigma|y|)$$

as $z \in \mathbb{C}_+$, $k \in \mathbb{N}$.

Proof. Since $\tau_k \in (0;1)$ and $\delta_k = \frac{2\tau_k}{1-\tau_k^2}$, the circles

$$\begin{aligned} U_k :=& \left\{\varsigma \in \mathbb{C} : \left|\frac{\varsigma-\lambda_k}{\varsigma+\overline{\lambda_k}}\right| < \tau_k\right\} \\ =& \left\{\varsigma = \xi + i\eta \in \mathbb{C} : \left(\xi - \frac{1+\tau_k^2}{1-\tau_k^2}\operatorname{Re}\lambda_k\right)^2 + (\eta - \operatorname{Im}\lambda_k)^2 < \left(\frac{2\tau_k \operatorname{Re}\lambda_k}{1-\tau_k^2}\right)^2\right\} \end{aligned}$$

are contained in $\mathbb{C}_+$. Then

$$\left|\tilde{\Omega}_k(z)\right| \leqslant \frac{|\Omega(z)|}{\tau_k} \leqslant \frac{c_1}{\tau_k}\exp(\sigma|y|) \quad \text{if} \quad \left|\frac{z-\lambda_k}{z+\overline{\lambda_k}}\right| \geqslant \tau_k.$$

If $\left|\frac{z-\lambda_k}{z+\overline{\lambda_k}}\right| < \tau_k$, then by the maximum principle we obtain

$$\left|\tilde{\Omega}_k(z)\right| \leqslant \max\left\{\frac{c_1 e^{\sigma|\operatorname{Im}\varsigma|}}{\tau_k} : \left|\frac{\varsigma-\lambda_k}{\varsigma+\overline{\lambda_k}}\right| = \tau_k\right\} \leqslant \frac{1}{\tau_k}e^{\sigma|y|+2\sigma\delta_k \operatorname{Re}\lambda_k}.$$

Since $\sigma\delta_k \operatorname{Re}\lambda_k \leqslant \sigma$, this completes the proof. □

We note that

$$\tau_k \geqslant \frac{1}{3}\operatorname{Re}\lambda_k$$

if $\sigma > 0$ and $\operatorname{Re}\lambda_k \geqslant 1$. Therefore, the proven lemma implies that the sequence $\lambda = (k)$ satisfies all assumptions of Theorem 3 for $\sigma = \pi$, and at that, we can take $\Omega(z) = \sin\pi z$. In addition, $l^\infty \subset l^\infty(h_0;\lambda)$ if $\lambda = (k)$. This is why Theorem 1 follows Theorem 3.

4. Proof of Theorem 2

Lemma 4.1. *Let* (λ_k) *be a sequence of different complex numbers in the half-plane* $\mathbb{C}_+ = \{z : \operatorname{Re} z > 0\}$ *such that inequalities (1.6), (1.8) hold and*

$$\sup\left\{S(r) - \frac{\sigma}{2\pi}\ln r : r \in [1;+\infty)\right\} < +\infty.$$

Let also (λ_k) *be a subsequence of zeroes of a holomorphic in* $\mathbb{C}_+$ *function* Ω *such that*

$$\left|\frac{\Omega(z)\left(z+\overline{\lambda_k}\right)}{(z-\lambda_k)\operatorname{Re}\lambda_k \Omega'(\lambda_k)}\right| \leqslant c_0 e^{rh_-(\varphi)}e^{-|\lambda_k|h_-(\varphi_n)}, \qquad z = x+iy = re^{i\varphi} \in \mathbb{C}_+, \qquad k \in \mathbb{N}.$$

Then for each sequence $(d_k) \in l^\infty(h_+;\lambda)$ *there exists a holomorphic in* $\mathbb{C}_+$ *function* $f \in H^\infty(\mathbb{C}_+;h_+)$ *satisfying condition (1.5).*

The proof of this lemma reproduces literally the proof of Theorem 3.

Lemma 4.2. *Let $\sigma \in [0;+\infty)$, a function $\Omega \in H^\infty(\mathbb{C}_+;h_+)$ has zeroes at the points $\lambda_k \in \mathbb{C}_+$ and*

$$\tilde{\Omega}_k(z) = \frac{\Omega(z)(z+\overline{\lambda_k})}{z-\lambda_k}.$$

Then

$$\left|\tilde{\Omega}_k(z)\right| \leqslant \frac{c_2}{\tau_k} \exp\left(rh_+(\varphi)\right) \quad as \quad z = x+iy = re^{i\varphi} \in \mathbb{C}_+.$$

The proof of this lemma is similar to the proof of Lemma 3.1.

We proceed to proving Theorem 2. We assume that $\sigma = \pi$. Let $\Omega(z) = e^{-i\frac{\pi}{2}(z-1)} \sin\frac{\pi}{2}(z-1)$. This functions has zeroes in $\mathbb{C}_+$ at the points $\lambda_k = 2k-1$, $k \in \mathbb{N}$, and $\Omega \in H^\infty(\mathbb{C}_+;h_+)$. At that, $|\Omega'(\lambda_k)| = \pi/2$, and according Lemma 4.2, the sequence $\lambda_k = 2k-1$ satisfies all assumptions of Lemma 4.1. Let $d_k = f(\lambda_k)$. Then $(d_k) \in l^\infty(h_+;\lambda)$. Hence, according Lemma 4.1, there exists a function $f_0 \in H^\infty(\mathbb{C}_+;h_+)$ such that $f_0(\lambda_k) = f(\lambda_k)$, $k \in \mathbb{N}$. Let $\tilde{f}(z) = \frac{f(z)-f_0(z)}{\Omega(z)}$. Since [10]

$$\left|\sin\frac{\pi}{2}(z-1)\right| \geqslant c_0 \exp\left(\frac{\pi}{2}|\mathrm{Im}\, z|\right)$$

outside the circles $|z-\lambda_k| \leqslant \varepsilon$ and therefore, outside these circles the estimate

$$\left|\tilde{f}(z)\right| \leqslant c_5 \exp\left(rh_-(\varphi)\right), \qquad z = x+iy = re^{i\varphi},$$

holds true. Now by the maximum principle we infer that $\tilde{f} \in H^\infty(\mathbb{C}_+;h_-)$. Moreover,

$$f(z) = \tilde{f}(z)\,\Omega(z) + f_0(z) = \frac{1}{2i}\tilde{f}(z) + f_0(z) - \frac{1}{2i}e^{-i\pi z}\tilde{f}(z).$$

Since $f_1(z) := \frac{1}{2i}\tilde{f}(z) \in H^\infty(\mathbb{C}_+;h_-)$ and $f_2(z) := f_0(z) - \frac{1}{2i}e^{-i\pi z}\tilde{f}(z) \in H^\infty(\mathbb{C}_+;h_+)$, this completes the proof of Theorem 2.

5. Addenda and remarks

Conditions (1.6) and (1.7) are necessary for the statement of Theorem 3. Indeed, let $Q(z) = f(z)\frac{z-\lambda_1}{z+\overline{\lambda_1}}$, where $f \in H^\infty(\mathbb{C}_+;h_0)$ is a function such that $f(\lambda_1) = 1$ and $f(\lambda_k) = 0$ if $k \neq 1$. Then $Q \in H^\infty(\mathbb{C}_+;h_0)$ and (λ_k) is a sequence of zeroes of the function Q. This is why, by the generalized Carleman formula [11] we obtain (1.6) and (1.7) [12]. If the sequence (λ_k) satisfies conditions (1.6) and (1.7), then [10] there exists a function $f \in H^\infty(\mathbb{C}_+;h_0)$, for which this is a sequence of its zeroes. Each function $f \in H^\infty(\mathbb{C}_+;h_0)$, $f \not\equiv 0$, is represented as [11]

$$f(z) = e^{ia_0+a_1z}\tilde{B}(z)\tilde{T}(z), \tag{5.1}$$

where $a_0 \in \mathbb{R}$ and $a_1 \in \mathbb{R}$ are constants,

$$Q_1(t;z) = \frac{(tz+i)^2}{(1+t^2)^2(t+iz)},$$

$$\tilde{T}(z) = \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{+\infty} Q_1(t;z)\left(\ln|f_0(it)|dt + dh(t)\right)\right\}, \quad \tilde{B}(z) = \prod_{j=1}^{\infty} W_j(z),$$

$f_0(it) = f(it)$ are angular boundary values of $f(z)$ on $\partial\mathbb{C}_+$, $h(t)$ is a non-increasing function (a singular boundary function of the function f), whose derivative vanishes everywhere,

$$W_j(z) = \frac{z-\lambda_j}{z+\overline{\lambda_j}} \quad \text{as} \quad |\lambda_j| \leqslant 1, \qquad W_j(z) = \frac{1-\frac{z}{\lambda_j}}{1+\frac{z}{\overline{\lambda_j}}}\exp\left(\frac{z}{\lambda_j}+\frac{z}{\overline{\lambda_j}}\right) \quad \text{as} \quad |\lambda_j| > 1.$$

In [13], the following statement was proved.

Proposition 1. *If* $f \in H^\infty(\mathbb{C}_+; h_0)$ *and* $f \not\equiv 0$, *then* $1_a)$ $\log|f_0| \in L_1^{loc}(i\mathbb{R})$, $2_a)$ $f_0(iy)\exp(-\sigma|y|) \in L^\infty(\mathbb{R})$, $1_b)$ $\sup\{K(r) : r \in [1; +\infty)\} < +\infty$, *and (1.6) holds, where*

$$K(r) := K_Z(r) + K_S(r) + K_B(r), \quad K_Z(r) := 2\sum_{1<|\lambda_k|\leqslant r}\left(\frac{1}{|\lambda_k|^2} - \frac{1}{r^2}\right)\mathrm{Re}\,\lambda_k,$$

$$K_S(r) := -\frac{1}{\pi}\int\limits_{1\leqslant|t|\leqslant r}\left(\frac{1}{|t|^2} - \frac{1}{r^2}\right)dh(t),$$

$$K_B(r) := -\frac{1}{\pi}\int\limits_{1\leqslant|t|\leqslant r}\left(\frac{1}{|t|^2} - \frac{1}{r^2}\right)\log|f_0(it)|\,dt.$$

Vice versa, if the sequence (λ_k) *of the points in the half-plane* $\mathbb{C}_+$, *a function* $f_0 : i\mathbb{R} \to \mathbb{C}$ *and a non-increasing function* $h : \mathbb{R} \to \mathbb{R}$, *whose derivative vanishes almost everywhere are such that conditions* $1_a)$, $2_a)$, $1_b)$ *and (1.6) hold, then the function* f *defined by identity (5.1) is holomorphic in* $\mathbb{C}_+$ *and satisfies the estimates* $|f(z)| \leqslant c_1\exp(\sigma|y| + c_1x)$. *At that, if in the product* $\tilde{B}(z)$ *we omit some of the factors, the above estimate remains true and the constant* c_1 *does not increase.*

Employing this statement and some ideas from the proof of necessary part of Carleson intrerpolation theorem (see [14]), we confirm that each of the following conditions

$$\prod_{j\in\mathbb{N},\, j\neq k}|W_j(\lambda_k)| \geqslant c_3\exp(-c_3\mathrm{Re}\,\lambda_k), \quad k \in \mathbb{N},$$

$$\sum_{j\in\mathbb{N}, j\neq k}\left(\frac{2\mathrm{Re}\,\lambda_k\mathrm{Re}\,\lambda_j}{\left|\lambda_k + \overline{\lambda_j}\right|^2} - \frac{2\mathrm{Re}\,\lambda_k\mathrm{Re}\,\lambda_j}{1 + |\lambda_j|^2}\right) \leqslant c_4\mathrm{Re}\,\lambda_k, \quad k \in \mathbb{N},$$

is necessary for the solvability of interpolation problem (1.5) in the class $H^\infty(\mathbb{C}_+; h_0)$ for each sequence $d \in l^\infty(h_0; \lambda)$. However, we fail in trying to prove the necessity of conditions (1.8) and (1.9). In view of this, it is useful to mention the inequality

$$\sum_{j=k}^{\infty}\mathrm{Re}\left(-\xi_j\frac{\lambda_k^2 - 1}{\lambda_k + \overline{\lambda_j}}\right) = \sum_{j=k}^{\infty}\left(\xi_j\frac{\left(1 + |\lambda_k|^2\right)\mathrm{Re}\,\lambda_j + \left(1 + |\lambda_j|^2\right)\mathrm{Re}\,\lambda_k}{\left|\lambda_k + \overline{\lambda_j}\right|^2} - \xi_j\mathrm{Re}\,\lambda_k\right)$$

$$\leqslant \sum_{j=k}^{\infty}\left(\frac{2\mathrm{Re}\,\lambda_k\mathrm{Re}\,\lambda_j}{\left|\lambda_k + \overline{\lambda_j}\right|^2} - \frac{\mathrm{Re}\,\lambda_k\mathrm{Re}\,\lambda_j}{1 + |\lambda_j|^2}\right).$$

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