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CHARACTERISTIC FUNCTION AND DEFICIENCY OF ALGEBROID FUNCTIONS ON ANNULI

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Abstract In this paper, we develop the value distribution theory for meromorphic functions with maximal deficiency sum for algebroid functions on annuli and we study the relationship between the deficiency of algebroid function on annuli and that of their derivatives. Let $W(z)$ be an ν-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$ with maximal deficiency sum and the order of $W(z)$ is finite. Then

i. $\limsup\limits_{r\to\infty} \frac{T_0(r,W')}{T_0(r,W)} = 2 - \delta_0(\infty, W) - \theta_0(\infty, W)$;

ii. $\limsup\limits_{r\to\infty} \frac{N_0(r,\frac{1}{W'})}{T_0(r,W')} = 0$;

iii. $\frac{1-\delta_0(\infty,W)}{2-\delta_0(\infty,W)} \leqslant K_0(W') \leqslant \frac{2(1-\delta_0(\infty,W))}{2-\delta_0(\infty,W)}$,

where

$$K_0(W') = \limsup_{r\to\infty} \frac{N_0(r,W') + N_0(r,\frac{1}{W'})}{T_0(r,W')}.$$

Keywords : Nevanlinna Theory, maximal deficiency sum, algebroid functions, the annuli, etc.

Subject Classification: 30D35

1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was first considered by Valiron, afterwards several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ were proved (see [3],[11]). In 2005, A.Ya. Khrystiyanyn and A.A. Kondratyuk have proposed the Nevanlinna theory for meromorphic functions on annuli (see [4], [5]) and after this work others have done lot of work in this area (see [8], [12], [13]–[36]). In 2009, Cao and Yi [1] studied the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Yang Tan [6], Yang Tan and Yue Wang [7] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. Thus, it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By doubly connected mapping theorem [10], each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \leqslant r < R \leqslant +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ and $0 \leqslant r < R \leqslant +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the

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annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in both cases each annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$. We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [2] and [9]).

Let $A_v(z), A_{v-1}(z), \ldots, A_0(z)$ be a group of analytic functions which have no common zeros and are defined on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, $1 < R_0 \leqslant +\infty$ and

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \ldots + A_1(z)W + A_0(z) = 0. \tag{1}$$

Then irreducible equation (1) defines a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$.

Let $W(z)$ be a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, we use the notations

$$m(r, W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w_j(re^{i\theta})|d\theta,$$

$$N_1(r, W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1} \frac{n_1(t, W)}{t}dt,$$

$$N_2(r, W) = \frac{1}{\nu}\int_{1}^{r} \frac{n_2(t, W)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_1\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_2\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r} \frac{\overline{n}_2\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_1^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_1^{k)}\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_2^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r} \frac{\overline{n}_2^{k)}\left(t, \frac{1}{w-a}\right)}{t}dt,$$

$$\overline{N}_1^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_1^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_2^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{1}^{r} \frac{\overline{n}_2^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r, W) = m(r, W) + m\left(\frac{1}{r}, W\right) - 2m(1, W),$$

$$N_0(r, W) = N_1(r, W) + N_2(r, W),$$

$$\overline{N}_0\left(r, \frac{1}{W-a}\right) = \overline{N}_1\left(r, \frac{1}{W-a}\right) + \overline{N}_2\left(r, \frac{1}{W-a}\right),$$

$$\overline{N}_0^k\left(r, \frac{1}{W-a}\right) = N_1^k\left(r, \frac{1}{W-a}\right) + N_2^k\left(r, \frac{1}{W-a}\right),$$

where $w_j(z)(j = 1, 2, \ldots, \nu)$ is a single-valued branch of $W(z)$, $n_1(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : t < |z| \leqslant 1\}$ and $n_2(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : 1 < |z| \leqslant t\}$ (both counting multiplicity). The symbol $\overline{n}_1\left(t, \frac{1}{W-a}\right)$ stands for the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \leqslant 1\}$ and

$\overline{n}_2\left(t, \frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \leqslant t\}$ (both ignoring multiplicity). By $n_1^k(t, a, W)$ we denote the number of zeros of $W - a$ in $\{z : t < |z| \leqslant 1\}$ and $n_2^k(t, a, W)$ is the number of zeros of $W - a$ in $\{z : 1 < |z| \leqslant t\}$, where zero of order $< k$ is counted according to its multiplicity and a zero of order $\geqslant k$ is counted exactly k times, respectively.

Let $W(z)$ be a v-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, as

$$a \in \mathbb{C}, \qquad n_0\left(r, \frac{1}{W-a}\right) = n_0\left(r, \frac{1}{\psi(z,a)}\right), \qquad N_0\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu} N_0\left(r, \frac{1}{\psi(z,a)}\right).$$

In particular, as $a = 0$, we have $N_0\left(r, \frac{1}{W}\right) = \frac{1}{\nu} N_0\left(r, \frac{1}{A_0}\right)$ and as $a = \infty$, the identity holds $N_0(r, W) = \frac{1}{\nu} N_0\left(r, \frac{1}{A_v}\right)$. Here $n_0\left(r, \frac{1}{W-a}\right)$ and $n_0\left(r, \frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z) - a$ and $\psi(z, a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, respectively.

Definition 1. [6] *Given an algebroid function $W(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, the function*

$$T_0(r, W) = m_0(r, W) + N_0(r, W), \qquad 1 \leqslant r < R_0$$

is called Nevanlinna characteristic of $W(z)$.

Definition 2. *Given an algebroid function $W(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, the order of $W(z)$ is defined by*

$$\sigma(W) = \limsup_{r \to +\infty} \frac{\log T_0(r, W)}{\log r}.$$

Definition 3. *Given an algebroid function $W(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, the value*

$$\delta_0(a, W) = \liminf_{r \to +\infty} \frac{m_0(r, \frac{1}{W-a})}{T_0(r, W)}$$

is called the deficiency of the function $W(z)$ for the value a. For $a = \infty$, we let

$$\delta_0(\infty, W) = \liminf_{r \to +\infty} \frac{m_0(r, W)}{T_0(r, W)} = 1 - \limsup_{r \to +\infty} \frac{N_0(r, W)}{T_0(r, W)}.$$

If $\delta_0, a \in \mathbb{C}_\infty$, we call a a deficient value of $W(z)$.

Definition 4. *Given an algebroid function $W(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, the value*

$$\Theta_0(a, W) = 1 - \limsup_{r \to +\infty} \frac{\overline{N}_0(r, \frac{1}{W-a})}{T_0(r, W)}$$

and

$$\theta_0(a, W) = \liminf_{r \to +\infty} \frac{N_0(r, \frac{1}{W-a}) - \overline{N}_0(r, \frac{1}{W-a})}{T_0(r, W)}$$

are called the reduced deficiencies of the function $W(z)$ for the value a.

2. Auxiliary lemmata

Lemma 1 (The first fundamental theorem on annuli [7]). *Let $W(z)$ be v-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, $a \in \mathbb{C}$. Then*

$$T_0\left(r, \frac{1}{W-a}\right) = m_0\left(r, \frac{1}{W-a}\right) + N_0\left(r, \frac{1}{W-a}\right) = T_0\left(r, W\right) + O(1).$$

Lemma 2 (The second fundamental theorem on annuli [13]). *Let $W(z)$ be ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$, a_k, $(k = 1, 2, .., p)$ are p distinct complex numbers (finite or infinite), then we have*

$$(p-2v)T_0\left(r, W\right) \leqslant \sum_{k=1}^{p} N_0\left(r, \frac{1}{W-a_k}\right) - N_1(r, W) + S_0(r, W) \tag{2}$$

where $N_1(r, W)$ is the density index of all multiple values including finite or infinite, every τ-multiple value is counted $\tau - 1$ times and

$$S_0(r, W) = m_0\left(r, \frac{W'}{W}\right) + \sum_{j=1}^{p} m_0\left(r, \frac{W'}{W-a_k}\right) + O(1).$$

The remainder satisfies the identity

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O(\log r),$$

outside a set of a finite linear measure if $r \to R_0 = +\infty$, while

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O\left(\log \frac{1}{R_0 - r}\right),$$

outside a set E such that

$$\int_E \frac{dr}{R_0 - r} < +\infty \quad as \quad r \to R_0 < +\infty.$$

Lemma 3. [7] *Let $W(z)$ be ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$. If the following conditions are satisfied*

$$\liminf_{r\to\infty} \frac{T_0\left(r, W\right)}{\log r} < \infty, \qquad R_0 = +\infty,$$

$$\liminf_{r\to R_0^-} \frac{T_0\left(r, W\right)}{\log \frac{1}{(R_0-r)}} < \infty, \qquad R_0 < +\infty,$$

then $W(z)$ is an algebraic function.

3. Main results

In the present paper we study the problem on the maximal deficiency sum for algebroid function on annuli as well as the relationship between the deficiency of algebroid function on annuli and that of their derivatives.

Theorem 1. *Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$. Then the set of all numbers $a \in \overline{\mathbb{C}}$ obeying $\Theta_0(a, W) > 0$ is countable and $\sum_{a\in\overline{\mathbb{C}}} \Theta_0(a, W) \leqslant 2$.*

Proof. By the second fundamental theorem for algebroid function on annuli we have

$$(q-2\nu)\,T_0(r,W) \leqslant \sum_{i=1}^{q} \overline{N}_0\left(r,\frac{1}{W-a_i}\right) + S_0(r,W),$$

and this implies

$$(q-2\nu) \leqslant \sum_{i=1}^{q} \frac{\overline{N}_0(r,\frac{1}{W-a_i})}{T_0(r,W)} + \frac{S_0(r,W)}{T_0(r,W)} \leqslant \sum_{i=1}^{q} \limsup_{r\to\infty} \frac{\overline{N}_0(r,\frac{1}{W-a_i})}{T_0(r,W)} + \limsup_{r\to\infty} \frac{S_0(r,W)}{T_0(r,W)}.$$

Since

$$\Theta_0(a,W) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_0(r,\frac{1}{W-ai})}{T_0(r,W)}$$

and

$$S_0(r,W) = O(T_0(r,W)),$$

we have

$$(q-2\nu) \leqslant \sum_{i=1}^{q}[1-\Theta_0(a_i,W)] \leqslant q - \sum_{i=1}^{q}\Theta_0(a_i,W).$$

Therefore,

$$\sum_{i=1}^{q}\Theta_0(a_i,W) \leqslant 2\nu,$$

and this holds for all extended complex numbers as $q \geqslant 3$. Hence,

$$\sum_{a\in\overline{\mathbb{C}}}\Theta_0(a,W) \leqslant 2,$$

and this is a defect relation for algebroid functions on annuli.

Let $E = \{a \in \overline{\mathbb{C}} : \Theta(a,W) > 0\}$ and we are going to show that that E is countable set. We denote

$$E_n = \left\{a \in \overline{\mathbb{C}} : \Theta(a,W) > \frac{1}{n}\right\},\ n \in \mathbb{N}.$$

Then, by the defect relation E_n contains at most $2n$ elements and hence $\cup_{n=1}^{\infty}E_n$ is a countable set. Let us show that

$$\cup_{n=1}^{\infty}E_n = E.$$

In order to do this, we take $a \in E$ and hence $\Theta_0(a,W) > 0$. By the Archimedian property, there exist $n \in \mathbb{N}$ such that

$$\Theta_0(a,W) > \frac{1}{n}.$$

Thus, $a \in E_n \subset \cup_{n=1}^{\infty}E_n$ and this yields

$$E \subset \cup_{n=1}^{\infty}E_n. \tag{3}$$

On the other hand, given $a \in \cup_{n=1}^{\infty}E_n$, we have $a \in E_n$ for some $n \in \mathbb{N}$. Therefore,

$$\Theta_0(a,W) > \frac{1}{n} > 0$$

and hence, $a \in E$. Now we infer that

$$\cup_{n=1}^{\infty} E_n \subset E. \tag{4}$$

By (3) and (4) we get $E = \cup_{n=1}^{\infty}E_n$ and therefore, E is a countable set. □

Remark 1. *By Theorem 1, the total deficiency of each algebroid function $W(z)$ on annuli satisfies the inequality*

$$\sum_{a\in\overline{\mathbb{C}}}\delta_0(a,W)+\delta_0(\infty,W)\leqslant 2. \tag{5}$$

If (5) holds, then we say that $W(z)$ has a maximal deficiency sum.

Theorem 2. *Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leqslant+\infty)$. Then*

$$\limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)}\leqslant 2\nu-\delta_0(\infty,W)-\delta_0(\infty,W).$$

Proof. We have

$$m_0(r,W')=m_0\left(r,W\frac{W'}{W}\right)\leqslant m_0\left(r,\frac{W'}{W}\right)+m_0(r,W)+O(1).$$

By Lemma 2, the identity

$$m_0\left(r,\frac{W^{(k)}}{W}\right)=S_0(r,W)$$

holds true and hence,

$$m_0(r,W')\leqslant m_0(r,W)+S_0(r,W). \tag{6}$$

We also have

$$N_0(r,W')=N_0(r,W)+\overline{N}_0(r,W)+N_x(r,W). \tag{7}$$

By (6) and (7) we conclude that

$$\begin{aligned}T_0(r,W')=&m_0(r,W')+N_0(r,W')\leqslant m_0(r,W)+m_0\left(r,\frac{W'}{W}\right)+N_0(r,W'),\\ \leqslant& m_0(r,W)+N_0(r,W)+\overline{N}_0(r,W)+N_x(r,W)+S_0(r,W)\\ \leqslant& T_0(r,W)+\overline{N}_0(r,W)+N_x(r,W)+S_0(r,W).\end{aligned}$$

This yields

$$\frac{T_0(r,W')}{T_0(r,W)}\leqslant 1+\frac{\overline{N}_0(r,W)}{T_0(r,W)}+\frac{N_x(r,W)}{T_0(r,W)}+\frac{S_0(r,W)}{T_0(r,W)}.$$

Therefore,

$$\begin{aligned}\limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)}\leqslant&1+\limsup_{r\to\infty}\frac{\overline{N}_0(r,W)}{T_0(r,W)}\\ \leqslant&1+1-\Theta_0(\infty,W)\leqslant 2-\Theta_0(\infty,W)\end{aligned}$$

But

$$\delta_0(\infty,W)+\theta_0(\infty,W)\leqslant\Theta_0(\infty,W)$$

and hence,

$$\limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)}\leqslant 2-\delta_0(\infty,W)-\theta_0(\infty,W).$$

The proof is complete. □

Theorem 2 yields the following corollary.

Corollary 1. *Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$. Then*

$$\limsup_{r\to\infty} \frac{T_0(r, W')}{T_0(r, W)} \leqslant 2\nu - \delta_0(\infty, W) - \delta_0(\infty, W).$$

Theorem 3. *Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leqslant +\infty)$ with a maximal deficiency sum and of a finite order. Then*

i. $\limsup\limits_{r\to\infty} \frac{T_0(r,W')}{T_0(r,W)} = 2 - \delta_0(\infty, W) - \theta_0(\infty, W)$;

ii. $\limsup\limits_{r\to\infty} \frac{N_0(r,\frac{1}{W'})}{T_0(r,W')} = 0$;

iii. $\frac{1-\delta_0(\infty,W)}{2-\delta_0(\infty,W)} \leqslant K_0(W') \leqslant \frac{2(1-\delta_0(\infty,W))}{2-\delta_0(\infty,W)}$, *where*

$$K_0(W') = \limsup_{r\to\infty} \frac{N_0(r, W') + N_0(r, \frac{1}{W'})}{T_0(r, W')}.$$

Proof. We have

$$\begin{aligned}
T_0(r, W') =& m_0(r, W') + N_0(r, W') \leqslant m_0(r, W) + m_0\left(r, \frac{W'}{W}\right) + N_0(r, W') \\
\leqslant& m_0(r, W) + N_0(r, W) + \overline{N}_0(r, W) + N_x(r, W) + S_0(r, W) \\
\leqslant& m_0(r, W) + N_0(r, W) + \overline{N}_0(r, W) + 2(\nu - 1)T_0(r, W) + S_0(r, W) \\
\leqslant& (2\nu - 1)T_0(r, W) + \overline{N}_0(r, W) + S_0(r, W) \leqslant 2\nu T_0(r, W) + S_0(r, W).
\end{aligned}$$

By Lemma 2 we get

$$S_0(r, W^{(k)}) = O(\log r T_0(r, W^{(k)})) = O(\log r T_0(r, W)) = S_0(r, W) \tag{8}$$

and hence

$$m_0\left(r, \frac{W^{(k)}}{W - a^{[i]}}\right) = S_0(r, W) \tag{9}$$

holds for each positive $a^{[i]}$. We let

$$F(z) = \sum_{i=1}^{p} \frac{1}{W(z) - a^{[i]}}.$$

Then, as in [11], we have

$$\begin{aligned}
m(r, F) + O(1) \geqslant& \sum_{i=1}^{p} m\left(r, \frac{1}{W(z) - a^{[i]}}\right), \\
m\left(\frac{1}{r}, F\right) \geqslant& \sum_{i=1}^{p} m\left(r, \frac{1}{W(z) - a^{[i]}}\right).
\end{aligned} \tag{10}$$

In fact, (10) holds if $p = 1$. If $p \geqslant 2$, we let

$$\delta = min_{i\neq j}|a^{[i]} - a^{[j]}|.$$

It is Obvious that $\delta > 0$. Given a fixed z, there exist $k \in \{1, 2, \ldots, \nu\}$ and $i \in \{1, 2, \ldots, q\}$ such that

$$|w_k - a^{[i]}| < \frac{\delta}{2q} \leqslant \frac{\delta}{4},$$

and the inequality

$$|w_k(z) - a^{[j]}| \geqslant |a^{[i]} - a^{[j]}| - |w_k(z) - a^{[i]}| \geqslant \frac{3\delta}{4}$$

holds true for $i \neq j$. Therefore, the set of points in $\partial\mathbb{C}_r$, where $\mathbb{C}_r = \{z : |z| = r\}(r = r \text{ or } r = \frac{1}{r})$ obeying $|w_k(z) - a^{[i]}| < \frac{\delta}{2q}$ is either empty or each two such sets are mutually disjoint for different i. In any case

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log^+ |F(re^{i\theta})|d\theta \geqslant & \frac{1}{2\pi}\sum_{i=1}^{q}\int_{|w_k-a^{[i]}|<\frac{\delta}{2q},|z|=r} \log^+ |F(re^{i\theta})|d\theta \\ \geqslant & \frac{1}{2\pi}\sum_{i=1}^{q}\int_{|w_k-a^{[i]}|<\frac{\delta}{2q},|z|=r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|}d\theta.\end{aligned}$$

Since

$$\begin{aligned}&\frac{1}{2\pi}\sum_{i=1}^{q}\int_{|w_k-a^{[i]}|<\frac{\delta}{2q},|z|=r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|}d\theta \\ &\quad = m\left(r, \frac{1}{W(z) - a^{[i]}}\right)\frac{1}{2\pi}\sum_{i=1}^{q}\int_{|w_k-a^{[i]}|\geqslant\frac{\delta}{2q},|z|=r} \log^+ \frac{1}{|w_k(re^{i\theta}) - a^{[i]}|}d\theta \\ &\quad \geqslant m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - q\log^+ \frac{2q}{\delta},\end{aligned}$$

we obtain that

$$\begin{aligned}m(r,F) =& \frac{1}{\nu}\sum_{k=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|F(re^{i\theta})|d\theta \geqslant \frac{1}{\nu}\sum_{k=1}^{\nu}\sum_{i=1}^{q} m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - \frac{1}{\nu}\log^+\frac{2q}{\delta} \\ =& \sum_{i=1}^{q} m\left(r, \frac{1}{W(z) - a^{[i]}}\right) - \frac{1}{\nu}\log^+\frac{2q}{\delta}.\end{aligned}$$

Now relation (10) follows the above inequality in the case $r = r$ and $r = \dfrac{1}{r}$. Since

$$\begin{aligned}m(r,F) =& m(r, W^{(k)}F) + m\left(r, \frac{1}{W^{(k)}}\right) \\ \leqslant & \sum_{i=1}^{p} m\left(r, \frac{W^{(k)}}{W - a^{[i]}}\right) + m\left(r, \frac{1}{W^{(k)}}\right)\end{aligned}$$

and

$$\begin{aligned}m\left(\frac{1}{r}, F\right) =& m\left(\frac{1}{r}, W^{(k)}F\right) + m\left(\frac{1}{r}, \frac{1}{W^{(k)}}\right) \\ \leqslant & \sum_{i=1}^{p} m\left(\frac{1}{r}, \frac{W^{(k)}}{W - a^{[i]}}\right) + m\left(\frac{1}{r}, \frac{1}{W^{(k)}}\right),\end{aligned}$$

we get

$$m_0(r,F) \leqslant \sum_{i=1}^{p} m_0\left(r, \frac{W^{(k)}}{W - a^{[i]}}\right) + m_0\left(r, \frac{1}{W^{(k)}}\right). \tag{11}$$

It follows from (8), (11) and Lemma 3.1 that

$$\begin{aligned}\sum_{i=1}^{p} m_0\left(r, \frac{1}{W(z) - a^{[i]}}\right) \leqslant & m_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W) \\ \leqslant & T_0(r, W^{(k)}) - N_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W).\end{aligned} \tag{12}$$

Thus,

$$pT_0(r,W) \leqslant \sum_{i=1}^{p} N_0\left(r, \frac{1}{W(z)-a^{[i]}}\right) + T_0(r,W^{(k)}) - N_0\left(r,\frac{1}{W^{(k)}}\right) + S_0(r,W). \tag{13}$$

By (13) we obtain

$$pT_0(r,W) \leqslant \sum_{i=1}^{p} N_0\left(r, \frac{1}{W(z)-a^{[i]}}\right) + T_0(r,W') - N_0\left(r,\frac{1}{W'}\right) + S_0(r,W). \tag{14}$$

and hence,

$$\begin{aligned} p \leqslant & \liminf_{r\to\infty} \frac{T_0(r,W')}{T_0(r,W)} + \sum_{i=1}^{p}(1-\delta_0(a^{[i]},W)) + \liminf_{r\to\infty}\frac{S_0(r,W)}{T_0(r,W)} \\ =& p + \limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)} - \sum_{i=1}^{p}\delta_0(a^{[i]},W). \end{aligned} \tag{15}$$

Therefore, we have

$$\sum_{i=1}^{p}\delta_0(a^{[i]},W) \leqslant \limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)}. \tag{16}$$

Hence, identity (8) and Lemma 2 imply that

$$\limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)} \leqslant 2\nu - \delta_0(\infty,W). \tag{17}$$

As p is arbitrary, we combine (17) and (16) to have

$$2\nu-\delta_0(\infty,W) \leqslant \liminf_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)} \leqslant \limsup_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)} \leqslant 2\nu-\delta_0(\infty,W),$$

that is,

$$\lim_{r\to\infty}\frac{T_0(r,W')}{T_0(r,W)} \leqslant 2\nu-\delta_0(\infty,W). \tag{18}$$

Given $\varepsilon>0$, we choose q sufficiently large so that

$$\sum_{i=1}^{p}\delta_0(a^{[i]},W) > 2\nu-\delta_0(\infty,W)-\varepsilon. \tag{19}$$

For these q, inequality (12) implies

$$\begin{aligned} \limsup_{r\to\infty}\frac{N_0(r,\frac{1}{W'})}{T_0(r,W')} + \liminf_{r\to\infty}\frac{T_0(r,W)}{T_0(r,W')}\sum_{i=1}^{p}\liminf_{r\to\infty}\frac{m_0\left(r,\frac{1}{W(z)-a^{[i]}}\right)}{T_0(r,W)} \\ \leqslant 1+\limsup_{r\to\infty}\frac{S_0(r,W)}{T_0(r,W')}. \end{aligned} \tag{20}$$

Thus, from (18)-(20) we deduce

$$\limsup_{r\to\infty}\frac{N_0(r,\frac{1}{W'})}{T_0(r,W')} \leqslant \frac{\varepsilon}{2\nu-\delta_0(\infty,W)}. \tag{21}$$

Since $\varepsilon>0$ is arbitrary, we have

$$\limsup_{r\to\infty}\frac{N_0(r,\frac{1}{W'})}{T_0(r,W')} = 0. \tag{22}$$

And since $N_0(r,W') \leqslant 2\nu N_0(r,W)$, we get

$$\frac{N_0(r,W')}{T_0(r,W')}\frac{T_0(r,W')}{T_0(r,W)} \leqslant 2\nu\frac{N_0(r,W)}{T_0(r,W)}. \tag{23}$$

Now it follows from (18) and (23) that

$$(2\nu-\delta_0(\infty,W))\limsup_{r\to\infty}\frac{N_0(r,W')}{T_0(r,W')} \leqslant 2\nu(1-\delta_0(\infty,W)). \tag{24}$$

Relations (22) and (24) imply

$$\limsup_{r\to\infty}\frac{N_0(r,W')+N_0(r,\frac{1}{W'})}{T_0(r,W')} \leqslant \frac{2\nu(1-\delta_0(\infty,W))}{(2\nu-\delta_0(\infty,W))}.$$

Therefore, we obtain

$$K_0(W') \leqslant \frac{2\nu(1-\delta_0(\infty,W))}{(2\nu-\delta_0(\infty,W))}. \tag{25}$$

Furthermore, we have $N_0(r,W) \leqslant N_0(r.W')$. By (18) we infer that

$$\frac{N_0(r,W)}{T_0(r,W)} \leqslant (2\nu-\delta_0(\infty,W))\frac{N_0(r,W')}{T_0(r,W')}.$$

Thus,

$$\limsup_{r\to\infty}\frac{N_0(r,W')}{T_0(r,W')} \geqslant \frac{1}{(2\nu-\delta_0(\infty,W))}\limsup_{r\to\infty}\frac{N_0(r,W)}{T_0(r,W)} = \frac{1-\delta_0(\infty.W)}{2\nu-\delta_0(\infty,W)}.$$

Hence,

$$K_0(W') \leqslant \frac{1-\delta_0(\infty.W)}{2\nu-\delta_0(\infty,W)}. \tag{26}$$

Due to (25) and (26) we have

$$\frac{1-\delta_0(\infty.W)}{2\nu-\delta_0(\infty,W)} \leqslant K_0(W') \leqslant \frac{2\nu(1-\delta_0(\infty,W))}{(2\nu-\delta_0(\infty,W))}.$$

□

Theorem 4. *Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leqslant+\infty)$ of finite order and $\delta_0(\infty,W)=1$. Then*

$$\sum_{a\in\mathbb{C}}\delta_0(a,W) \leqslant \delta_0(0,W').$$

Proof. If

$$\sum_{a\in\mathbb{C}}\delta_0(a,W)=0,$$

Theorem 3 is valid in this case. In what follows we assume that $\sum\limits_{a\in\mathbb{C}}\delta_0(a,W)>0$. Let $\{a_\mu\}$ be a sequence of distinct complex complex numbers in $\mathbb{C}$ containing all the finite deficient values of $W(z)$. For each positive integer q the inequality

$$\sum_{\mu=1}^{q} m_0\left(r,\frac{1}{W(z)-a_\mu}\right)+N_0\left(r,\frac{1}{W'}\right) \leqslant T_0(r,W')+S_0(r,W)$$

holds for any q finite complex numbers in a_μ. Therefore, we have

$$\frac{N_0\left(r,\frac{1}{W'}\right)}{T_0(r,W')}+\frac{T_0\left(r,W\right)}{T_0(r,W')}\left(\frac{\sum\limits_{\mu=1}^{q} m_0\left(r,\frac{1}{W(z)-a_\mu}\right)}{T_0(r,W)}-o(1)\right)\leqslant 1,$$

as $r\to\infty$. Hence, by inequality (18) we obtain that

$$\begin{aligned}1 &\geqslant \limsup_{r\to\infty}\left[\frac{N_0\left(r,\frac{1}{W'}\right)}{T_0(r,W')}+\frac{T_0\left(r,W\right)}{T_0(r,W')}\left(\frac{\sum\limits_{\mu=1}^{q} m_0\left(r,\frac{1}{W(z)-a_\mu}\right)}{T_0(r,W)}-o(1)\right)\right]\\ &\geqslant \limsup_{r\to\infty}\frac{N_0\left(r,\frac{1}{W'}\right)}{T_0(r,W')}+\liminf_{r\to\infty}\frac{T_0\left(r,W\right)}{T_0(r,W')}\left(\frac{\sum\limits_{\mu=1}^{q} m_0\left(r,\frac{1}{W(z)-a_\mu}\right)}{T_0(r,W)}-o(1)\right)\\ &\geqslant \limsup_{r\to\infty}\frac{N_0\left(r,\frac{1}{W'}\right)}{T_0(r,W')}+\liminf_{r\to\infty}\frac{T_0\left(r,W\right)}{T_0(r,W')}\liminf_{r\to\infty}\frac{\sum\limits_{\mu=1}^{q} m_0\left(r,\frac{1}{W(z)-a_\mu}\right)}{T_0(r,W)}\\ &\geqslant \limsup_{r\to\infty}\frac{N_0\left(r,\frac{1}{W'}\right)}{T_0(r,W')}+\frac{\sum\limits_{\mu=1}^{q}\delta_0(a_\mu,W)}{2\nu-\delta_0(\infty,W)}.\end{aligned}$$

Since q is arbitrary and $\delta_0(\infty,W)=1$, we have

$$\sum_{a\in\mathbb{C}}\delta_0(a,W)\leqslant\delta_0(0,W').$$

□

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