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# ON ISOMORPHISM OF SOME FUNCTIONAL SPACES UNDER ACTION OF INTEGRO-DIFFERENTIAL OPERATORS

**S.B. KLIMENTOV**

**Abstract.** In the paper we consider representations of the second kind for solutions to the linear general uniform first order elliptic system in the unit circle $D = \{z : |z| \leqslant 1\}$ written in terms of complex functions:

$$\mathcal{D}w \equiv \partial_{\bar{z}} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar{z}}\overline{w} + A(z)w + B(z)\overline{w} = R(z),$$

where $w = w(z) = u(z) + iv(z)$ is the sought complex function, $q_1(z)$ and $q_2(z)$ are given measurable complex functions satisfying the uniform ellipticity condition of the system:

$$|q_1(z)| + |q_2(z)| \leqslant q_0 = \text{const} < 1,\ z \in \overline{D},$$

and $A(z)$, $B(z)$, $R(z) \in L_p(\overline{D})$, $p > 2$, are also given complex functions.

The representation of the second kind is based on the well–known Pompeiu's formula: if $w \in W_p^1(\overline{D})$, $p > 2$, then

$$w(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi}\iint\limits_{D} \frac{\partial w}{\partial \bar{z}} \cdot \frac{d\xi d\eta}{\zeta - z},$$

where $w(z) \in W_p^1(\overline{D})$, $p > 2$. Then for the solution $w(z)$ we can write the representation

$$\Omega(w) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w(\zeta)}{\zeta - z} d\zeta + TR(z)$$

where

$$\Omega(w) \equiv w(z) + T(q_1(z)\partial_z w + q_2(z)\partial_{\bar{z}}\overline{w} + A(z)w + B(z)\overline{w}).$$

Under appropriate assumptions about on coefficients we prove that $\Omega$ is the isomorphism of the spaces $C_\alpha^k(\overline{D})$ and $W_p^k(\overline{D})$, $k \geqslant 1$, $0 < \alpha < 1$, $p > 2$. These results develop and complete B.V. Boyarsky's works, where representations of the first kind were obtained. Also this work complete author's results on representations of the second kind with more difficult operators. As an implication of the properties of the operator $\Omega$, we obtain apriori estimates for the norms $\|w\|_{C_\alpha^{k+1}(\overline{D})}$ and $\|w\|_{W_p^k(\overline{D})}$.

**Keywords:** general elliptic first order system, representation of the second kind.

**Mathematics Subject Classification:** 35C15

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## 1. Introduction and formulation of results

We denote by $D = \{z : |z| < 1\}$ the unit circle in the complex $z$-plane $E_z$, $z = x + iy$, $i^2 = -1$, $\Gamma = \partial D$ is the boundary of the circle $D$, $\overline{D} = D \cup \Gamma$.

In the paper we use the following functional spaces with standard norms: $L_p(\overline{D})$ is the space of functions summable in $\overline{D}$ with power $p \geqslant 1$; $W_p^k(\overline{D})$, $k = 0, 1, \ldots$, $p \geqslant 1$, is the class of functions possessing generalized Sobolev derivatives in $\overline{D}$ up to $k$th order summable with power $p$, $W_p^0(\overline{D}) \equiv L_p(\overline{D})$; $C_\alpha^k(\overline{D})$, $k = 0, 1, \ldots$, $0 < \alpha < 1$, is the space of functions with continuous in $\overline{D}$ partial derivatives up to order $k$ obeying the Hölder condition with the exponent $\alpha$, $C_\alpha^0(\overline{D}) \equiv C_\alpha(\overline{D})$. The spaces $L_p(\Gamma)$ and $C_\alpha^k(\Gamma)$ are introduced in the same way but for the functions defined on $\Gamma$. Detailed definitions of these spaces and their norms can be found in [1].

A closed subspace of holomorphic functions of the space $C_\alpha^k(\overline{D})$ (respectively, $W_p^k(\overline{D})$) is denoted by $A_\alpha^k(\overline{D}) = A_\alpha^k$ (respectively, $A_p^k(\overline{D}) = A_p^k$). The notations $C_\alpha^k(\overline{G})$, $A_\alpha^k(\overline{G})$ and others are clear if $G$ is another domain in the complex plane. We also employ the space $W_p^{k-\frac{1}{p}}(\Gamma)$ of the traces of the functions in $W_p^k(\overline{D})$, for details see Section 2.3.

In $\overline{D}$ we consider a general elliptic first order system in the complex writing:

$$\mathcal{D}w \equiv \partial_{\bar{z}} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar{z}}\overline{w} + A(z)w + B(z)\overline{w} = R(z), \tag{1}$$

where $w = w(z) = u(z) + iv(z)$ is an unknown complex function,

$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

are derivative in the Sobolev sense, $q_1(z)$ and $q_2(z)$ are given measurable complex functions satisfying the uniform ellipticity condition (1)

$$|q_1(z)| + |q_2(z)| \leqslant q_0 = \mathrm{const} < 1, \qquad z \in \overline{D}, \tag{2}$$

$A(z)$, $B(z)$, $R(z) \in L_p(\overline{D})$, $p > 2$, are also given complex functions.

While studying solutions of equation (1), various representations making correspondence between holomorphic functions and the solutions play an important role. In the particular case $q_1(z) = q_2(z) \equiv 0$, for continuous in $\overline{D}$ solutions, various authors obtained two representations: the representation of the first kind for the homogeneous system

$$w(z) = \Phi(z) \exp\left\{-T\left(A + B\frac{\overline{w}}{w}\right)\right\} \tag{3}$$

and the representation of the second kind for the inhomogeneous system

$$w(z) + T(Aw + B\overline{w}) = \Phi(z) + TR(z). \tag{4}$$

In both representations, $\Phi(z)$ is some holomorphic function and

$$Tf(z) = T_D f(z) = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta, \qquad \partial_{\bar{z}} Tf(z) = f(z), \tag{5}$$

see [1] and the references therein. Of course, the interesting cases are when these representations allow one to recover the solution $w(z)$ by a given holomorphic function $\Phi(z)$, that is, when representations (3), (4) can be employed as a tool for constructing solutions to system (1) in various functional spaces.

In [1, Ch. 3, Sect. 7], for representation (3), there was proved the bijection between solutions $w(z) \in W_p^1(\overline{D})$, $p > 2$, and holomorphic functions in the class $A_p^1(\overline{D})$. At that, if a holomorphic function $\Phi(z)$ has various singularities, representation (3) also allows one to recover a solution $w(z)$ with corresponding singularities. At the same time, if the coefficients $A$ and $B$ are smooth enough and the holomorphic function $\Phi(z)$ is in some class smaller than $A_p^1(\overline{D})$, for instance, is in $A^1_\alpha(\overline{D})$, then for the recovered solution one can ensure at most the belonging to the class $W^1_p(\overline{D})$, where $p > 2$ can be arbitrary large.

For representation (4) of continuous in $\overline{D}$ solutions the picture is more complete, namely, under appropriate assumptions on the coefficients $A$ and $B$ (now we do not dwell on them), it was proved in [1] that the operator $I + T$ is an isomorphism of the Banach space $W^k_p(\overline{D})$ or $C^k_\alpha(\overline{D})$).

The first fundamental study of solutions to general system (1) was made by B.V. Boyarskii [2]. He obtained a generalization of first kind representation (3) and proved the invertibility of this representation under a given holomorphic function $\Phi(z)$. The disadvantages remained the same and also for a sufficiently nice holomorphic function $\Phi(z)$ he can only stated that $w(z) \in W^1_s(\overline{D})$, where $s > 2$ is sufficiently close to.

Representation of second kind (4) is based on well-known Pompeiu's formulae [1]: if $w(z) \in W^1_p(\overline{D})$, $p > 2$, then

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint\limits_D \frac{\partial w}{\partial \bar z}\cdot\frac{d\xi d\eta}{\zeta - z}, \qquad z \in D,\\ w(z) &= -\frac{1}{2\pi i}\int\limits_\Gamma \frac{w(\zeta)}{\bar\zeta - \bar z}d\bar\zeta - \frac{1}{\pi}\iint\limits_D \frac{\partial w}{\partial z}\cdot\frac{d\xi d\eta}{\bar\zeta - \bar z}, \qquad z \in D. \end{aligned} \tag{6}$$

In view of this, as $A(z)$, $B(z)$, $R(z) \in L_p(\overline{D})$, for a solution $w(z) \in W^1_p(\overline{D})$, $p > 2$, to general equation (1) we can write the representation of second kind:

$$\Omega(w) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(\zeta)}{\zeta - z}d\zeta + TR(z) \tag{7}$$

where

$$\Omega(w) \equiv w(z) + T(q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} + A(z)w + B(z)\overline{w}). \tag{8}$$

Here, the issue on invertibility of the operator $\Omega$ arises naturally.

The main results of the present work are the following statements.

**Theorem 1.** *If $q_1(z)$, $q_2(z) \in C(\overline{D})$, $A(z)$, $B(z) \in L_p(\overline{D})$, $p > 2$, then $\Omega$ is a real linear isomorphism of the Banach space $W^1_p(\overline{D})$.*

**Theorem 2.** *If $q_1(z)$, $q_2(z)$, $A(z)$, $B(z) \in C^k_\alpha(\overline{D})$, $k \geqslant 0$, $0 < \alpha < 1$, then $\Omega$ is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{D})$.*

**Theorem 3.** *If $q_1(z)$, $q_2(z)$, $A(z)$, $B(z) \in W^k_p(\overline{D})$, $k \geqslant 1$, $p > 2$, then $\Omega$ is a real linear isomorphism of the Banach space $W^{k+1}_p(\overline{D})$.*

The representations of the second kind can be written basing on any operator "like $T$", that is, on the right inverse to the Cauchy-Riemann operator $\partial_{\bar z}$. Analogues of Theorems 1–3 based on a more complicated than (5) operator $T_n f(z)$ such that $\mathrm{Re}\{z^{-n}T_n f(z)\} = 0$, $z \in \Gamma$, $T_n f(z_m) = 0$, $z_m \in \Gamma$, $m = 1, \dots, n$, were obtained by the author in [3]. The proof of the isomorphic property of the operator $\Omega$ based on a simpler operator $T$ turned out to be more complicated.

The operator $T_n f(z)$ and the results of work [3] employed essentially the fact that $D$ is the unit circle. An approach proposed here, allows us to extend Theorems 1–3 to the case, when $D$ is a simply-connected bounded domain with the boundary of an appropriate smoothness. In order not to make the paper too bulky, here we do not make such extension.

The construction of solutions to system (1) in the class $W^1_p(\overline{D})$ for arbitrary $p > 2$ under the assumptions on the coefficients as in Theorem 1 was also made in works by V.S. Vinogradov [4], [5]. Instead of representations of second kind (7), in these works, as in [2], only two-dimensional singular integrable equations are employed; these equations turned out to be unavoidable in the present work, too. The results of work [5] on the Riemann-Hilbert boundary problem with a canonical boundary condition reproduces corresponding results in [3].

We provide extra two simple but important corollaries of Theorems 1–3.

**Theorem 4.** *Under the assumptions of Theorem 2, for each function* $w(z) \in C^{k+1}_\alpha(\overline{D})$, $k \geqslant 0$, *we have the apriori estimate:*

$$\|w\|_{C^{k+1}_\alpha(\overline{D})} \leqslant \text{const}\left\{\|\mathcal{D}w\|_{C^k_\alpha(\overline{D})} + \|w\|_{C^{k+1}_\alpha(\Gamma)}\right\},$$

*where* const *depends only on* $k$, $\alpha$ *and the norms in* $C^k_\alpha(\overline{D})$ *of the coefficients of the operator* $\mathcal{D}$.

**Theorem 5.** *Under the assumptions of Theorems 1, 3, for each function* $w(z) \in W^k_p(\overline{D})$, $k \geqslant 1$, *the apriori estimates*

$$\|w\|_{W^k_p(\overline{D})} \leqslant \text{const}\left\{\|\mathcal{D}w\|_{W^{k-1}_p(\overline{D})} + \|w\|_{W_p^{k-\frac{1}{p}}(\Gamma)}\right\}$$

*holds true, where* const *depends only on* $k$, $p$ *and* $W^{k-1}_p(\overline{D})$*-norms of the coefficients of the operator* $\mathcal{D}$ *as* $k > 1$, *while as* $k = 1$, *it depends on* $p$ *and the norms* $\|q_1\|_{C(\overline{D})}$, $\|q_2\|_{C(\overline{D})}$, $\|A\|_{L_p(\overline{D})}$, $\|B\|_{L_p(\overline{D})}$.

## 2. Auxiliary statements

**2.1. Operators $T$ and $\Pi$.** We denote

$$\Pi f(z) = \partial_z T f(z) = -\frac{1}{\pi}\iint\limits_D \frac{f(\zeta)}{(\zeta - z)^2}d\xi d\eta,$$

where the integral is understood in the sense of the Cauchy principal. The following lemma is true.

**Lemma 1.** *A singular operator* $\Pi$ *maps continuously the spaces* $C^k_\alpha(\overline{D})$, $k \geqslant 0$, $0 < \alpha < 1$, *and* $W^k_p(\overline{D})$, $k \geqslant 0$, $p > 2$, *into themselves. At that,* $\|\Pi\|_{L_2} = 1$ *and for each* $q_0$ *such that* $0 < q_0 < 1$ *there exists* $s_0 = s_0(q_0) > 2$ *such that* $q_0\|\Pi\|_{L_s} < 1$ *as* $2 < s < s_0$.

**Remark 1.** *The statement on the boundedness of* $\Pi$ *in* $C^k_\alpha(\overline{D})$, $k \geqslant 0$, $0 < \alpha < 1$, *and the properties of its norm in* $L_s$ *were proved in* [1, Ch. 1, Sect. 8, 9]. *A careful proof of the boundedness of* $\Pi$ *in* $W^k_p(\overline{D})$, $k \geqslant 0$, $p > 2$, *can be found in* [6].

Formula (5) and Lemma 1 imply immediately the next lemma.

**Lemma 2.** *The operator* $T$ *maps continuously* $C^k_\alpha(\overline{D})$, $k \geqslant 0$, $0 < \alpha < 1$, *into* $C^{k+1}_\alpha(\overline{D})$ *and* $W^k_p(\overline{D})$, $k \geqslant 0$, $p > 2$, *into* $W^{k+1}_p(\overline{D})$.

It is obvious that a similar statement is true for the operator

$$\overline{T}f(z) = \overline{\left(T\overline{f(z)}\right)},\ \ \partial_{\bar z}\overline{T}f(z) = f(z).$$

**2.2. Shifts.** We shall say that a contour $\mathcal{L}$ belongs to $C^k_\alpha$, $k \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, if there exists a homeomorphic mapping $\zeta = f(z)$ of the circumference $\Gamma$ on $\mathcal{L}$ in the class $C^k_\alpha(\Gamma)$, at that, the inverse mapping $z = f^{-1}(\zeta)$ is in the class $C^k_\alpha(\mathcal{L})$. In this case the mapping $\zeta = f(z)$ and the inverse one are called diffemorphism of contours $\Gamma$ and $\mathcal{L}$ of class $C^k_\alpha$.

We mention that if $s$ is the arc length on $\Gamma$, and $\sigma$ is the arc length on $\mathcal{L}$, then under the diffeomorphism the relations hold: $\zeta'_t(t) \neq 0$, where $\zeta'_t = \zeta'_s \cdot s'_t = \zeta'_s \cdot \overline{t'}_s$, $t$ is the affix of a point in the contour $\Gamma$; and, respectively, $z'_\tau(\tau) \neq 0$, where $z'_\tau = z'_\sigma \cdot \sigma'_\tau = z'_\sigma \cdot \overline{\tau'}_\sigma$, $\tau$ is the affix of a point in the contour $\mathcal{L}$.

Generalizing [7], for a function $\varphi(\zeta)$ defined on $\mathcal{L}$, we introduce the operator $\mathcal{W}\varphi(t) = \varphi(\zeta(t))$, where $\zeta = \zeta(t)$ is a diffeomorphic mapping of the contour $\Gamma$ on the contour $\mathcal{L} \in C^k_\alpha$ of class $C^k_\alpha$, $k \geqslant 1$. It is obvious that $\mathcal{W}$ is a linear bounded continuously invertible operator acting from $C^k_\alpha(\mathcal{L})$ into $C^k_\alpha(\Gamma)$.

Let $\zeta(t)$ be a diffeomorphism of the contour $\Gamma$ on the contour $\mathcal{L}$ of class $C^k_\alpha(\Gamma)$, $k \geqslant 1$, $0 \leqslant \alpha \leqslant 1$. We denote by

$$S_\Gamma\varphi(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau, \quad t \in \Gamma, \tag{9}$$

a one-dimensional singular integral operator. In the same way we define the operator $S_\mathcal{L}$.

We shall need certain properties of the superposition

$$\Psi\varphi(t) = \left(\mathcal{W}S_\mathcal{L}\mathcal{W}^{-1} - S_\Gamma\right)\varphi(t) = \frac{1}{\pi i}\int\limits_\Gamma \left[\frac{\zeta'(\tau)}{\zeta(\tau) - \zeta(t)} - \frac{1}{\tau - t}\right]\varphi(\tau)d\tau$$

established in [8].

**Theorem 6.** *If $\zeta(t) \in C^1_\alpha(\Gamma)$, $0 < \alpha \leqslant 1$, $\varphi(t) \in C_\beta(\Gamma)$, $0 < \beta \leqslant 1$, $\mu = \alpha + \beta \leqslant 2$, then as $\mu < 1$ we have $\Psi\varphi(t) \in C_\mu(\Gamma)$ and*

$$\|\Psi\varphi(t)\|_{C_\mu(\Gamma)} \leqslant \mathrm{const}\|\varphi(t)\|_{C_\beta(\Gamma)}, \tag{10}$$

*where the constant depends only on $\|\zeta\|_{C^1_\alpha(\Gamma)}$.*

*If $\mu = 1$, then $\Psi\varphi(t) \in C_{\mu-\varepsilon}(\Gamma)$ for all $\varepsilon$ such that $0 < \varepsilon < \mu$ with an estimate similar to* (10)*.*

*If $\mu > 1$, then $\Psi\varphi(t) \in C^1_{\mu-1}(\Gamma)$ and*

$$\|\Psi\varphi(t)\|_{C^1_{\mu-1}(\Gamma)} \leqslant \mathrm{const}\|\varphi(t)\|_{C_\beta(\Gamma)}, \tag{11}$$

*where the constant depends only on $\|\zeta\|_{C^1_\alpha(\Gamma)}$.*

**Corollary 1.** *If $\zeta(t) \in C^1_\alpha(\Gamma)$, $0 < \alpha \leqslant 1$, $\varphi(t) \in C^1_\beta(\Gamma)$, $0 < \beta \leqslant 1$, then $\Psi\varphi(t) \in C^1_\alpha(\Gamma)$ and*

$$\|\Psi\varphi(t)\|_{C^1_\alpha(\Gamma)} \leqslant \mathrm{const}\|\varphi(t)\|_{C^1_\beta(\Gamma)}, \tag{12}$$

*where the constant depends only on $\|\zeta\|_{C^1_\alpha(\Gamma)}$.*

**Remark 2.** *In work* [8]*, in Theorem 6, the unit circumference served as the curve $\mathcal{L}$ but the proof did not employed this. It is obvious that in the above formulation we can assume that $\mathcal{L}$ is a circumference and $\Gamma$ is its diffeomorphic image; on the change of the variable in a singular integral see, for instance,* [13]*.*

### 2.3. Space $W_p^{k-\frac{1}{p}}(\Gamma)$.

Following [9, Ch. 5], we denote by $W_p^{k-\frac{1}{p}}(\Gamma)$, $k \geqslant 1$, $p > 2$, the set of traces of the functions in the space $W^k_p(\overline{D})$. The norm of a function $f \in W_p^{k-\frac{1}{p}}(\Gamma)$ is defined as a norm in the space $W^k_p(\overline{D})$ of a harmonic in $D$ function with the boundary values $f$ on $\Gamma$. This is a Banach norm and singular operator (9) is bounded in the Banach space $W_p^{k-\frac{1}{p}}(\Gamma)$, see [10, Ch. 6, Sect. 1].

Since by Sobolev-Kondrashev embedding theorem $W^k_p(\overline{D}) \subset C^{k-1}_\beta(\overline{D})$, $\beta = \frac{p-2}{p}$, we have

$$W_p^{k-\frac{1}{p}}(\Gamma) \subset C^{k-1}_\beta(\Gamma), \qquad k \geqslant 1, \quad p > 2. \tag{13}$$

By $\mathcal{K}f(z)$ we denote the Cauchy type integral

$$\mathcal{K}f(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(t)}{t - z}dt, \qquad z \in D. \tag{14}$$

The following inequality

$$\|\mathcal{K}f(z)\|_{W_p^k(\overline{D})} \leqslant \text{const}\|f(t)\|_{W_p^{k-\frac{1}{p}}(\Gamma)}, \qquad k \geqslant 1, \quad p > 2, \tag{15}$$

holds, where const is independent of $f$, see [10, Ch. 6, Sect. 1].

### 2.4. Localization tools.

*2.4.1. Vicinity of boundary.* The following lemma holds.

**Lemma 3** ([3])**.** *Let $f(x)$ be an even monotonically increasing as $x > 0$ function of a real variable $x$, $f(x) \in C_\gamma^1[-\varepsilon, \varepsilon]$, $0 < \gamma < 1$, $\varepsilon > 0$, and $\|f\|_{C_\gamma^1[-\varepsilon,\varepsilon]} \to 0$ as $\varepsilon \to 0$.*

*Given a constant $\delta > 0$, there exists $\varepsilon > 0$ such that $f(x)$ can be continued on the segment $[-1, 1]$ keeping the parity, monotonicity as $x > 0$ and the belonging to the class $C_\gamma^1$; the function $f$ satisfies $f(x) \equiv \text{const} > 0$ in the vicinity of the point $x = \pm 1$ and*

$$\|f^*\|_{C_\gamma^1[-\varepsilon,\varepsilon]} < \delta,$$

*where $f^*$ is the continuation of $f$ on $[-1, 1]$. At that, the inequality holds:*

$$\left\|\frac{df^*}{dx}\right\|_{C[\varepsilon, 1]} \leqslant \left.\frac{df^*}{dx}\right|_{x=\varepsilon}.$$

Let $G$ be the unit circle $|\zeta| < 1$ in the complex $\zeta$-plane $E_\zeta$. In this plane we construct a special domain $G^*$ with the boundary $\mathcal{L}$ defined by the equation

$$\zeta(\theta) = \rho(\theta)e^{i\theta}, \ \ -\infty < \theta < \infty,$$

where $\rho(\theta) \in C_\gamma^1(-\infty, \infty)$ is a $2\pi$-periodic function, $0 < \gamma < 1$.

We define a function $\rho(\theta)$ so that $\rho(0) = 1$, $\rho(\theta) \geqslant 1$, and for $\theta \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, the function $\zeta(\theta)$ defines a circumference of a radius $r > 1$,

$$\rho'(\varepsilon) \geqslant |\rho'(\theta)|, \ \ \theta \in [\varepsilon, 2\pi - \varepsilon], \qquad \|\rho(\theta) - 1\|_{C_\gamma^1[0,2\pi]} < \delta, \tag{16}$$

where $\delta > 0$ is some constant, which will be specified later.

By Lemma 3, such function $\rho(\theta)$ exists. By $t$ we denote a point in the domain $G^*$ and we let $\overline{G}^* = G^* \cup \mathcal{L}$.

The following statement holds true [11].

**Lemma 4.** *Let $f(t) = \zeta$ be a conformal mapping of the domain $G^*$ onto the circle $G$, $f(0) = 0$, $f'(0) > 0$. There exists a sufficiently small number $\delta > 0$ such that once the second inequality in* (16) *is satisfied then $f \in C_\gamma^1(\overline{G}^*)$ and*

$$\|f(t) - t\|_{C_\gamma^1(\overline{G}^*)} \leqslant M \cdot \delta, \tag{17}$$

*where the constant $M$ is independent of $\gamma$.*

In what follows, the number $\delta$ in (16) is chosen less than $\frac{1}{2M}$ and appropriate for applying Lemma 4. By (17), these assumptions allow us to estimate as follows:

$$|1 - |f'(t)|| \leqslant \|f'(t) - 1\|_{C_\gamma(\overline{G}^*)} \leqslant \|f(t) - t\|_{C_\gamma^1(\overline{G}^*)} \leqslant \frac{1}{2},$$

which yields

$$\frac{1}{2} \leqslant |f'(t)| \leqslant \frac{3}{2}, \ \ \forall t \in \overline{G}^*, \qquad \frac{1}{2} \leqslant \|f'(t)\|_{C_\gamma(\overline{G}^*)} \leqslant \frac{3}{2}. \tag{18}$$

We identify the planes $E_z$ and $E_\zeta$ and hence, the unit circles $G$ and $D$.

Let $z_0 \in \Gamma = \partial D$. We apply the similarity transformation

$$z = \frac{1}{r} \cdot t \tag{19}$$

to $\overline{G}^*$ and by moving, we locate the image of a closed domain $\overline{G}^*$ so that the image of the point $t = 1$ is located at $z_0$, and the corresponding part of the boundary of the image $\overline{G}^*$ lay on $\Gamma$. By the first inequality in (16), the domain $G^*$ is mapped into the subdomain of class $C^1_\gamma$ of the circle $D$; this subdomain is denoted by $D_r$. The described mapping is denoted by $z = g(t)$. By Lemma 4, the conformal mapping

$$\varphi = g \circ f^{-1} : G \to G^* \to D_r, \qquad z = \varphi(\zeta),$$

belongs to the class $C^1_\gamma(\overline{G})$ and by (18), (19) we have:

$$\min_{\zeta\in\overline{G}} |\varphi'(\zeta)| \geqslant \frac{2}{3r}, \qquad \max_{\zeta\in\overline{G}} |\varphi'(\zeta)| \leqslant \frac{2}{r} \to 0, \quad r \to +\infty. \tag{20}$$

By (18) and (20), for $\zeta = \psi(z) = \varphi^{-1}(z)$ we get

$$\max_{\zeta\in\overline{D}_r} |\psi'(z)| \leqslant \frac{3r}{2}, \qquad \min_{\zeta\in\overline{D}_r} |\psi'(z)| \geqslant \frac{r}{2}, \qquad \|\psi'(z)\|_{C_\gamma(\overline{D}_r)} \leqslant 2r. \tag{21}$$

It is clear that as $r \to +\infty$, we have $\varepsilon \to 0$ and $\operatorname{diam} D_r \to 0$.

**Lemma 5.** *The function* $\omega(z) = \dfrac{\psi'(z)}{\overline{\psi'(z)}}$ *belongs to* $C_\gamma(\overline{D}_r)$ *and* $\|\omega(z)\|_{C_\gamma(\overline{D}_r)} \leqslant \mathrm{const}$*, where the constant is independent of* $r$*.*

*Proof.* Since $|\omega(z)| \equiv 1$, we need to estimate the Hölder constant for this function. Given $z_1$, $z_2 \in \overline{D}_r$, we obviously have the relation:

$$|\omega(z_1) - \omega(z_2)| = \frac{|(\psi'(z_1) - \psi'(z_2))\overline{\psi'(z_2)} + \psi'(z_2)(\overline{\psi'(z_2)} - \overline{\psi'(z_1)})|}{|\psi'(z_1)| \cdot |\psi'(z_2)|}.$$

In view of (21), this implies:

$$|\omega(z_1) - \omega(z_2)| \leqslant \frac{12}{r} |\psi'(z_1) - \psi'(z_2))| \leqslant 24 \cdot |z_1 - z_2|^\gamma.$$

□

**Remark 3.** *In the above constructions, the number* $\gamma$, $0 < \gamma < 1$*, can be arbitrary and not related with the Hölder exponent in the formulation of Theorem 2.*

**Remark 4.** *It is clear that if as the domain* $D_r$ *we choose a circle of radius* $1/r$ *contained in* $D$*, then estimates* (20)*,* (21) *and Lemma 5 remain true.*

*2.4.2. Partition of the unity.* We denote by $\mathcal{U} = \{U_l\}$ a finite covering of a closed circle $\overline{D}$, $U_l \subset \overline{D}$ consisting of open circles of a fixed radius $1/r$ and contained in $D$ and of domains of type $D_r$ adjoining the boundary $\Gamma$ and described in the previous section. The number $r$ is assumed to be large enough; this will be specified later while using the covering $\mathcal{U}$.

By $\mathcal{H} = \{h_l\}$ we denote the partition of the unity of class $C^\infty$ on $\overline{D}$ associated with the covering $\mathcal{U}$, that is, all functions $h_l$ are non-negative, $h_l \in C^\infty(\overline{D})$, the support of each function obeys $\operatorname{supp} h_l \subset U_l$ and $\sum_l h_l(z) = 1$ for all $z \in \overline{D}$. The existence of such partition of the unity is due to [12, Ch. II, Sect. 4].

**Lemma 6.** *Let* $f(t) \in C_\beta(\Gamma)$, $0 < \beta < 1$, $\mathcal{K}f(z)$ *be the integral of Cauchy type* (14)*,* $h_l$ *is an element of the partition of the unity* $\mathcal{H}$*.*

*If* $\mathcal{K}f(z) \in A^k_\alpha$, $k \geqslant 0$, $0 < \alpha < 1$ *(*$A^{k+1}_p$, $p > 2$*), then* $\mathcal{K}(h_l f)(z) \in A^k_\alpha$ *(*$A^{k+1}_p$*).*

*Proof.* It is clear that as $k = 0$, we have $\alpha \leqslant \beta$, and as $\operatorname{supp} h_l \cap \Gamma = \varnothing$, the statement becomes trivial. Let $\operatorname{supp} h_l \cap \Gamma \neq \varnothing$.

We consider the expression

$$h_l(z)\mathcal{K}f(z) - \mathcal{K}(h_l f)(z) = P(z) \in C_\beta(\overline{D}).$$

By the Sokhotski-Plemelj formulae [13, Ch. 1, Sect. 4], the limiting values of the function $P(z)$ as $z \to \tau \in \Gamma$ read as

$$P^+(\tau) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{h_l(\tau) - h_l(t)}{t - \tau} \cdot f(t) \cdot dt.$$

Since $h_l \in C^\infty(\Gamma)$, by the Taylor formula with the residual error in the integral form we immediately get that

$$\frac{h_l(\tau) - h_l(t)}{t - \tau} \in C^\infty(\Gamma),$$

where the quotient is regarded as a function of $\tau$. Hence, $P^+(\tau) \in C^\infty(\Gamma)$.

At the same time, $h_l(\tau)\mathcal{K}^+ f(\tau) \in C^k_\alpha(\Gamma)$ $(W^{k+1-\frac{1}{p}}(\Gamma))$, and this implies $\mathcal{K}^+(h_l f)(\tau) \in C^k_\alpha(\Gamma)$ $(W^{k+1-\frac{1}{p}}(\Gamma))$. In view of the properties of the Cauchy type integral, [1, Ch. 1, Sect. 3] and (15), and the Cauchy formula for holomorphic functions, we arrive at the statement of the lemma. The proof is complete. □

### 2.5. Regularity and uniqueness of solutions.

**Lemma 7.** *If under ellipticity condition* (2)*, the coefficients in equation* (1) *satisfy*

$$q_1(z),\, q_2(z) \in C^{k+1}_\alpha(D), \qquad k \geqslant 0, \quad 0 < \alpha < 1 \qquad (W^{k+1}_p(D), \quad k \geqslant 0, \quad p > 2),$$
$$A(z),\, B(z), R(z) \in C^k_\alpha(D), \qquad k \geqslant 0, \quad 0 < \alpha < 1 \qquad (W^k_p(D), \quad k \geqslant 0, \quad p > 2),$$

*then each solution of this equations $w(z) \in W^1_s(D)$, $s > 2$, belongs to $C^{k+1}_\alpha(D)$ $(W^{k+1}_p(D))$.*

This lemma follows the results in [1, Ch. 2, Sect. 7; Ch. 4, Sect. 7].

**Lemma 8.** *If under ellipticity condition* (2)*, the functions $q_1(z)$, $q_2(z)$ are measurable, $A(z)$, $B(z) \in L_p(\overline{D})$, $p > 2$, $R(z) \equiv 0$, and the trace of a solution $w(z) \in W^1_s(\overline{D})$, $s > 2$, on the boundary vanishes on a set of a positive linear measure on $\Gamma$, then $w(z) \equiv 0$.*

The lemma is implied immediately by Theorem 4.4 in work [2], see also [1, Ch. 3, Sect. 17].

## 3. Proof of main results

The scheme of the proof is as follows. We first prove the unique solvability of the equation

$$\Omega(w) = F \in W^1_p(\overline{D}), \qquad p > 2, \tag{22}$$

in the space $W^1_s(\overline{D})$, where $2 < s \leqslant p$ and $s$ is sufficiently close to 2. Then Theorems 1, 2 are proved for constant coefficients $q_1$ and $q_2$ and $A(z) = B(z) \equiv 0$. Then by a version of a method of local regularization with freezing coefficients developed in [3] we obtain general statements of these theorems. The proof of Theorem 3 is a minor modification of the proof of Theorem 2.

### 3.1. Solvability of equation (22) in $W^1_s(\overline{D})$.

**Lemma 9.** *If $q_1(z)$, $q_2(z)$ are measurable functions obeying condition* (2)*, and $A(z)$, $B(z) \in L_s(\overline{D})$, $s > 2$, then the equation*

$$\Omega(w) = 0 \tag{23}$$

*has only the trivial solution in the class $W^1_s(\overline{D})$.*

*Proof.* We argue by contradiction assuming that a solution $w(z) \in W^1_s(\overline{D}) \subset C_\beta(\overline{D})$, $\beta = \dfrac{s-2}{s}$, to equation (23) does not vanish identically. It is known that as $f(z) \in L_s(\overline{D})$, the function $Tf(z) \in C_\beta(E)$ is holomorphic outside $\overline{D}$ and $Tf(\infty) = 0$ [1]. Thus, it follows from (23) that $w(z)$ can be holomorphically continued outside $\overline{D}$ and vanishes at infinity.

On the other hand, differentiating (23) with respect to $\bar{z}$, we obtain that $w(z)$ solves homogeneous equation (1) (as $F(z) \equiv 0$). By (6) this implies that $w(z)$ satisfies the equation

$$\Omega(w) = \Phi(z), \qquad \Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{w(t)}{t-z} dt, \tag{24}$$

that is, the Cauchy type integral in the right hand side in (24) is continuous on the entire plane and vanishes at infinity. This is possible only as $w(t) \equiv 0$, $t \in \Gamma$ [13]. At the same time, a continuous in $\overline{D}$ solution of homogeneous equation (1) vanishing on $\Gamma$ is identically zero, see Lemma 8. □

**Lemma 10.** *If $q_1(z)$, $q_2(z)$ are measurable functions obeying condition* (2)*, and $A(z)$, $B(z) \in L_p(\overline{D})$, $p > 2$, then equation* (22) *possesses a unique solution $w(z) \in W^1_s(\overline{D})$, where $2 < s \leqslant p$ and $s$ is sufficiently close to* 2.

*Proof.* The uniqueness is due to Lemma 9. Let us show that the operator

$$\Omega_1(w) = w + T(q_1 \partial_z w + q_2 \partial_{\bar{z}} \overline{w}) \tag{25}$$

has a bounded inverse in $W^1_s(\overline{D})$ if $2 < s \leqslant p$ and $s$ is sufficiently close to 2.

We consider the equations

$$\Omega_1(w) = \omega \in W^1_s(\overline{D}), \tag{26}$$

$$\lambda + \Pi(q_1\lambda + q_2\overline{\lambda}) \equiv \lambda + \sigma\lambda = \partial_z \omega. \tag{27}$$

Equation (27) is obtained by differentiating (26) in $z$ by replacing $\partial_z w$ by $\lambda(z)$. By Lemma 1, there exists $2 < s < s_0(q_0) \leqslant p$ such that $q_0 \|\Pi\|_{L_s} < 1$. Hence, by the contracting mapping principle, equation (27) is uniquely solvable in $L_s(\overline{D})$:

$$\lambda(z) = (I + \sigma)^{-1} \partial_z \omega(z) \tag{28}$$

and the norm of the linear operator $(I + \sigma)^{-1} : L_s(\overline{D}) \to L_s(\overline{D})$ is bounded by a constant depending on $q_0$ only.

We seek a solution to equation (26) as

$$w(z) = \overline{T\overline{\lambda}} + \Psi(\bar{z}), \tag{29}$$

where $\Psi(z)$ is a holomorphic in $D$ function belonging to $W^1_s(\overline{D})$. Substituting (29) into (26), we obtain

$$\Psi(\bar{z}) = \omega(z) - \overline{T\overline{\lambda}} - T(q_1\lambda + q_2\overline{\lambda}). \tag{30}$$

Since $Tf(z) \in W^1_s(\overline{D})$ as $f(z) \in L_s(\overline{D})$ [1, Ch. 1, Sect. 6], by (5), (27) we get $\partial_z \Psi(\bar{z}) = 0$, that is, the function $\Psi(z)$ defined by formula (30) is holomorphic and belongs to $W^1_s(\overline{D})$. Thus, the formula

$$w(z) = \omega(z) - T\left(q_1(z)(I+\sigma)^{-1}\partial_z \omega + q_2(z)\overline{(I+\sigma)^{-1}\partial_z \omega}\right) \tag{31}$$

provides a solution to equation (26). Since by the properties of the operator $T$ (see Lemma 2) the operator $\Omega_1 : W^1_s(\overline{D}) \to W^1_s(\overline{D})$ is continuous, by the Banach theorem, the inverse operator $\Omega_1^{-1} : W^1_s(\overline{D}) \to W^1_s(\overline{D})$ is also continuous; this can also obtained straightforwardly from (31).

We rewrite equation (22) as

$$w + \Omega_1^{-1} \circ Pw = \Omega_1^{-1} F, \tag{32}$$

where $Pw = T(Aw + B\overline{w})$. Since the operator $P$ is completely continuous in $C(\overline{D})$ and maps $C(\overline{D})$ into $W^1_p(\overline{D})$ [1, Ch. 1, Sect. 6], then the operator $\Omega_1^{-1} \circ P$ is completely continuous in $C(\overline{D})$ and maps $C(\overline{D})$ into $W^1_s(\overline{D})$.

In view of the mentioned properties of the operators involved in (32), a continuous in $\overline{D}$ solution of homogeneous equation (32) belongs to $W^1_s(\overline{D})$ and by Lemma 9 it vanishes identically.

Thus, by Fredholm theorem, equation (32) is uniquely solvable in $C(\overline{D})$ and its solution $w(z)$ belongs to $W^1_s(\overline{D})$. □

**Remark 5.** *It follows from* (31) *and the properties of the operator $T$ that the norm of the linear operator $\Omega_1^{-1} : W^1_s(\overline{D}) \to W^1_s(\overline{D})$ is bounded by a number depending on $q_0$ only* [1, Ch. 1, Sect. 6]*.*

### 3.2. Case of constant coefficients $q_1$ and $q_2$ of the operator $\Omega_1$.

**Lemma 11.** *If $q_1(z) = \text{const}$, $q_2(z) = \text{const}$, then the operator $\Omega_1$ defined by formula* (25) *is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{D})$, $k \geqslant 0$, $0 < \alpha < 1$ ($W^{k+1}_p(\overline{D})$, $k \geqslant 0$, $p > 2$).*

*Proof.* It follows from the properties of the operator $T$ (see Lemma 2) that the mapping

$$\Omega_1 : C^{k+1}_\alpha(\overline{D}) \to C^{k+1}_\alpha(\overline{D}) \qquad \big(W^{k+1}_p(\overline{D}) \to W^{k+1}_p(\overline{D})\big)$$

is continuous. Thanks to the Banach theorem, it is sufficient to show that it is surjective.

We consider the equation

$$\Omega_1(w) = F(z) \in C^{k+1}_\alpha(\overline{D}) \quad \big(W^{k+1}_p(\overline{D})\big)\,. \tag{33}$$

By Lemma 10, equation (33) possesses the only solution $w(z) \in W^1_s(\overline{D})$, where $s > 2$ is sufficiently close to 2. Let us show that this solution belongs to the class $C^{k+1}_\alpha(\overline{D})$ $\big(W^{k+1}_p(\overline{D})\big)$.

Employing (6), we rewrite equation (33) as

$$w(z) + T(q_1\partial_z w + q_2\partial_{\bar z}\overline{w}) = \Phi(z) + T\partial_{\bar z}F(z), \tag{34}$$

where

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{F(t)}{t-z}dt \in A^{k+1}_\alpha \qquad \big(A^{k+1}_p\big)\,.$$

We apply the operator

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{\bullet}{z-\zeta}dz, \qquad \zeta \in D,$$

to (34), we obtain that $\Phi(z)$ is also represented by the formula

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(t)}{t-z}dt. \tag{35}$$

First we suppose that $q_1 = 0$, that is, we consider the equation

$$w + T(q_2\partial_{\bar z}\overline{w}) = \Phi(z) + T\partial_{\bar z}F. \tag{36}$$

Differentiating (36) with respect to $\bar z$, we obtain

$$\partial_{\bar z}(w + q_2\overline{w}) = \partial_{\bar z}F,$$

that is, by (6), the function $w(z)$ satisfies the relation

$$w(z) + q_2\overline{w(z)} = \Phi(z) + q_2\Psi(z) + T\partial_{\bar z}F, \tag{37}$$

where $\Phi(z)$ is given by formula (35), and

$$\Psi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\overline{w(t)}}{t-z}dt. \tag{38}$$

Passing to the limit as $z \to \tau \in \Gamma$, by the Sokhotski-Plemelj formulae [13], for the limiting values $\Psi^+(\tau)$ of the function $\Psi(z)$ we obtain the following expression:

$$\Psi^+(\tau) = \frac{1}{2}\overline{w(\tau)} + \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\overline{w(t)}}{t-\tau}dt.$$

Since on $\Gamma$ the identities $t \cdot \bar{t} = 1$, $\tau \cdot \bar{\tau} = 1$ hold, the expression for $\Psi^+(\tau)$ can be transformed as follows:

$$\Psi^+(\tau) = \overline{w(\tau)} - \overline{\Phi^+(\tau)} - C, \tag{39}$$

where

$$\Phi^+(\tau) = \frac{1}{2}w(\tau) + \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w(t)}{t-\tau}dt, \quad C = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\overline{w(t)}}{\bar{t}}d\bar{t}. \tag{40}$$

Thus, by (37) we get

$$w(\tau) = \Phi^+(\tau) - q_2\overline{\Phi^+(\tau)} - q_2 C + T\partial_{\bar{z}}F(\tau) \in C^{k+1}_{\alpha}(\Gamma)\ \left(W_p^{k+1-\frac{1}{p}}(\Gamma)\right).$$

But in this case $\overline{w(\tau)} \in C^{k+1}_{\alpha}(\Gamma)\ \left(W_p^{k+1-\frac{1}{p}}(\Gamma)\right)$, and hence, $\Psi(z) \in C^{k+1}_{\alpha}(\overline{D})\ \left(W^{k+1}_p(\overline{D})\right)$ (see [1], (15)) and

$$w(z) + q_2\overline{w(z)} = F_0(z) \in C^{k+1}_{\alpha}(\overline{D}) \qquad \left(W^{k+1}_p(\overline{D})\right).$$

This implies

$$w(z) = \frac{1}{1-|q_2|^2}\left(F_0(z) - q_2\overline{F_0(z)}\right) \equiv \Xi(\Phi,\, F)(z) \in C^{k+1}_{\alpha}(\overline{D}) \quad \left(W^{k+1}_p(\overline{D})\right) \tag{41}$$

and the particular case $q_1 = 0$ is complete.

**Remark 6.** *We note that this yields that if $q_1 = 0$ and $w(z) \in W^1_s(\overline{D})$, $s > 2$, $\Phi^+(t) \in C^n_\gamma(\Gamma)$, where $0 \leqslant n \leqslant k+1$, $0 < \gamma \leqslant \alpha < 1$, then $w(z) \in C^n_\gamma(\overline{D})$.*

The following lemma is true.

**Lemma 12.** *If $\Phi(z) \in A^n_\gamma(\overline{D})$, where $0 \leqslant n \leqslant k+1$, $0 < \gamma \leqslant \alpha < 1$, then formula (41) determines uniquely in $D$ the solution of the equation*

$$\partial_{\bar{z}}w + q_2\partial_{\bar{z}}\overline{w} = \partial_{\bar{z}}F$$

*in the class $C^n_\gamma(\overline{D})$, for which the function $\Phi(z)$ can be represented by formula (35).*

Now let $q_1 \neq 0$. We shall argue by induction. First let us show that the lemma is true for $k = 0$.

We denote

$$\mu = \frac{2q_1}{1 + |q_1|^2 - |q_2|^2 + \sqrt{\Delta}} = \text{const},$$

where

$$\Delta = (1 + |q_1|^2 - |q_2|^2)^2 - 4|q_1|^2 \geqslant (1 - q_0^2)^2 > 0,$$

$q_0$ is the ellipticity constant in (2). It is easy to see that $|\mu| < 1$.

We denote by $\zeta = \zeta(z)$ the principal homeomorphism of the Beltrami equation

$$\partial_{\bar{z}}\zeta + \mu\partial_z\zeta = 0$$

mapping the unit circle $\overline{D}_z = \{z : |z| \leqslant 1\}$ onto the unit circle $\overline{D}_\zeta = \{\zeta : |\zeta| \leqslant 1\}$ with the normalization $\zeta(0) = 0$, $\zeta(1) = 1$. As it is known, $\zeta(z) \in C^\infty(\overline{D}_z)$ [14].

In the equation

$$\partial_{\bar{z}}w + q_1\partial_z w + q_2\partial_{\bar{z}}\overline{w} = \partial_{\bar{z}}F(z) \tag{42}$$

we pass to the argument $\zeta = \zeta(z)$ and we denote $w(z(\zeta)) = w(\zeta)$, where $z = z(\zeta) \in C^\infty(\overline{D}_\zeta)$ is the mapping inverse to $\zeta = \zeta(z)$. Equation (42) is transformed to form [15]:

$$\partial_{\bar\zeta} w + a\partial_{\bar\zeta}\overline{w} = R_1(\zeta) \in C^k_\alpha(\overline{D}_\zeta) \ \left(W^k_p(\overline{D}_\zeta)\right), \tag{43}$$

where

$$a = \frac{q_2(1-|\mu|^2)}{|1-q_1\bar\mu|^2 - |q_2\mu|^2} = \text{const}, \qquad |a| < 1.$$

Similar to (36), the function $w = w(\zeta)$ satisfies the integro-differential equation

$$w(\zeta) + aT(\partial_{\bar\zeta}\overline{w}) = \Phi_1(\zeta) + T\partial_{\bar\zeta}R_1, \tag{44}$$

where

$$\Phi_1(\zeta) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(z(\omega))}{\omega - \zeta}d\omega = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(t)}{\zeta(t)-\zeta}\zeta'(t)dt, \tag{45}$$

on the change of the variable in a singular integral see [13].

We observe that by the Sobolev-Kondrashev embedding theorem we have $w(z) \in W^1_s(\overline{D}_z) \subset C_\beta(\overline{D}_z)$, where $\beta = (s-2)/s$ [1], that is, $w(t) \in C_\beta(\Gamma_z)$ and $w(\zeta) = w(z(\zeta)) \in C_\beta(\Gamma_\zeta)$.

We consider the formula

$$\Psi w(t) = \frac{1}{2\pi i}\int\limits_{\Gamma_z}\left[\frac{\zeta'(\tau)}{\zeta(\tau)-\zeta(t)} - \frac{1}{\tau - t}\right]w(\tau)d\tau = \Phi_1^+(\zeta(t)) - \Phi^+(t),$$

where $\Phi^+(t)$ are defined by formula (40), and

$$\Phi_1^+(\zeta) = \frac{1}{2}w(\zeta) + \frac{1}{2\pi i}\int\limits_{\Gamma_\zeta}\frac{w(\omega)}{\omega-\zeta}d\omega.$$

Since $\zeta(\tau) \in C^\infty(\Gamma_z)$, by Theorem 6 $\Psi w(t) \in C^1_\beta(\Gamma_z)$, and since $\Phi^+(t) \in C^1_\alpha(\Gamma_z)$ $(W_p^{1-\frac{1}{p}}(\Gamma_z))$, then, if $\beta < \alpha$, we have $\Phi_1^+(\zeta(t)) \in C^1_\beta(\Gamma_z)$ $(W_p^{1-\frac{1}{p}}(\Gamma_z))$ and $\Phi_1^+(\zeta) \in C^1_\beta(\Gamma_\zeta)$ $(W_p^{1-\frac{1}{p}}(\Gamma_\zeta))$. This yields $w(\zeta) \in C^1_\beta(\overline{D}_\zeta)$, see Remark 6, and respectively, $w(\zeta) \in W^1_p(\overline{D}_\zeta))$ and $w(z) \in C^1_\beta(\overline{D}_z)$ $(w(z) \in W^1_p(\overline{D}_z))$, while $w(t) \in C^1_\beta(\Gamma_z)$.

If $\beta < \alpha$, by Corollary 1 we get $\Psi(t) \in C^1_\alpha(\Gamma_z)$ and $\Phi_1^+(\zeta) \in C^1_\alpha(\Gamma_\zeta)$. Hence, $w(\zeta) \in C^1_\alpha(\overline{D}_\zeta)$ and $w(z) \in C^1_\alpha(\overline{D}_z)$. If $\beta \geqslant \alpha$, the arguing obviously becomes simpler. Thus, we have proved the lemma in the case $k = 0$.

Now assume that Lemma 11 is true for some $k = m-1$, $m \geqslant 1$, and let us show that it is valid also for $k = m$. By the induction assumption, if the right hand side in equation (33) belongs to the class $C^m_\alpha(\overline{D})$ $(W^m_p(\overline{D}))$, then the solution of this equation satisfies $w(z) \in C^m_\alpha(\overline{D})$ $(W^m_p(\overline{D}))$.

Suppose that the right hand side of equation (33) belongs to the class $C^{m+1}_\alpha(\overline{D})$ $(W^{m+1}_p(\overline{D}) \subset C^m_\beta(\overline{D})$, $\beta = (p-2)/p)$. We denote $z = re^{is}$ and we introduce the function

$$\frac{\partial w(re^{is})}{\partial s} \equiv w_s(z) = i(zw_z - \bar z w_{\bar z}). \tag{47}$$

By Lemma 7, $w(z) \in C^{m+1}_\alpha(D)$ $(W^{m+1}_p(D))$, that is,

$$w_s(z) \in C^m_\alpha(D) \cap C^{m-1}_\alpha(\overline{D}), \qquad (W^m_p(D) \cap C^{m-1}_\beta(\overline{D})),$$

and this is why the functions

$$\frac{\partial w_s}{\partial \bar z} = i(zw_{z\bar z} - w_{\bar z} - \bar z w_{\bar z\bar z}), \qquad \frac{\partial w_s}{\partial z} = i(w_z + zw_{zz} - \bar z w_{\bar z z}),$$
$$\frac{\partial \overline{w}_s}{\partial \bar z} = -i(\overline{w}_{\bar z} + \overline{z}\overline{w}_{\bar z\bar z} - z\overline{w}_{z\bar z}) \tag{48}$$

are well-defined in $D$. By (48), (33) we have

$$\begin{aligned}\frac{\partial w_s}{\partial \bar z}+q_1\frac{\partial w_s}{\partial z}+q_2\frac{\partial \overline{w}_s}{\partial \bar z}=&i(-w_{\bar z}+q_1w_z-q_2\overline{w}_{\bar z})-i\bar z\frac{\partial}{\partial \bar z}(w_{\bar z}+q_1w_z+q_2\overline{w}_{\bar z})\\&+iz\frac{\partial}{\partial z}(w_{\bar z}+q_1w_z+q_2\overline{w}_{\bar z})\\=&2iq_1w_z-i\bar zF_{\bar z\bar z}+izF_{\bar z z}-iF_{\bar z}.\end{aligned}$$

Thus, $w_s(z)$ solves the equation

$$\frac{\partial w_s}{\partial \bar z}+q_1\frac{\partial w_s}{\partial z}+q_2\frac{\partial \overline{w}_s}{\partial \bar z}=R_s(z)\in C^{m-1}_\alpha(\overline{D})\ (W^{m-1}_p(\overline{D})) \tag{49}$$

in $D$. Our next aim is to prove that $w_s(z)\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D}))$.

We denote

$$\Phi_s(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{w_s(t)}{t-z}dt.$$

The relation holds [1]:

$$\Phi'(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{w_s(t)\bar t'_s}{t-z}dt.$$

Taking into consideration that on $\Gamma$ the identities $t\cdot\bar t=1$, $\bar t'(s)=-i\bar t$ hold, we get

$$\Phi_s(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{w_s(t)}{t}dt+iz\Phi'(z)\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D})). \tag{50}$$

As $m=1$, by (50) and Lemma 13 we have $w_s(z)\in C^1_\alpha(\overline{D})\ (W^1_p(\overline{D}))$. If $m>1$, then $w_s(z)$ solves the equation

$$\Omega_1 w_s=TR_s+\Phi_s\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D})),$$

and therefore, by the induction assumption, $w_s(z)\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D}))$.

By (47) and (42) we obtain

$$w_z=\frac{-i[w_s(\bar z+z\bar q_1)+\bar zq_2\overline{w}_s]+\bar zF_{\bar z}(\bar z+z\bar q_1)-|z|^2q_2\overline{F}_z}{|z+\bar zq_1|^2-|z|^2|q_2|^2}. \tag{51}$$

Since $|q_1|+|q_2|<1$, the denominator of the quotient in (51) does not vanish as $z\neq 0$.

By (51), $w_s(z)\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D}))$ and $w(z)\in C^{m+1}_\alpha(D)\ (W^{m+1}_p(D))$ we obtain $w_z(z)\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D}))$. By (42) this yields $w_{\bar z}\in C^m_\alpha(\overline{D})\ (W^m_p(\overline{D}))$. Thus, $w(z)\in C^{m+1}_\alpha(\overline{D})$ $(W^{m+1}_p(\overline{D}))$ and this completes the proof. □

We note that the arguing in the above proof in the case $k=0$, formula (41) and Lemma 12 implies the following statement.

**Lemma 13.** *If*

$$\Phi(z)=\frac{1}{2\pi i}\int\limits_\Gamma\frac{w(t)}{t-z}dt\in A^1_\alpha,\ \ 0<\alpha<1,\ \ (A^1_p,\ p>2),$$

$w(z)\in C^1(D)\cap C_\alpha(\overline{D})$ *(*$W^1_p(D)\cap C_\beta(\overline{D})$, $0<\beta<1$*) is a solution of equation* (42), *then the function* $w(z)$ *can be represented by the formula*

$$w(z)=\Xi(\Phi,\,R_1)(\zeta(z)),$$

*and therefore,* $w(z)\in C^1_\alpha(\overline{D})$ *(*$W^1_p(\overline{D})$*).*

**3.3. Proof of Theorem 1.** First we prove the theorem for a particular case, for operator (25). By Lemma 10, the equation

$$\Omega_1(w) = F \in W_p^1(\overline{D}) \tag{52}$$

has the unique solution $w(z) \in W_s^1(\overline{D})$, $2 < s \leqslant p$. Since the operator $\Omega_1$ maps continuously $W_p^1(\overline{D})$ into $W_p^1(\overline{D})$, in view of the Banach theorem it is sufficient to show that $w(z) \in W_p^1(\overline{D})$.

On $\overline{D}$ we consider the partition of the unity $\mathcal{H} = \{h_l\}$ described in Section 2.4.2 and a set of functions $\{w_l(z)\}$, $w_l = h_l \cdot w \in W_s^1(\overline{D})$. Since

$$\sum_l w_l(z) = w(z) \quad \text{for all} \quad z \in \overline{D},$$

it is sufficient to show that $w_l(z) \in W_p^1(\overline{D})$ for all $l$, or, equivalently, $w_l(z) \in W_p^1(\overline{U}_l)$.

The solution $w(z)$ of equation (52) satisfies the differential equation

$$\partial_{\bar z} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} = \partial_{\bar z}F(z) \in L_p(\overline{D}),$$

this is why the function $w_l(z)$ satisfies the differential equation

$$\partial_{\bar z} w_l + q_1(z)\partial_z w_l + q_2(z)\partial_{\bar z}\overline{w}_l = F_l(z) \in L_p(\overline{D}), \tag{53}$$

where

$$F_l(z) = \partial_{\bar z}F + \partial_{\bar z}h_l \cdot (w_l + q_2\overline{w}_l) + q_1 \cdot w_l \cdot \partial_z h_l.$$

By (6) this implies that $w_l(z)$ satisfies the integro-differential equation

$$w_l(z) + T_{U_l}(q_1\partial_z w_l + q_2\partial_{\bar z}\overline{w}_l) = T_{U_l}F_l + \Phi_l(z), \tag{54}$$

where

$$\Phi_l(z) = \frac{1}{2\pi i}\int\limits_{\partial U_l} \frac{w_l(t)dt}{t-z}.$$

We note that if $\operatorname{supp} h_l \cap \Gamma = \varnothing$, then $\Phi_l(z) \equiv 0$, and if $\operatorname{supp} h_l \cap \Gamma \neq \varnothing$, then

$$\Phi_l(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w_l(t)dt}{t-z}, \tag{55}$$

that is, formula (55) is valid in each case.

In (52), similar to (34), we have

$$F(z) = T_D\partial_{\bar z}F(z) + \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w(t)dt}{t-z} \in W_p^1(\overline{D}),$$

that is,

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{w(t)dt}{t-z} \in A_p^1. \tag{56}$$

By Lemma 6 this implies $\Phi_l(z) \in A_p^1 \subset C_\beta(\overline{D})$, $\beta = (p-2)/p$.

We denote by $\zeta = \psi(z)$ the conformal mapping of the domain $D_r = U_l$ onto the unit circle $G = \{\zeta : |\zeta| < 1\|$, $\partial G = \Gamma^*$, described in Section 2.4.1. If $U_l \cap \Gamma = \varnothing$, this is a similarity transformation, see Remark 4. We choose the number $\gamma : 0 < \gamma < 1$, $\partial D_r \in C_\gamma^1$, such that $\beta + \gamma > 1$, see Remark 3.

In differential equation (53) we pass to the argument $\zeta$ and we denote $w_l(\zeta) = w_l(\varphi(\zeta))$, where $\varphi = \psi^{-1}$. Equation (53) then casts into the form

$$\partial_{\bar\zeta} w_l + q_1^*(\zeta)\partial_\zeta w_l + q_2^*(\zeta)\partial_{\bar\zeta}\overline{w}_l = F_l^*(\zeta), \tag{57}$$

where

$$q_1^*(\zeta)=q_1(\varphi(\zeta))\cdot\frac{\psi'(\varphi(\zeta))}{\overline{\psi'(\varphi(\zeta))}},\qquad q_2^*(\zeta)=q_2(\varphi(\zeta)),\qquad F_l^*(\zeta)=\overline{\varphi'(\zeta)}\cdot F_l(\varphi(\zeta)),\tag{58}$$

$$|q_1^*(\zeta)|+|q_2^*(\zeta)|\leqslant q_0=\mathrm{const}<1,\tag{59}$$

$$q_1^*(\zeta),\, q_2^*(\zeta)\in C(\overline{G}),\qquad F_l^*(\zeta)\in L_p(\overline{G}).$$

Similar to (54), the function $w_l(\zeta)$ satisfies the integro-differential equation

$$w_l(\zeta)+T_G(q_1^*\partial_\zeta w_l+q_2^*\partial_{\bar\zeta}\overline{w}_l)=T_GF_l^*(\zeta)+\Phi_l^*(\zeta),\tag{60}$$

where

$$\Phi_l^*(\zeta)=\frac{1}{2\pi i}\int\limits_{\Gamma^*}\frac{w_l(\tau)d\tau}{\tau-\zeta}=\frac{1}{2\pi i}\int\limits_{\partial U_l}\frac{w_l(t)\psi'(t)dt}{\psi(t)-\zeta}.\tag{61}$$

By Theorem 6 and the inequality $\beta+\gamma>1$ we obtain

$$\Psi w_l(t)=\frac{1}{\pi i}\int\limits_{\partial U_l}\left[\frac{\psi'(\lambda)}{\psi(\lambda)-\psi(t)}-\frac{1}{\lambda-t}\right]w_l(\lambda)d\lambda\in C^1_{\beta+\gamma-1}(\partial U_l).$$

Since by the Sokhotski-Plemelj formulae [13, Ch. 1, Sect. 4] we have

$$\Psi w_l(t)=[\Phi_l^*(\psi(t))]^+-\Phi_l^+(t),\ \ t\in\partial U_L,\tag{62}$$

and $\psi(z)\in C^1_\gamma(\overline{U}_l)$ and $\Phi_l^+(z)\in A^1_p(\overline{D})$, by (62) we get $[\Phi_l^*(\psi(t))]^+\in W_p^{1-\frac{1}{p}}(\partial U_l)$, $\Phi_l^*(\psi(z))\in A^1_p(\overline{U}_l)$ and $\Phi_l^*(\zeta)\in A^1_p(\overline{G})$. Thus, for the right hand side of equation (60) we have

$$F_l^{**}(\zeta)=T_GF_l^*(\zeta)+\Phi_l^*(\zeta)\in W^1_p(\overline{G}).\tag{63}$$

We fix a point $z_l$ in $\overline{U}_l=\overline{D}_r$. To be definite, we suppose that if $U_l$ is a circle, then $z_l$ is its center and if $U_l$ is an adjoining to $\Gamma$ domain then $z_l\in\operatorname{supp}h_l\cap\Gamma$. We denote $\zeta_l=\psi(z_l)$ and $\tilde q_1=q_1^*(\zeta_l)$, $\tilde q_2=q_2^*(\zeta_l)$. Let us show that the quantity

$$\varepsilon_l=\max_{\zeta\in\overline{G}}\{|q_1^*(\zeta)-\tilde q_1|+|q_2^*(\zeta)-\tilde q_2|\}$$

tends to zero as $r\to\infty$ uniformly in $l$; we shall need this later.

In the notations of Lemma 5 we have

$$\begin{aligned}|q_1^*(\zeta)-\tilde q_1|=&|[q_1(z)-q_1(z_l)]\omega(z)+q_1(z_l)|\omega(z)-\omega(z)]|\\ \leqslant&|q_1(z)-q_1(z_l)|+24|z-z_l|^\gamma,\qquad z=\varphi(\zeta).\end{aligned}\tag{64}$$

It follows from $q_1(z)$, $q_2(z)\in C(\overline{D})$, (58) and (64) that

$$\varepsilon_l\to0,\qquad r\to\infty,\tag{65}$$

uniformly in $l$.

We rewrite equation (60) as

$$\Omega_1^*(w_l)+\Omega_2^*(w_l)=F_l^{**}(\zeta),\tag{66}$$

where

$$\Omega_1^*(w_l)=w_l+T_G(\tilde q_1\partial_\zeta w_l+\tilde q_2\partial_{\bar\zeta}\overline{w}_l),\qquad \Omega_2^*(w_l)=T_G[(q_1^*(\zeta)-\tilde q_1)\partial_\zeta w_l+(q_2^*(\zeta)-\tilde q_2)\partial_{\bar\zeta}\overline{w}_l].$$

By Lemma 11, the operator $\Omega_1^*$ is a linear morphism of the space $W^1_p(\overline{G})$ and obviously, it depends continuously on the numbers $\tilde q_1$, $\tilde q_2$. Since the passage to the inverse operator is a continuous operation [16, Ch. 2, Sect. 9], then $[\Omega_1^*]^{-1}$ is a continuous operator-valued function of the variables $\tilde q_1$, $\tilde q_2$ defined on the compact set determined by inequality (59). Therefore, the norm of the inverse operator $[\Omega_1^*]^{-1}$ is bounded uniformly in $\tilde q_1$, $\tilde q_2$.

For the operator $\Omega_2^*$ in Lemma 2 we have the estimate

$$\|\Omega_2^*(w_l)\|_{W_p^1(\overline{G})} \leqslant \text{const} \cdot \varepsilon_l \cdot \|\partial_\zeta w_l\|_{L_p(\overline{G})} \leqslant \text{const} \cdot \varepsilon_l \cdot \|w_l\|_{W_p^1(\overline{G})}, \tag{67}$$

where const is independent of $r$ and $l$.

We rewrite equation (66) as

$$w_l + [\Omega_1^*]^{-1} \circ \Omega_2^*(w_l) = [\Omega_1^*]^{-1} F_l^{**} \tag{68}$$

and fix $r$ large enough so that

$$\| [\Omega_1^*]^{-1} \| \cdot \text{const} \cdot \varepsilon_l < 1,$$

uniformly in $l$, that is, so that the operator $[\Omega_1^*]^{-1} \circ \Omega_2^*$ in equation (68) becomes contracting from $W_p^1(\overline{G})$ into $W_p^1(\overline{G})$. Then we obtain that $w_l(\zeta) \in W_p^1(\overline{G})$, and therefore, $w_l(z) \in W_p^1(\overline{U}_l)$ for each $l$ and this proves the theorem for operator (25).

We proceed to the general case. According Lemma 10, equation (22) has the unique solution $w(z) \in W_s^1(\overline{D})$, $s > 2$. As above, it is sufficient to show that $w(z) \in W_p^1(\overline{D})$. We rewrite equation (22) as

$$\Omega_1(w) = F_2(z),$$

where

$$F_2(z) = F(z) - T_D(Aw + B\overline{w}) \in W_p^1(\overline{D}).$$

By the above proven particular case, this implies

$$w(z) = [\Omega_1]^{-1} F_2(z) \in W_p^1(\overline{D})$$

and this completes the proof.

### 3.4. Proof of Theorem 2.

First we consider the case $k = 0$.

*3.4.1. Case $k = 0$.* Here we slightly modify the arguing in Section 3.3. We begin with proving the theorem for the operator $\Omega_1$. We consider the equation

$$\Omega_1(w) = F(z) \in C_\alpha^1(\overline{D}), \tag{69}$$

and by Theorem 1 it possesses the unique solution $w(z) \in W_s^1(\overline{D}) \subset C_\beta(\overline{D})$, where $s > 2$ is arbitrarily large and hence, $\beta = \dfrac{s-2}{s}$ is arbitrarily close to 1. As in Section 3.3, in view of the Banach theorem, it is sufficient to show that $w(z) \in C_\alpha^1(\overline{D})$.

As in (34),

$$F(z) = T_D \partial_{\bar{z}} F(z) + \Phi(z),$$

where $\Phi(z)$ is represented by formula (35) and $\Phi(z) \in A_\alpha^1(\overline{D})$.

As in Section 3.3, we consider the partition of the unity $\mathcal{H} = \{h_l\}$, the function $w_l(z) = h_l(z)w(z)$, and we are going to show that $w_l(z) \in C_\alpha^1(\overline{D})$ for each $l$. We assume that the adjoining domains $U_l = D_r$ are enveloped by curves of class $C_\gamma^1$, where $\beta + \gamma > 1 + \alpha$, see Remark 3.

The function $w_l(z)$ satisfies equation (53), where $F_l \in C_\alpha(\overline{D})$, and also equation (54). Similar to (56), we employ Lemma 6 and the belonging $\Phi(z) \in A_\alpha^1(\overline{D})$ to obtain $\Phi_l(z) \in A_\alpha^1(\overline{D})$. Then we proceed to equation (57), in which $q_1^*(\zeta)$, $q_2^*(\zeta) \in C_\alpha(\overline{G})$, $F_l^*(\zeta) \in C_\alpha(\overline{G})$, and to equation (60), in which, similar to Section 3.3, by $\beta + \gamma > 1 + \alpha$ and Theorem 6, we have $\Phi_l^*(\zeta) \in A_\alpha^1(\overline{G})$, $F_l^{**}(\zeta) \in C_\alpha^1(\overline{G})$.

As in Section 3.3, we rewrite equation (60) as (66). By Lemma 11, the operator $\Omega_1^*$ is a linear isomorphism of the space $C_\alpha^1(\overline{G})$ and the inverse operator $[\Omega_1^*]^{-1}$ is bounded uniformly in $\tilde{q}_1$ and $\tilde{q}_2$. Similar to (67), by Lemma 2, for the operator $\Omega_2^*$ we have

$$\|\Omega_2^*(w_l)\|_{C_\alpha^1(\overline{G})} \leqslant \text{const} \cdot \tilde{\varepsilon}_l \cdot \|w_l\|_{C_\alpha^1(\overline{G})}, \tag{70}$$

where

$$\tilde{\varepsilon}_l = \|q_1^*(\zeta) - \tilde{q}_1\|_{C_\alpha(\overline{G})} + \|q_2^*(\zeta) - \tilde{q}_2\|_{C_\alpha(\overline{G})},$$

and const is independent of $r$ and $l$.

Let us show that

$$\tilde{\varepsilon}_l \to 0, \qquad r \to \infty, \tag{71}$$

uniformly in $l$. First, by (65),

$$\|q_i^*(\zeta) - \tilde{q}_i\|_{C(\overline{G})} \to 0, \qquad r \to \infty, \quad i = 1, 2,$$

uniformly in $l$. Let us estimate the Hölder constants for these differences. By $q_i(z) \in C_\alpha(\overline{D})$, (64) and (20) we obtain that

$$|q_i^*(\zeta_1) - q_i^*(\zeta_2)| \leqslant \text{const} \cdot |\varphi(\zeta_1) - \varphi(\zeta_2|^\alpha \leqslant \frac{\text{const}}{r^\alpha} \cdot |\zeta_1 - \zeta_2|^\alpha, \ i = 1, 2,$$

where $\zeta_1, \zeta_2 \in \overline{G}$, and const is independent of $r$ and $l$. Thus, as $r \to \infty$, the Hölder constants of the functions $q_i^*(\zeta) - \tilde{q}_i$, $i = 1, 2$, tends to zero uniformly in $l$ and this proves relation (71).

We proceed to equation (68). By (70), (71) and the uniform boundedness of the operator $[\Omega_1^*]^{-1}$, we can fix a large $r$ such that for each $l$ the operator $[\Omega_1^*]^{-1} \circ \Omega_2^*$ is contracting as acting from $C_\alpha^1(\overline{G})$ into $C_\alpha^1(\overline{G})$. Having fixed such $r$, we have $w_l(z) \in C_\alpha^1(\overline{D})$ for all $l$ and $w(z) \in C_\alpha^1(\overline{D})$ and this completes the case $k = 0$ for the operator $\Omega_1$.

We proceed to the operator $\Omega$ as $k = 0$. We write this operator as

$$\Omega(w) = \Omega_1(w) + P(w),$$

where $P(w) = T_D(Aw + B\overline{w})$. By Lemma 2, the operator $P$ maps continuously $C_\alpha(\overline{D})$ into $C_\alpha^1(\overline{D})$ and is completely continuous in $C_\alpha(\overline{D})$. Thus, the operator $\Omega$ maps continuously the space $C_\alpha^1(\overline{D})$ into itself and thanks to the Banach theorem, it is sufficient to show that the equation

$$\Omega(w) = F \in C_\alpha^1(\overline{D})$$

is uniquely solvable in $C_\alpha^1(\overline{D})$. We rewrite this equation as

$$w + [\Omega_1]^{-1} \circ P(w) = [\Omega_1]^{-1} F. \tag{72}$$

The operator $[\Omega_1]^{-1} \circ P$ is completely continuous in $C_\alpha(\overline{D})$ and maps this space into $C_\alpha^1(\overline{D})$. Hence, a solution $w \in C_\alpha(\overline{D})$ of homogeneous equation (72) belongs to the class $C_\alpha^1(\overline{D})$ and by Lemma 9, it vanishes. By the Fredholm theorem, equation (72) possesses the unique solution in the class $C_\alpha(\overline{D})$, and this solution belongs to $C_\alpha^1(\overline{D})$. This completes the case $k = 0$.

*3.4.2. Regularity of solutions in interior points.* To proceed to the case $k \geqslant 1$, we study first the regularity of solution in interior points of the circle $D$.

**Lemma 14.** *If $w(z)$ is a compactly supported in $D$ function (*$\operatorname{supp} w(z) \subset D$*) of class $W_s^1(\overline{D})$, $s > 2$, satisfying equation* (1) *with the coefficients obeying*

$$q_1(z),\ q_2(z),\ A(z),\ B(z),\ R(z) \in C_\alpha^k(\overline{D}), \qquad k \geqslant 0, \quad 0 < \alpha < 1,$$

*then $w(z) \in C_\alpha^{k+1}(\overline{D})$.*

**Remark 7.** *It is clear under the assumptions of the above lemma, the function $R(z)$ is compactly supported in $D$.*

*Proof of Lemma 14.* By (6), the function $w(z)$ satisfies the equation

$$\Omega(w) = T_D R \in C_\alpha^{k+1}(\overline{D})$$

and by Lemma 9 this function is well-defined. The statement of the lemma in the case $k = 0$ is implied by the proven isomorphic property of the operator $\Omega$ in $C_\alpha^1(\overline{D})$.

We consider the case $k = 1$. First we assume that $A(z) = B(z) \equiv 0$, that is, the function $w(z)$ satisfies the equation

$$\partial_{\bar{z}} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar{z}}\overline{w} = R(z). \tag{73}$$

We differentiate formally (73) with respect to $z$ and we rewrite the result denoting $\partial_z w(z) = W(z)$:

$$\partial_{\bar z} W + q_1 \partial_z W + q_2 \partial_z \overline{W} + A_1(z) W + B_1(z) \overline{W} = \partial_z R(z) \in C_\alpha(\overline{D}), \tag{74}$$

where

$$A_1(z) = \partial_z q_1(z),\ A_2(z) = \partial_z q_2(z) \in C_\alpha(\overline{D}).$$

Excluding $\partial_z \overline{W}$ from (74) and complex conjugate identity, we obtain

$$\partial_{\bar z} W + Q_1 \partial_z W + Q_2 \partial_{\bar z} \overline{W} + A_2(z) W + B_2(z) \overline{W} = R_2(z), \tag{75}$$

where

$$\begin{gathered} Q_1(z) = \frac{q_1(z)}{1 - |q_2(z)|^2}, \quad Q_2(z) = -\frac{\bar q_1 \cdot q_2}{1 - |q_2(z)|^2}, \\ A_2(z) = \frac{A_1 - q_2 \cdot \overline{B}_1}{1 - |q_2|^2}, \quad B_2(z) = \frac{B_1 - q_2 \cdot \overline{A}_1}{1 - |q_2|^2}, \qquad R_2(z) = \partial_z R - q_2 \cdot \partial_{\bar z} \overline{R}, \end{gathered} \tag{76}$$

$Q_1$, $Q_2 \in C^1_\alpha(\overline{D})$, $A_2$, $B_2$, $R_2 \in C_\alpha(\overline{D})$, and

$$|Q_1| + |Q_2| \leqslant \text{const} < 1.$$

We consider the integro-differential equation

$$W + T_D(Q_1 \partial_z W + Q_2 \partial_{\bar z} \overline{W} + A_2 W + B_2 \overline{W}) = T_D R_2. \tag{77}$$

By the proven case $k = 0$, this equation has the unique solution $W(z) \in C^1_\alpha(\overline{D})$. Let us show that the function $W(z)$ is compactly supported in $D$.

By (77) and (6) we infer that function $W(z)$ is continuous, has a holomorphic continuation from $\overline{D}$ into entire complex plane and the Cauchy type integral satisfies the identity

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{W(t)dt}{t - z} \equiv 0, \qquad z \in \overline{D}.$$

As it was mentioned in the proof of Lemma 9, this implies

$$W(t) \equiv 0, \qquad t \in \Gamma. \tag{78}$$

Since the function $R_2(z)$ is compactly supported in $D$, in some annulus $1 - \varepsilon \leqslant |z| \leqslant 1$, $\varepsilon > 0$, the function $W(z)$ satisfies homogeneous differential equation (77). By (78) this yields that in this annulus $W(z) \equiv 0$, see [1, Ch. 3, Sect. 17]. Excluding $\partial_{\bar z} \overline{W}$ from (75) and the complex conjugate identity, we obtain that the solution $W(z)$ of equation (77) solves (74).

We let

$$w(z) = \overline{T}_D W(z) \in C^2_\alpha(\overline{D}). \tag{79}$$

As above, we establish that the function $w(z)$ is compactly supported in $D$. Since $\partial_z w = W(z)$, relation (74) can be rewritten as

$$\partial_z \left[\partial_{\bar z} w + q_1 \partial_z w + q_2 \partial_{\bar z} \overline{w} - R\right] = 0,\ z \in \overline{D}. \tag{80}$$

Thus, the expression in square brackets in (80) is an anti-holomorphic function vanishing on $\Gamma$ and hence, it vanishes identically and (79) is the considered unique solution of equation (73).

In the case $k = 1$ for general equation (1), we rewrite it as

$$\partial_{\bar z} w + q_1(z) \partial_z w + q_2(z) \partial_{\bar z} \overline{w} = R_3(z),$$

where $R_3 = R - Aw - B\overline{w} \in C^1_\alpha(\overline{D})$, since as it was proven for $k = 0$, we have $w \in C^1_\alpha(\overline{D})$. By the arguing for equation (73), this implies $w(z) \in C^2_\alpha(\overline{D})$.

We make the induction in $k$. Assume that the lemma is true for $k = n - 1 \geqslant 1$. Let us show that it is true for $k = n$. We have

$$q_1(z),\ q_2(z),\ A(z),\ B(z),\ R(z) \in C^n_\alpha(\overline{D}),$$

and by the induction assumption $w(z) \in C^n_\alpha(\overline{D})$, $n \geqslant 2$. We differentiate equation (1) with respect to $z$, we denote $\partial_z w = W$ and exclude $\partial_z \overline{W}$ from the obtained identity. As a result, we obtain the equation

$$\partial_{\bar z} W + Q_1(z)\partial_z W + Q_2(z)\partial_{\bar z}\overline{W} + A_3(z)W + B_3(z)\overline{W} = R_4(z),$$

where $Q_1(z)$, $Q_2(z) \in C^n_\alpha(\overline{D})$ are defined by formulae (76), and $A_3(z)$, $B_3(z)$, $R_4(z) \in C^{n-1}_\alpha(\overline{D})$. The function $W$ is compactly supported and is the derivative of the considered solution. It is also easy to write exact formulae for $A_3$, $B_3$ and $R_4$, but we do not need them.

By Lemma 7, $W(z) = \partial_z w(z) \in C^n_\alpha(D)$. Hence, by (1) we obtain $\partial_{\bar z} w(z) \in C^n_\alpha(D)$. Since it is compactly supported, we get $w(z) \in C^{n+1}_\alpha(\overline{D})$. The proof is complete. □

**Lemma 15.** *If the coefficients of equation* (1) *obey*

$$q_1(z),\ q_2(z),\ A(z),\ B(z),\ R(z) \in C^k_\alpha(\overline{D}), \qquad k \geqslant 0, \quad 0 < \alpha < 1,$$

*then each solution* $w(z) \in W^1_s(\overline{D})$ *belongs also to* $C^{k+1}_\alpha(D)$.

*Proof of Lemma 15.* We fix arbitrarily a point $z_0 \in D$ and a circle $D_\varepsilon = \{z : |z - z_0| < \varepsilon\} \subset D$, $\varepsilon > 0$. We denote by $h_\varepsilon(z) \in C^\infty(\overline{D})$ a compactly supported function determined by the following conditions:

1. $h_\varepsilon(z) \geqslant 0$;
2. $h_\varepsilon(z) = \begin{cases} 1, & |z - z_0| \leqslant \dfrac{\varepsilon}{2}, \\ 0, & |z - z_0| \geqslant \dfrac{3\varepsilon}{4}. \end{cases}$

The existence of such function was discussed, for instance, in [12, Ch. II, Sect. 4].

It is obvious that it is sufficient to show that a compactly supported in $D$ function $w_\varepsilon(z) = h_\varepsilon(z) \cdot w(z)$ belongs to $C^{k+1}_\alpha(\overline{D})$. But the function $w_\varepsilon(z)$ satisfies the differential equation

$$\partial_{\bar z} w_\varepsilon + q_1(z)\partial_z w_\varepsilon + q_2(z)\partial_{\bar z}\overline{w}_\varepsilon + A_\varepsilon(z)w_\varepsilon + B_\varepsilon(z)\overline{w}_\varepsilon = R_\varepsilon(z),$$

whose coefficients and the free term in the class $C^k_\alpha(\overline{D})$. Then we apply Lemma 14 and this completes the proof. □

**Remark 8.** *It is easy to see that under the assumptions of Lemma 15, it is sufficient to suppose that the coefficients and the free term belong to* $C^k_\alpha(D)$ *and to consider a solution* $w(z) \in W^1_s(D)$.

*3.4.3. Case* $k = 1$. As above, we need to prove that a solution $w(z)$ of the equation

$$\Omega(w) = F^*(z) \in C^2_\alpha(\overline{D})$$

belongs to the class $C^2_\alpha(\overline{D})$. By the results of Section 3.4.1, we have $w(z) \in C^1_\alpha(\overline{D})$. We rewrite the equation as

$$\Omega_1(w) = F(z) \in C^2_\alpha(\overline{D}), \tag{81}$$

where $F(z) = F^*(z) - T_D(A(z)w + B(z)\overline{w}) \in C^2_\alpha(\overline{D})$. Since $w(z) \in C^1_\alpha(\overline{D})$, function (47) is well-defined in $\overline{D}$:

$$\frac{\partial w(re^{is})}{\partial s} \equiv w_s(z) = i(zw_z - \bar z w_{\bar z}).$$

By Lemma 15 we have $w(z) \in C^2_\alpha(D)$ and hence, $w_s(z) \in C^1_\alpha(D) \cap C_\alpha(\overline{D})$.

Since by (81), the function $w(z)$ solves the differential equation

$$\partial_{\bar z} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} = \partial_{\bar z} F(z) \in C^1_\alpha(\overline{D}), \tag{82}$$

similar to (49), in $D$ the function $w_s(z)$ solves the differential equation

$$\frac{\partial w_s}{\partial \bar z} + q_1(z)\frac{\partial w_s}{\partial z} + q_2(z)\frac{\partial \overline{w}_s}{\partial \bar z} = \widetilde{R}_s(z), \tag{83}$$

where

$$\widetilde{R}_s(z) = 2iq_1(z) + i\bar{z}\left(\frac{\partial q_1}{\partial \bar{z}} w_z + \frac{\partial q_2}{\partial \bar{z}} \overline{w}_{\bar{z}}\right) - iz\left(\frac{\partial q_1}{\partial z} w_z + \frac{\partial q_2}{\partial z} \overline{w}_{\bar{z}}\right) - i\bar{z}F_{z\bar{z}} + izF_{\bar{z}\bar{z}} - iF_{\bar{z}} \in C_\alpha(\overline{D}).$$

As in (50),

$$\Phi_s(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w_s(t)}{t-z}dt \in C^1_\alpha(\overline{D}).$$

Now we consider the integro-differential equation

$$\Omega_1(W_s) = T_D\widetilde{R}_s + \Phi_s \in C^1_\alpha(\overline{D}).$$

By the results of Section 3.4.1, this equations has the unique solution $W_s(z) \in C^1_\alpha(\overline{D})$ obeying

$$\Phi_s(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{W_s(t)}{t-z}dt.$$

Thus, the function $W(z) = w_s(z) - W_s(z) \in C_\alpha(\overline{D}) \cap C^1_\alpha(D)$ satisfies homogeneous equation (83) and

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{W(t)}{t-z}dt = 0 \quad \text{for all} \quad z \in \overline{D}. \tag{84}$$

It follows from (84) that $W(t)$, $t \in \Gamma$, is the boundary value of a holomorphic as $|z| > 1$ function vanishing at infinity, see [13, Ch. 1, Sect. 4.3]. This means that $W(z) \equiv 0$ [2, Thm. 4.5], and $w_s(z) \in C^1_\alpha(\overline{D})$. By formula (51), which is obviously valid for non-constant coefficients $q_1(z)$, $q_2(z)$, this yields $w_z \in C^1_\alpha(\overline{D} \setminus \{0\})$, and since $w_z \in C^1_\alpha(D)$, then $w_z \in C^1_\alpha(\overline{D})$ and by (82), $w_{\bar{z}} \in C^1_\alpha(\overline{D})$. The case $k = 1$ is complete.

*3.4.4. Case* $k > 1$. We make the induction in $k$. Assume that the theorem is true for $k = n - 1 \geqslant 1$. The proof of its validity as $k = n$ reproduces literally the arguing of Section 3.4.3, just $C^2_\alpha(\overline{D})$ and $C^1_\alpha(\overline{D})$ are to be replaced by $C^{n+1}_\alpha(\overline{D})$ and $C^n_\alpha(\overline{D})$ and stating $W_s(z) \in C^n_\alpha(\overline{D})$, we refer to the induction assumption. The proof is complete.

**3.5. Proof of Theorem 3.** The proof almost literally reproduces the proof of Theorem 2. We mention just few differences.

1) The classes $C^n_\alpha(\overline{D})$ are to be replaced by $W^n_p(\overline{D})$.

2) The case $k = 0$ is absent.

3) Instead of Lemmata 14 and 15 we employ the following statements.

**Lemma 16.** *If $w(z)$ is a compactly supported in $D$ function (*$\operatorname{supp} w(z) \subset D$*) of class $W^1_s(\overline{D})$, $s > 2$, satisfying equation* (1)*, in which*

$$q_1(z),\ q_2(z),\ A(z),\ B(z),\ R(z) \in W^k_p(\overline{D}), \quad k \geqslant 1, \quad p > 2,$$

*then $w(z) \in W^{k+1}_p(\overline{D})$.*

**Lemma 17.** *If the coefficients of equation* (1) *obey*

$$q_1(z),\ q_2(z),\ A(z),\ B(z),\ R(z) \in W^k_p(\overline{D}), \quad k \geqslant 1, \quad p > 2,$$

*then each its solution $w(z) \in W^1_s(\overline{D})$ belongs to the class $W^{k+1}_p(D)$.*

The proofs of these lemmata reproduce literally those of Lemmata 14 and 15 with the mentioned replacement of the space and we also note that the existence of solution $w(z) \in W^1_p(\overline{D})$ to equation (77) is due to Theorem 1; hereafter the reference to the considered case $k = 0$ should be replaced by the reference to Theorem 1.

4) While considering the case $k = 1$, we need to take into consideration that $W_p^n(\overline{D}) \subset C_\beta^{n-1}(\overline{D})$, where $n \geqslant 1$, $\beta = \dfrac{p-2}{p}$, and this is why, by Theorem 2, the solution $w(z)$ to equation (81) belongs to $C_\beta^1(\overline{D})$ and $W(z) = w_s(z) - W_s(z) \in W_p^1(\overline{D}) \cap C_\beta(\overline{D})$.

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Sergey Borisov Klimentov,
Southern Federal University,
Milchakov str. 8-a,
344090, Rostov-on-Don, Russia

Southern Mathematical Institute,
Vatutin str. 53,
362027, Vladikavkaz, Russia
E-mail: `sbklimentov@sfedu.ru`