ISSN 2304-0122 Ufa Mathematical Journal. Vol. 10. No 4 (2018). P. 92-102.

doi:10.13108/2018-10-4-92 UDC 517.925

SOLUTIONS TO ANALOGUES OF NON-STATIONARY SCHRÖDINGER EQUATIONS DEFINED BY ISOMONODROMIC HAMILTON SYSTEM $H^{2+1+1+1}$

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Abstract. We construct simultaneous solutions to two analogues of time-dependent solutions to Schrödinger equations defined by the Hamiltonians $H^{2+1+1+1}_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$ $(k = 1, 2)$ to system $H^{2+1+1+1}$. This system is the first representative in a famous degenerations hierarchy of the Garnier system described in 1986 by H. Kimura. By an explicit symplectic transformation, this system reduces to a symmetric Hamilton system. In the constructions of this paper we rely mostly on linear systems of equations in the method of isomonodromic deformations for the system $H^{2+1+1+1}$ written out in 2012 in a paper by A. Kavakami, A. Nakamura and H. Sakai. These analogues of the non-stationary Schrödinger equations are evolution equations with times s_1 and s_2, which depend of two spatial variables. From the canonical non-stationary Schrödinger equations, these analogues arise as a result of the formal replacement of the Planck constant by $-2\pi i$. We construct the exact solutions to the two evolution equations in terms of the solutions to corresponding linear ordinary differential equations in the method of isomonodromic deformations. We discuss further prospects for constructing similar solutions to analogues of the non-stationary Schrödinger equations corresponding to the Hamiltonians of the entire degeneracy hierarchy of the Garnier system.

Keywords: Hamilton systems, Schrödinger equation, Painlevé equations, method of isomonodromic deformations.

Mathematics Subject Classification: 34M56, 35Q41

1. Introduction

Together with six classical ordinary differential (ODE) Painlevé equations of form $q''_{tt} = f_j(s, q, q'_s)$, $(j = 1, \dots, 6)$, which are integrable by isomonodromic deformations method (IDM) [5], [27], there is an increasing interest of scientists to nonlinear higher order ODEs admitting an application of IDM.

At present, a finite list of compatible pair of Hamiltonian systems of ODEs is known [8], [9], [21]:

$$(q_j)'_{s_k} = (H_{s_k})'_{p_j}, \quad (p_j)'_{s_k} = -(H_{s_k})'_{q_j} \quad (k = 1, 2), \quad (j = 1, 2), \tag{1}$$

with Hamiltonians $H_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$, each being the compatibility condition of two linear system of ODEs of form

$$V'_{s_k} = L_{s_k} V, \tag{2}$$

$$V'_{\eta} = AV, \tag{3}$$

where square matrices L_{s_k} and A are of the same dimension and are rational in the variable η; the matrix A is the same for both Hamiltonian systems (1). Treating each of systems (1) as

V.A. Pavlenko, B.I. Suleimanov, Solutions to analogues of non-stationary Schrödinger equations defined by isomonodromic Hamilton system $H^{2+1+1+1}$.

Submitted August 1, 2018.

such compatibility condition is a base of application of IDM to them [27]. The corresponding pairs of linear systems (2), (3) are called matrix $L-A$ IDM pairs and the solutions of ODEs being the compatibility conditions are called isomonodromic.

The solutions to the hierarchy of Hamiltonian degenerations of Garnier systems written out in a known paper by H. Kimura [10] are also among such solutions. The representatives of this hierarchy can be written in several equivalent forms including compatible pairs of Hamiltonian systems (1) defined various pairs of Hamiltonians $H_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$ square in momenta p_1, p_2 and polynomial in coordinates q_1, q_2 [9], [21], [22].

It was shown in [31] that for one of polynomial forms of the Garnier system, there can be constructed solutions to two compatible linear evolution equations and this can be done by explicit changes in terms of the solutions to IDM system (2), (3) written in papers [9], [21] and in Section 3.3 in paper [31]. These evolution equations can be symbolically represented as ($\varepsilon = 1$)

$$\varepsilon \frac{\partial \Psi}{\partial s_k} = H_{s_k}(s_1, s_2, r, \rho, -\varepsilon \frac{\partial}{\partial r}, -\varepsilon \frac{\partial}{\partial \rho})\Psi \qquad (k = 1, 2), \tag{4}$$

where the right hand sides are determined by the Hamiltonians $H_{s_k} = H_{Gar,s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$ of a polynomial form for the Garnier system, whose isomonodromic solutions are exactly the compatibility conditions of matrix $L-A$ pairs (2), (3) written out in [9], [21] and Section 3.3 in [31]. From the corresponding quantum mechanical non-stationary Schrödinger equations of form (4) depending on the Planck constant $h = 2\pi\hbar = -2\pi i\varepsilon$, these evolution equations are obtained by a formal replacement of the parameter ε by 1. Earlier, in paper [36], the solutions to such compatible analogues of Schrödinger equations with $\varepsilon = 5/54$ were constructed for compatible Hamiltonian systems, which are equivalent to the last representative in the hierarchy of the degenerate Garnier system in paper [10], a so-called system $H^{9/2}$. An assumption seemed to be natural is that such constructing of solutions to evolution equations of form (4) with some particular values of the parameter ε can be made for all representatives of this hierarchy of degenerations.

In the present paper we construct such solutions as $\varepsilon = 1$ for first of the degenerations of the Garnier system called system $H^{2+1+1+1}$. One of the equivalent forms of this system is a pair of compatible Hamiltonian systems (1) defined by Hamiltonians (γ, κ, κ_j, θ_j are arbitrary constants):

$$\begin{aligned} s_1^2 H_{s_1} =& q_1^2(q_1 - s_1)p_1^2 + 2q_1^2 q_2 p_1 p_2 + q_1 q_2 (q_2 - s_2) p_2^2 \\ &- p_1[(\kappa_0 + \theta_2 - 1)q_1^2 + \kappa_1 q_1 (q_1 - s_1) + \gamma s_1 q_2 + \gamma(q_1 - s_1)] \\ &- p_2[(\kappa_1 + \kappa_0 - 1)q_1 q_2 + \theta_2 q_1 (q_2 - s_2) - \gamma(s_2 - 1)q_2] + \kappa q_1, \end{aligned} \tag{5}$$

$$\begin{aligned} s_2(s_2 - 1)H_{s_2} =& q_1^2 q_2 p_1^2 + 2q_1 q_2 (q_2 - s_2) p_1 p_2 + \left(q_2(q_2 - 1)(q_2 - s_2) + \frac{q_1 q_2 s_2 (s_2 - 1)}{s_1} \right) p_2^2 \\ &- p_1[(\kappa_1 + \kappa_0 - 1)q_1 q_2 + \theta_2 q_1 (q_2 - s_2) - \gamma(s_2 - 1)q_2] \\ &- p_2\Big((\kappa_0 - 1)q_2(q_2 - 1) + \kappa_1 q_2 (q_2 - s_2) + \theta_2 (q_2 - 1)(q_2 - s_2) \\ &+ \frac{s_2(s_2 - 1)}{s_1} (\theta_2 q_1 + \gamma q_2) \Big) + \kappa q_2. \end{aligned} \tag{6}$$

The compatible solutions of the pair of equations (4) with $\varepsilon = 1$ provided in the present paper correspond exactly to this pair of Hamiltonians. These solutions are explicitly written in terms of solutions to matrix $L-A$ pairs (2), (3) from paper [9], whose compatibility condition are exactly Hamiltonian ODE systems (1) with Hamiltonians (5), (6).

Remark 1. *For six Painlevé ODEs, it was shown in* [33], [34] *that one can construct explicitly solutions to the six evolution equations*

$$\frac{\partial \Psi}{\partial \tau} = H(\tau, \eta, \frac{\partial}{\partial \eta})\Psi.$$

in terms of solutions V of the corresponding pairs of linear equations of IDM

$$V''_{\eta\eta} = P(\eta, \tau, \lambda, \lambda'_\tau)V, \quad V'_\tau = B(\eta, \tau, \lambda, \lambda'_\tau)V'_\eta - \frac{B_\eta(\eta, \tau, \lambda, \lambda'_\tau)}{2}V$$

written by R. Garnier in classical work [5].

The right hand sides of these evolution equations are determined by Hamiltonians $H_j(\tau, q, p)$ of Hamiltonian systems with one degree of freedom, whose coordinates q are defined by solutions of a corresponding Painlevé equation. In last 10 years the issue on relation of IDM equations with evolution equations in quantum mechanics (and in quantum field theory after work [29]*) was developed in many works, see, for instance,* [1]–[4], [6], [7], [11]–[20], [24], [25], [26], [28]–[31], [35], [37]–[39].

Remark 2. *In work* [31], *there was formulated an opinion that after a generalization of a known procedure of step-by-step degenerations of the terms in Garnier hierarchy on the quantum level* [9], [10], *by the results in* [31] *one should get immediately compatible solutions to pair of equations (4) for all representative of this hierarchy. However, as it was noted in* [32], *in this way there are troubles related to the fact that in some of step-by-step degenerations given in* [9], [10] *combinations of coordinate and momenta were employed.*

2. Various forms of system $H^{2+1+1+1}$ and equations of IDM for this system

2.1. System $H^{2+1+1+1}$ in paper [10] was written out in two forms: in the aforementioned form of two compatible Hamiltonian systems (1) with polynomial Hamiltonians (5), (6) and in the form of compatible Hamiltonian systems

$$\frac{\partial \lambda_k}{\partial \tau_j} = \frac{\partial K_j}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial \tau_j} = -\frac{\partial K_j}{\partial \lambda_k} \quad (j, k = 1, 2), \tag{7}$$

where the Hamiltonians $K_i(\tau_1, \tau_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$ are given by the formulae

$$\begin{aligned} K_1 = \frac{(\lambda_1 - 1)(\lambda_2 - 1)}{\tau_1(\tau_2 - 1)(\lambda_2 - \lambda_1)} \sum_{k=1}^{2} (-1)^k \lambda_k(\lambda_k - 1)(\lambda_k - \tau_2) \\ \cdot \Big[\mu_k^2 - (\frac{\kappa_0}{\lambda_k} - \frac{\gamma\tau_1}{(\lambda_k - 1)^2} + \frac{\kappa_1 - 1}{\lambda_k - 1} + \frac{\theta_2}{\lambda_k - \tau_2})\mu_k + \frac{\kappa}{\lambda_k(\lambda_k - 1)}\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} K_2 = \frac{(\lambda_1 - \tau_2)(\lambda_2 - \tau_2)}{\tau_2(\tau_2 - 1)^2(\lambda_1 - \lambda_2)} \sum_{k=1}^{2} (-1)^k \lambda_k(\lambda_k - 1)^2 \\ \Big[\mu_k^2 - (\frac{\kappa_0}{\lambda_k} - \frac{\gamma\tau_1}{(\lambda_k - 1)^2} + \frac{\kappa_1}{\lambda_k - 1} + \frac{\theta_2 - 1}{\lambda_k - \tau_2})\mu_k + \frac{\kappa}{\lambda_k(\lambda_k - 1)}\Big]; \end{aligned} \tag{9}$$

there is a misprint in paper [10]: the Hamiltonian K_2 is written with the opposite sign. These two pairs of Hamiltonian systems are mutually related by the symplectic transform [10]

$$q_1 = -\frac{(\lambda_1 - 1)(\lambda_2 - 1)}{\tau_1(\tau_2 - 1)}, \quad q_2 = \frac{(\lambda_1 - \tau_2)(\lambda_2 - \tau_2)}{\tau_2(\tau_2 - 1)^2}, \qquad s_1 = \frac{1}{\tau_1}, \quad s_2 = \frac{\tau_2}{\tau_2 - 1}. \tag{10}$$

Later Y. Sasano in paper [22] provided a bi-rational symplectic transform

$$P_1 = \frac{1}{q_1}, \quad P_2 = -\frac{q_2}{q_1}, \quad Q_1 = q_1(p_1 q_1 + p_2 q_2 - \nu), \quad Q_2 = p_2 q_1, \quad t_1 = \frac{1}{s_1}, \quad t_2 = \frac{s_2}{s_1}, \tag{11}$$

where
$$\nu=-\frac{\kappa_0+\kappa_1+\theta_2+\alpha-1}{2},\qquad \nu(\nu+\alpha)=\kappa.$$
This transform reduces the pair of Hamiltonian system (1), (5), (6) reduces to the pair of compatible Hamiltonian systems
$$\frac{\partial Q_k}{\partial t_j}=\frac{\partial H_{t_j}}{\partial P_k},\quad \frac{\partial P_k}{\partial t_j}=-\frac{\partial H_{t_j}}{\partial Q_k}\quad (j,k=1,2) \tag{12}$$
with other polynomial Hamiltonians H_{t_j}. As the constant γ is non-zero, by means of the rescaling
$$P_i\to\gamma P_i,\quad Q_i\to\frac{Q_i}{\gamma},\quad t_i\to\gamma t_i$$
in all above formulae we can get
$$\gamma=1.$$
Under such value of γ the Hamiltonians H_{t_j} are given by the formulae
$$\begin{aligned} t_1H_{t_1}=&Q_1(Q_1-1)^2P_1^2+t_1Q_1P_1+(\theta_2^\infty-\theta^1)Q_1(Q_1-1)P_1-(\theta^0+\theta_2^\infty)(Q_1-1)P_1\\ &-\theta^1\theta_2^\infty(Q_1-1)+P_2Q_2(Q_1-1)(P_1Q_1-P_1-\theta^1)\\ &-\frac{t_1}{t_1-t_2}(P_1(Q_1-Q_2)-\theta^1)(P_2(Q_1-Q_2)+\theta^t),\end{aligned} \tag{13}$$
$$\begin{aligned} t_2H_{t_2}=&Q_2(Q_2-1)^2P_2^2+t_2Q_2P_2+(\theta_2^\infty-\theta^t)Q_2(Q_2-1)P_2-(\theta^0+\theta_2^\infty)(Q_2-1)P_2\\ &-\theta^t\theta_2^\infty(Q_2-1)+P_1Q_1(Q_2-1)(P_2Q_2-P_2-\theta^t)\\ &+\frac{t_2}{t_1-t_2}(P_1(Q_1-Q_2)-\theta^1)(P_2(Q_1-Q_2)+\theta^t),\end{aligned} \tag{14}$$
where the constants θ^0, θ^1, θ^t, θ_1^∞, θ_2^∞ satisfy the so-called Fuchs-Hukuhara relation:
$$\theta^0+\theta^1+\theta^t+\theta_1^\infty+\theta_2^\infty=0.$$
These Hamiltonians are related by the changes $t_1\leftrightarrow t_2$, $Q_1\leftrightarrow Q_2$, $P_1\leftrightarrow P_2$, $\theta^1\leftrightarrow\theta^t$.

2.2. On solutions to equations (12) with Hamiltonians (13), (14), the system of linear ODEs
$$\begin{cases}\dfrac{\partial Y}{\partial\eta}=\left(\dfrac{A_0}{\eta}+\dfrac{A_1}{\eta-1}+\dfrac{A_t}{\eta-\frac{t_2}{t_1}}+A_\infty\right)Y=AY,\\[2ex] \dfrac{\partial Y}{\partial t_1}=\left(E_2\eta+B_1+\dfrac{\frac{t_2}{t_1^2}A_t}{\eta-\frac{t_2}{t_1}}\right)Y=UY,\\[2ex] \dfrac{\partial Y}{\partial t_2}=-\dfrac{\frac{1}{t_1}A_t}{\eta-\frac{t_2}{t_1}}Y=VY\end{cases} \tag{15}$$
is compatible [9]. The coefficients of this system are
$$A_0=\begin{pmatrix}P_1Q_1+P_2Q_2+\theta_0+\theta_2^\infty & -u(P_1Q_1+P_2Q_2+\theta_2^\infty)\\ \frac{1}{u}(P_1Q_1+P_2Q_2+\theta_0+\theta_2^\infty) & -P_1Q_1-P_2Q_2-\theta_2^\infty\end{pmatrix},$$
$$A_1=\begin{pmatrix}\theta^1-P_1Q_1 & uP_1\\ \frac{Q_1}{u}(\theta^1-P_1Q_1) & P_1Q_1\end{pmatrix},\quad A_t=\begin{pmatrix}\theta^t-P_2Q_2 & uP_2\\ \frac{Q_2}{u}(\theta^t-P_2Q_2) & P_2Q_2\end{pmatrix},$$
$$A_\infty=\begin{pmatrix}0&0\\0&t_1\end{pmatrix},\quad E_2=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad B_1=\frac{1}{t_1}\begin{pmatrix}0 & (A_0+A_1+A_t)_{12}\\ (A_0+A_1+A_t)_{21} & 0\end{pmatrix}$$
depending also on a simultaneous solution to following linear ODEs:
$$u'_{t_1}=\frac{u}{t_1}(P_1(Q_1-1)^2-\theta^1(Q_1-1)+\theta_1^\infty-\theta_2^\infty),\quad u'_{t_2}=\frac{u}{t_2}(P_2(Q_2-1)^2-\theta^t(Q_2-1)).$$

It is easy to see that the change

$$Z = \exp((\eta t_1 + \theta^0 \ln \eta + \theta^1 \ln(\eta - 1) + \theta^t \ln(\eta t_1 - t_2) - \theta^t \ln t_1)/2) Y$$

maps compatible systems of IDM (15) into equivalent compatible systems

$$\begin{cases} \dfrac{\partial Z}{\partial \eta} = \left(\dfrac{\hat{A}_0}{\eta} + \dfrac{\hat{A}_1}{\eta - 1} + \dfrac{\hat{A}_t}{\eta - \frac{t_2}{t_1}} + \hat{A}_\infty\right) Y = \hat{A} Z, \\ \dfrac{\partial Z}{\partial t_1} = \left(\hat{E}_2 \eta + B_1 + \dfrac{\frac{t_2}{t_1^2}\hat{A}_t}{\eta - \frac{t_2}{t_1}}\right) Y = \hat{U} Y, \\ \dfrac{\partial Z}{\partial t_2} = -\dfrac{\frac{1}{t_1}\hat{A}_t}{\eta - \frac{t_2}{t_1}} Y = \hat{V} Y, \end{cases} \tag{16}$$

with matrix coefficients

$$\hat{A}_0 = \begin{pmatrix} P_1Q_1 + P_2Q_2 + \frac{\theta^0}{2} + \theta_2^\infty & -u(P_1Q_1 + P_2Q_2 + \theta_2^\infty) \\ \frac{1}{u}(P_1Q_1 + P_2Q_2 + \theta^0 + \theta_2^\infty) & -P_1Q_1 - P_2Q_2 - \frac{\theta^0}{2} - \theta_2^\infty \end{pmatrix},$$

$$\hat{A}_1 = \begin{pmatrix} \frac{\theta^1}{2} - P_1Q_1 & uP_1 \\ \frac{\lambda_1}{u}(\theta^1 - P_1Q_1) & P_1Q_1 - \frac{\theta^1}{2} \end{pmatrix}, \quad \hat{A}_t = \begin{pmatrix} \frac{\theta^t}{2} - P_2Q_2 & uP_2 \\ \frac{Q_2}{u}(\theta^t - P_2Q_2) & P_2Q_2 - \frac{\theta^t}{2} \end{pmatrix},$$

$$\hat{A}_\infty = \begin{pmatrix} -\frac{t_1}{2} & 0 \\ 0 & \frac{t_1}{2} \end{pmatrix}, \quad \hat{E}_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

having zero trace. Exactly this matrix form of equations of IDM for system $H^{2+1+1+1}$ will be employed in this system for constructing solutions to corresponding evolution equations of form (4).

3. Construction of solutions to analogue of non-stationary Schrödinger equations

3.1. First of all by the simultaneous fundamental solution Z of linear systems of ODEs (16) we form the 2×2 matrix

$$M = Z^{-1}(t_1, t_2, \zeta) Z(t_1, t_2, \eta). \tag{17}$$

This matrix satisfies simultaneously two scalar spatial two-dimensional evolution equations: the equation with the time variable t_1

$$\begin{aligned} M'_{t_1} =& \frac{\eta(\eta-1)(\zeta-1)(\eta t_1 - t_2)}{t_1(t_1-t_2)(\zeta-\eta)} M''_{\eta\eta} - \frac{\zeta(\zeta-1)(\zeta t_1 - t_2)(\eta-1)}{t_1(t_1-t_2)(\zeta-\eta)} M''_{\zeta\zeta} + \\ &+ \frac{b(t_1,t_2,\zeta,\eta)M'_\eta + c(t_1,t_2,\zeta,\eta)M'_\zeta}{t_1(t_1-t_2)(\zeta-\eta)} + g_1(t_1,t_2,\zeta,\eta,u,P_1,P_2,Q_1,Q_2)M, \end{aligned} \tag{18}$$

in which the functions b, c and g_1 are as follows:

$$b(\zeta-\eta) = \eta t_1(\eta^2 - \zeta^2 - 4\zeta\eta + 2\zeta^2\eta + 2\zeta) - t_2(\eta^3 + 2\zeta^2\eta - \zeta\eta^2 - \zeta^2 - \eta^2 + \zeta + \eta - 2\zeta\eta),$$

$$c(\zeta-\eta) = \zeta t_1(\zeta^2 - \eta^2 - 4\zeta\eta + 2\zeta\eta^2 + 2\eta) - t_2(\zeta^3 + 2\zeta\eta^2 - \zeta^2\eta - \zeta^2 - \eta^2 + \zeta + \eta - 2\zeta\eta),$$

$$\begin{aligned} g_1 =& \frac{(\theta^0)^2 t_2(\zeta-1)(\eta-1)}{4\zeta\eta t_1(t_1-t_2)} + \frac{(\theta^1)^2(\zeta\eta t_2 - 2\zeta\eta t_1 + t_1(\zeta+\eta) - t_2)}{4(\zeta-1)(\eta-1)t_1(t_1-t_2)} \\ &- \frac{(\theta^t)^2(\zeta-1)(\eta-1)t_1t_2}{4(\zeta t_1 - t_2)(\eta t_1 - t_2)(t_1-t_2)} + \frac{t_1(\zeta-1)(\eta-1)(t_1(\zeta+\eta)-t_2)}{4(t_1-t_2)} \\ &- \frac{t_1(\theta^0 + \theta^t + 2\theta_2^\infty)(\zeta-1)(\eta-1)}{2(t_1-t_2)} + \frac{\theta^1((\zeta+\eta-\zeta\eta)t_1 - t_2)}{2(t_1-t_2)} \end{aligned}$$

$$
-\frac{2(\hat{A}_0)_{11}(\hat{A}_1)_{11}+(\hat{A}_0)_{12}(\hat{A}_1)_{21}+(\hat{A}_0)_{21}(\hat{A}_1)_{12}}{t_1}
-\frac{2(\hat{A}_t)_{11}(\hat{A}_1)_{11}+(\hat{A}_t)_{12}(\hat{A}_1)_{21}+(\hat{A}_t)_{21}(\hat{A}_1)_{12}}{t_1-t_2}-P_1Q_1,
$$

and the equation with time variable t_2:

$$
\begin{aligned}
M'_{t_2}=&\frac{\eta(\eta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)}M''_{\eta\eta}-\frac{\zeta(\zeta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)}M''_{\zeta\zeta}\\
&+\frac{(2\zeta\eta-\zeta-\eta)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)^2}(M'_\zeta+M'_\eta)+g_2(t_1,t_2,\zeta,\eta,u,P_i,Q_i)M
\end{aligned}
\tag{19}
$$

with the coefficient

$$
\begin{aligned}
g_2=&\frac{(\theta^0)^2(\zeta t_1-t_2)(\eta t_1-t_2)}{4\zeta\eta(t_2-t_1)t_1t_2}-\frac{(\theta^1)^2(\zeta t_1-t_2)(\eta t_1-t_2)}{4(\zeta-1)(\eta-1)(t_2-t_1)t_1t_2}\\
&+\frac{(\theta^t)^2(t_1^2\zeta\eta-2t_1t_2\zeta\eta+t_2^2(\zeta+\eta-1))t_1}{4(\zeta t_1-t_2)(\eta t_1-t_2)t_2(t_2-t_1)}+\frac{t_1(\zeta+\eta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{4t_2(t_2-t_1)}\\
&-\frac{(\theta^0+\theta^1+2\theta_2^\infty)(\zeta t_1-t_2)(\eta t_1-t_2)}{2t_2(t_2-t_1)}+\frac{\theta^t t_1(t_2(\zeta+\eta-1)-t_1\zeta\eta)}{2t_2(t_2-t_1)}\\
&-\frac{2(\hat{A}_0)_{11}(\hat{A}_t)_{11}+(\hat{A}_0)_{12}(\hat{A}_t)_{21}+(\hat{A}_0)_{21}(\hat{A}_t)_{12}}{t_2}\\
&-\frac{2(\hat{A}_t)_{11}(\hat{A}_1)_{11}+(\hat{A}_t)_{12}(\hat{A}_1)_{21}+(\hat{A}_t)_{21}(\hat{A}_1)_{12}}{t_2-t_1}-P_2Q_2.
\end{aligned}
$$

3.2. We make the change

$$W=e^{S(t_1,t_2)}M,$$

where the function S satisfies two compatible identities

$$
\begin{aligned}
S'_{t_1}=&\frac{2(\hat{A}_0)_{11}(\hat{A}_1)_{11}+(\hat{A}_0)_{12}(\hat{A}_1)_{21}+(\hat{A}_0)_{21}(\hat{A}_1)_{12}}{t_1}\\
&+\frac{2(\hat{A}_t)_{11}(\hat{A}_1)_{11}+(\hat{A}_t)_{12}(\hat{A}_1)_{21}+(\hat{A}_t)_{21}(\hat{A}_1)_{12}}{t_1-t_2}+P_1Q_1,\\
S'_{t_2}=&\frac{2(\hat{A}_0)_{11}(\hat{A}_t)_{11}+(\hat{A}_0)_{12}(\hat{A}_t)_{21}+(\hat{A}_0)_{21}(\hat{A}_t)_{12}}{t_2}\\
&+\frac{2(\hat{A}_t)_{11}(\hat{A}_1)_{11}+(\hat{A}_t)_{12}(\hat{A}_1)_{21}+(\hat{A}_t)_{21}(\hat{A}_1)_{12}}{t_2-t_1}+P_2Q_2.
\end{aligned}
$$

This change relates solutions of equations (18), (19) with solutions of evolution equations

$$
\begin{aligned}
W'_{t_1}=&\frac{\eta(\eta-1)(\zeta-1)(\eta t_1-t_2)}{t_1(t_1-t_2)(\zeta-\eta)}W''_{\eta\eta}-\frac{\zeta(\zeta-1)(\zeta t_1-t_2)(\eta-1)}{t_1(t_1-t_2)(\zeta-\eta)}W''_{\zeta\zeta}\\
&+\frac{b(t_1,t_2,\zeta,\eta)W'_\eta+c(t_1,t_2,\zeta,\eta)W'_\zeta}{t_1(t_1-t_2)(\zeta-\eta)}+g_3(t_1,t_2,\zeta,\eta)W,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
W'_{t_2}=&\frac{\eta(\eta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)}W''_{\eta\eta}-\frac{\zeta(\zeta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)}W''_{\zeta\zeta}\\
&+\frac{(2\zeta\eta-\zeta-\eta)(\zeta t_1-t_2)(\eta t_1-t_2)}{t_1t_2(t_2-t_1)(\zeta-\eta)^2}(W'_\zeta+W'_\eta)+g_4(t_1,t_2,\zeta,\eta)W,
\end{aligned}
\tag{21}
$$

which depend on the functions

$$\begin{aligned}
g_3 =&\frac{(\theta^0)^2 t_2(\zeta-1)(\eta-1)}{4\zeta\eta t_1(t_1-t_2)}+\frac{(\theta^1)^2(\zeta\eta t_2-2\zeta\eta t_1+t_1(\zeta+\eta)-t_2)}{4(\zeta-1)(\eta-1)t_1(t_1-t_2)}\\
&-\frac{(\theta^t)^2(\zeta-1)(\eta-1)t_1t_2}{4(\zeta t_1-t_2)(\eta t_1-t_2)(t_1-t_2)}+\frac{t_1(\zeta-1)(\eta-1)(t_1(\zeta+\eta)-t_2)}{4(t_1-t_2)}\\
&-\frac{t_1(\theta^0+\theta^t+2\theta_2^\infty)(\zeta-1)(\eta-1)}{2(t_1-t_2)}+\frac{\theta^1((\zeta+\eta-\zeta\eta)t_1-t_2)}{2(t_1-t_2)},\\
g_4 =&\frac{(\theta^0)^2(\zeta t_1-t_2)(\eta t_1-t_2)}{4\zeta\eta(t_2-t_1)t_1t_2}-\frac{(\theta^1)^2(\zeta t_1-t_2)(\eta t_1-t_2)}{4(\zeta-1)(\eta-1)(t_2-t_1)t_1t_2}\\
&+\frac{(\theta^t)^2(t_1^2\zeta\eta-2t_1t_2\zeta\eta+t_2^2(\zeta+\eta-1))t_1}{4(\zeta t_1-t_2)(\eta t_1-t_2)t_2(t_2-t_1)}+\frac{t_1(\zeta+\eta-1)(\zeta t_1-t_2)(\eta t_1-t_2)}{4t_2(t_2-t_1)}\\
&-\frac{(\theta^0+\theta^1+2\theta_2^\infty)(\zeta t_1-t_2)(\eta t_1-t_2)}{2t_2(t_2-t_1)}+\frac{\theta^t t_1(t_2(\zeta+\eta-1)-t_1\zeta\eta)}{2t_2(t_2-t_1)}.
\end{aligned}$$

The latter pair of equations is independent of Q_i and P_i.

3.3. In equations (20), (21) we pass to the independent variables

$$\tau_1=t_1,\quad \tau_2=\frac{t_2}{t_2-t_1},\quad x=\frac{\zeta}{\zeta-1},\quad y=\frac{\eta}{\eta-1}$$

and make the change

$$W=(y-x)((x-1)(y-1))^{c_1}(xy)^{c_2}((x-\tau_2)(y-\tau_2))^{c_3}(\tau_2)^{c_3(2c_2+1)}(\tau_2-1)^{2c_3(c_1+c_3)}e^{f(\tau_1,\tau_2,x,y)}\Psi,$$

where c_i are constants and

$$\begin{aligned}
f(\tau_1,\tau_2,x,y) =&\frac{\tau_1}{2(x-1)}+\frac{\tau_1}{2(y-1)}+\frac{\theta^1+\theta^t-2(c_1+c_2)\tau_1\tau_2}{2(\tau_2-1)}+\frac{(c_1+c_2+c_3)\tau_1}{\tau_2-1}\\
&+\frac{(c_1^2-c_2^2-c_3^2-(\theta^1)^2/4)\tau_2\ln\tau_1}{\tau_2-1}+\frac{(-c_1^2+c_3^2+(\theta^0)^2/4+(\theta^1)^2/4)\ln\tau_1}{\tau_2-1}.
\end{aligned}$$

This gives the pair of equations

$$\begin{aligned}
\tau_1(\tau_2-1)\Psi'_{\tau_1} =&\frac{y(y-1)^2(x-1)(y-\tau_2)}{y-x}\\
&\cdot\left(\Psi''_{yy}+\Psi'_y\left(\frac{2c_1+1}{y-1}+\frac{2c_2+1}{y}+\frac{2c_3+1}{y-\tau_2}-\frac{\tau_1}{(y-1)^2}\right)\right)\\
&-\frac{x(x-1)^2(y-1)(x-\tau_2)}{y-x}\\
&\cdot\left(\Psi''_{xx}+\Psi'_x\left(\frac{2c_1+1}{x-1}+\frac{2c_2+1}{x}+\frac{2c_3+1}{x-\tau_2}-\frac{\tau_1}{(x-1)^2}\right)\right)\\
&+g_5(\tau_1,\tau_2,x,y)\Psi,
\end{aligned}\tag{22}$$

$$
\begin{aligned}
\tau_2(\tau_2-1)^2\Psi'_{\tau_2} =& \frac{x(x-1)^2(y-\tau_2)(x-\tau_2)}{y-x} \\
&\cdot \left(\Psi''_{xx}+\Psi'_x\left(\frac{2c_1+2}{x-1}+\frac{2c_2+1}{x}+\frac{2c_3}{x-\tau_2}-\frac{\tau_1}{(x-1)^2}\right)\right) \\
&- \frac{y(y-1)^2(x-\tau_2)(y-\tau_2)}{y-x} \\
&\cdot \left(\Psi''_{yy}+\Psi'_y\left(\frac{2c_1+2}{y-1}+\frac{2c_2+1}{y}+\frac{2c_3}{y-\tau_2}-\frac{\tau_1}{(y-1)^2}\right)\right) \\
&+ g_6(\tau_1,\tau_2,x,y)\Psi.
\end{aligned}
\tag{23}
$$

Here the functions g_5 and g_6 read as

$$
\begin{aligned}
g_5 =& \left((c_1+c_2+c_3+1)^2-\frac{(\theta^1)^2}{4}\right)(x-1)(y-1)+\frac{(c_2^2-\frac{(\theta^0)^2}{4})\tau_2(x+y-1)}{4xy} \\
&+\frac{(\theta_2^\infty-\theta_1^\infty-2c_1)\tau_1(\tau_2(x+y)-xy-2\tau_2+1)}{2(x-1)(y-1)}+\frac{(\frac{(\theta^t)^2}{4}-c_3^2)(\tau_2-1)^2(xy-\tau_2)}{(x-\tau_2)(y-\tau_2)}, \\
g_6 =& \frac{(\gamma^2-1)(\tau_1^2(xy-1)(x-\tau_2)(y-\tau_2))}{4(x-1)^2(y-1)^2}-\left((c_1+c_2+c_3+1)^2-\frac{(\theta^1)^2}{4}\right)(x-\tau_2)(y-\tau_2) \\
&+\frac{(\theta_2^\infty-\theta_1^\infty-2\gamma c_1)\tau_1(x-\tau_2)(y-\tau_2)}{2(x-1)(y-1)}+\frac{(\frac{(\theta^0)^2}{4}-c_2^2)\tau_2(x+y-\tau_2)}{xy} \\
&+c_2^2-\frac{(\theta^0)^2}{4}+\frac{(c_3^2-\frac{(\theta^t)^2}{4})(\tau_2-1)^2(xy-\tau_2^2)}{(x-\tau_2)(y-\tau_2)}.
\end{aligned}
$$

Letting now

$$
\begin{aligned}
&c_2=\frac{\theta^0}{2}, \quad 2c_1=\theta_2^\infty-\theta_1^\infty, \quad c_3=\frac{\theta^t}{2}, \quad \kappa=(c_1+c_2+1)^2, \\
&c_1=\frac{\kappa_1-2}{2}, \quad c_2=\frac{\kappa_0-1}{2}, \quad c_3=\frac{\theta_2-1}{2},
\end{aligned}
$$

we obtain that thanks to the relations

$$
\frac{\partial}{\partial x}x-x\frac{\partial}{\partial x}=1, \qquad \frac{\partial}{\partial y}y-y\frac{\partial}{\partial y}=1,
$$

the pair of evolutions equations (22), (23) can be symbolically written as the following analogues of non-stationary Schrödinger equations ($\varepsilon=1$):

$$
\varepsilon\frac{\partial\Psi}{\partial\tau_j}=K_{\tau_j}(\tau_1,\tau_2,x,y,-\varepsilon\frac{\partial}{\partial x},-\varepsilon\frac{\partial}{\partial y})\Psi \qquad (j=1,2),
\tag{24}
$$

defined by Hamiltonians (8), (9) of Hamiltonian system (7).

3.4. If in this pair of evolution equations we make the changes

$$
r=-\frac{(x-1)(y-1)}{\tau_1(\tau_2-1)}, \qquad \rho=\frac{(x-\tau_2)(y-\tau_2)}{(\tau_2-1)^2},
\tag{25}
$$

being quantum analogues of two first parts of symplectic transform (10) and we pass from times τ_j to times s_j according transform (10), these analogues of Schrödinger equations transform to the equations

$$
\begin{aligned}
s_1^2\Psi'_{s_1} =& r^2(r-s_1)\Psi''_{rr}+2r^2\rho\Psi''_{r\rho}+r\rho(\rho-s_2)\Psi''_{\rho\rho} \\
&+\Psi'_r[(\kappa_0+\theta_2-1)r^2+\kappa_1 r(r-s_1)+s_1\rho+(r-s_1)]
\end{aligned}
$$

$$
\begin{aligned}
&+\Psi'_\rho[(\kappa_1+\kappa_0-1)r\rho+\theta_2 r(\rho-s_2)-(s_2-1)\rho]+\kappa r\Psi,\\
s_2(s_2-1)\Psi'_{s_2}=&r^2\rho\Psi''_{rr}+2r\rho(\rho-s_2)\Psi''_{r\rho}+\left(\rho(\rho-1)(\rho-s_2)+\frac{r\rho s_2(s_2-1)}{s_1}\right)\Psi''_{\rho\rho}\\
&+\Psi'_r[(\kappa_1+\kappa_0-1)r\rho+\theta_2 r(\rho-s_2)-(s_2-1)\rho]\\
&+\Psi'_\rho\Big((\kappa_0-1)\rho(\rho-1)+\kappa_1\rho(\rho-s_2)+\theta_2(\rho-1)(\rho-s_2)\\
&+\frac{s_2(s_2-1)}{s_1}(\theta_2 r+\rho)\Big)+\kappa\rho\Psi.
\end{aligned}
$$

Thanks to the operator relations

$$\frac{\partial}{\partial r}r-r\frac{\partial}{\partial r}=1,\qquad \frac{\partial}{\partial \rho}\rho-\rho\frac{\partial}{\partial \rho}=1$$

the same pair of equations can be symbolically written as analogues of non-stationary Schrödinger equations (4) with $\varepsilon=1$ defined by polynomial Hamiltonians (5) and (6).

4. Conclusions

The constructed solutions to analogues of non-stationary Schrödinger (4) and (24) are expressed via solutions Z to matrix $L-A$ pairs of IDM (16) depending explicitly on solutions to nonlinear Hamiltonian systems of ODEs (12) with Hamiltonians (13), (14). By means of known symplectic transforms (10) and (11), solutions of these Hamiltonian systems can be expressed both via solutions of Hamiltonian systems (1) with polynomial Hamiltonians (5), (6), and via solutions of Hamiltonian systems (7) with Hamiltonians (8), (9). Thus, the described solutions to these analogues of Schrödinger equations are related in two ways with corresponding classical Hamiltonian systems.

It should be stressed that the issue on similar analogues to Schrödinger equation corresponding to Hamiltonian systems presented by various forms of systems $H^{2+1+1+1}$ is not completely solved by the results of the present paper. In particular, the authors did not succeed to construct solutions of any analogues of non-stationary Schrödinger equations corresponding to Hamiltonians (13),(14). The same concerns a series of equivalent Hamiltonian systems described in paper [22].

In constructions of the present paper, change (17) plays an important role. Earlier, the same change was successfully applied in papers [31], [32] and [36], in which there were constructed solutions to analogues of non-stationary Schrödinger equations defined by the Hamiltonians of Garnier system as well as by some of its degenerations. Earlier, for other purposes, this change was employed by D.P. Novikov in [29]. D.P. Novikov observed a similarity of this change with formula (2.3.36) in [23]. For system $H^{2+1+1+1}$, this justifies an assumption of paper [32] that this change can be useful in constructing analogues of non-stationary Schrödinger equations defined by the Hamiltonians of all degenerations of Garnier system.

Apart of such change, it is useful to have in mind the changes being quantum analogues of known classical transformations. For instance, such change turned out to be rather useful in constructions in [31], see equations (46) and (56). In the present work, a similar change is provided by formula (25).

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