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ON QUALITATIVE PROPERTIES OF SOLUTIONS TO QUASILINEAR PARABOLIC EQUATIONS ADMITTING DEGENERATIONS AT INFINITY

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Abstract. We consider the Cauchy problem for a quasilinear parabolic equations $\rho(x)u_t = \Delta u + g(u)|\nabla u|^2$, where the positive coefficient ρ degenerates at infinity, while the coefficient g either is a continuous function or have singularities of at most first power. These nonlinearities called Kardar–Parisi–Zhang nonlinearities (or KPZ-nonlinearities) arise in various applications (in particular, in modelling directed polymer and interface growth). Also, they are of an independent theoretical interest because they contain the second powers of the first derivatives: this is the greatest exponent such that Bernstein-type conditions for the corresponding elliptic problem ensure apriori L_∞-estimates of first order derivatives of the solution via the L_∞-estimate of the solution itself. Earlier, the asymptotic properties of solutions to parabolic equations with nonlinearities of the specified kind were studied only for the case of uniformly parabolic linear parts. Once the coefficient ρ degenerates (at least at infinity), the nature of the problem changes qualitatively, which is confirmed by the presented study of qualitative properties of (classical) solutions to the specified Cauchy problem. We find conditions for the coefficient ρ and the initial function guaranteeing the following behavior of the specified solutions: there exists a (limit) Lipschitz function $A(t)$ such that, for any positive t, the generalized spherical mean of the solution tends to the specified Lipschitz function as the radius of the sphere tends to infinity. The generalized spherical mean is constructed as follows. First, we apply a monotone function to a solution; this monotone function is determined only by the coefficient at the nonlinearity (both in regular and singular cases). Then we compute the mean over the $(n-1)$-dimensional sphere centered at the origin (in the linear case, this mean naturally is reduced to a classical spherical mean). To construct the specified monotone function, we use the Bitsadze method allowing us to express solutions of the studied quasilinear equations via solutions of semi-linear equations.

Keywords: parabolic equations, KPZ-nonlinearities, long-time behavior, degeneration at infinity.

Mathematics Subject Classification: 35K59, 35K65

1. Introduction

Nowadays, the equations of the form $\rho(x)\frac{\partial u}{\partial t} = \Delta u$, where $\rho(x) \geqslant \rho_0 > 0$, as well as parabolic equations of more general form can be regarded as a rather classical object: their complete theory is constructed and deeply developed and this theory shows that these equations inherit the

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properties of their model object, the heat equation. This concerns both issues on well posedness of boundary value problems for the mentioned equations and the qualitative properties of their solutions including their specific properties, due to which there are principal differences between these equations and equations of any other type.

The situation becomes essentially much complicated if we weaken the condition imposed for the coefficient ρ: even if we replace the positive definiteness by a global strict positivity, the coefficient gets a chance to degenerate at infinity being strictly positive at each point. In this case, the equation can be parabolic at each point and even uniformly parabolic in each bounded domain, but, for instance, the Cauchy problem in the half-space, the most important type of problems in the parabolic theory, is no longer classical: the equation loses its type at infinity. Due to the same reason one cannot guarantee the asymptotic properties of solutions specific for classical parabolic theory.

At present, the study of parabolic equations with coefficients degenerating at infinity is far from being complete, but for the linear case, important results were obtained in [1]. In particular, there were obtained analogues of classical results on stabilization of solutions although these results have principal differences from the case of uniformly parabolic equations; for instance, the stabilization is proven for the spherical means of solutions instead for solutions themselves.

In the present work, the mentioned results are extended for some quasilinear equations, namely, the equations involving the square of the gradient of the uknown functions. Such nonlinearities called Kardar-Parisi-Zhang nonlinearities (KPZ-nonlinearities) arise in various applications (see, for instance, [2]–[14]) and they are also of independent theoretical interest. The reason is that they involve the square of the derivative of the unknown function and as it is known, see, for instance, [15]-[16], this is the maximal (limiting) exponent, under which Bernstein-type conditions for the associated elliptic problem ensure apriori L_∞-estimates for the derivatives of the solution in terms of the L_∞-estimate of the solution itself.

Asymptotic properties of solutions to parabolic and elliptic equations with KPZ-nonlinearity including singular equations were studied in [17]–[24], but the case of the degeneration at infinity was not studied before.

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2. Regular coefficients

Let $n \geqslant 3$, while ρ and u_0 be functions defined in $\mathbb{R}^n$ such that ρ is positive and the function g is continuous on the real axis. We assume that a bounded function $u(x,t)$ satisfies the equation

$$\rho(x)\frac{\partial u}{\partial t} = \Delta u + g(u)|\nabla u|^2, \qquad x \in \mathbb{R}^n, \quad t \in (0, +\infty), \tag{1}$$

and the initial condition

$$u\Big|_{t=0} = u_0(x), \qquad x \in \mathbb{R}^n, \tag{2}$$

in the classical sense. Following [25], on the real line, we define a function f as follows:

$$f(s) = \int\limits_0^s e^{\int\limits_0^x g(\tau)d\tau} dx. \tag{3}$$

Then

$$f'(s) = e^{\int\limits_0^s g(\tau)d\tau} > 0,$$

that is, the introduced function f is strictly monotone and

$$f''(s) = g(s)e^{\int\limits_0^s g(\tau)d\tau}, \qquad \text{that is,} \qquad g(s) = \frac{f''(s)}{f'(s)}.$$

We denote the function $f[u(x,t)]$ by $v(x,t)$ and we calculate

$$\begin{aligned}
\rho(x)\frac{\partial v}{\partial t} - \Delta v =& \rho(x)f'(u)\frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f'(u)\frac{\partial u}{\partial x_j}\right] \\
=& \rho(x)f'(u)\frac{\partial u}{\partial t} - \sum_{j=1}^{n} f''(u)\left(\frac{\partial u}{\partial x_j}\right)^2 - \sum_{j=1}^{n} f'(u)\frac{\partial^2 u}{\partial x_j^2} \\
=& f'(u)\rho(x)\frac{\partial u}{\partial t} - f''(u)|\nabla u|^2 - f'(u)\Delta u \\
=& f'(u)\left[\rho(x)\frac{\partial u}{\partial t} - \frac{f''(u)}{f'(u)}|\nabla u|^2 - \Delta u\right] \\
=& f'(u)\left[\rho(x)\frac{\partial u}{\partial t} - \Delta u - g(u)|\nabla u|^2\right] = 0.
\end{aligned}$$

Hence, the function $v(x,t)$ satisfies the equation

$$\rho(x)\frac{\partial v}{\partial t} = \Delta v, \qquad x \in \mathbb{R}^n, \quad t \in (0,+\infty), \tag{4}$$

in the classical sense. Moreover, the function $v(x,t)$ is bounded thanks to the continuity of the function g and the boundedness of the function u, and its trace on the hyperplane $\{t=0\}$ is equal to a function $f[u_0(x)] \stackrel{\text{def}}{=} v_0(x)$ bounded on the real axis.

Following [1], in $\mathbb{R}^n \times (0,+\infty)$ we define the function $V(x,t) = \int\limits_0^t v(x,\tau)d\tau$. We impose the following conditions on the functions ρ and u_0:

- The Poisson equation with the right hand side $-\rho(x)$ possesses a solution bounded in $\mathbb{R}^n$;
- There exists a constant $\varkappa$, $0 < \varkappa < 1$, such that $\rho \in C^{\varkappa+1}_{\text{loc}}(\mathbb{R}^n)$ and $f(u_0) \in C^{\varkappa}_{\text{loc}}(\mathbb{R}^n)$.

Then, by [1, Th 1.1], there exists a Lipschitz function A on $[0,+\infty)$ such that the relation

$$\lim_{R\to\infty} \frac{1}{R^{n-1}} \int\limits_{|x|=R} V(x,t)d\sigma_x = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}A(t)$$

holds for each positive t and the relation

$$\lim_{R\to\infty} \frac{1}{R^{n-1}} \int\limits_{|x|=R} [V(x,t) - A(t)]d\sigma_x = 0$$

is satisfied uniformly in $t \in [0,T]$ for each positive T.

Thus, the following statement is true.

Theorem 1. *Let $u(x,t)$ be a classical bounded solution of the Cauchy problem for equation (1), where the coefficient g is continuous and the coefficient $\rho(x)$ and the initial function $u_0(x)$ satisfy the following conditions:*

(i) *The equation $\Delta w + \rho(x) = 0$ possesses a bounded in $\mathbb{R}^n$ solution;*
(ii) *There exists a constant $\varkappa$, $0 < \varkappa < 1$, such that $\rho \in C^{\varkappa+1}_{\text{loc}}(\mathbb{R}^n)$ and $f(u_0) \in C^{\varkappa}_{\text{loc}}(\mathbb{R}^n)$.*

Then there exists a Lipschitz function A on $[0,+\infty)$ such that the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\int\limits_0^t f[u(x,\tau)]d\tau d\sigma_x=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}A(t)$$

holds for each positive t and the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\left(\int\limits_0^t f[u(x,\tau)]d\tau-A(t)\right)d\sigma_x=0$$

is satisfied uniformly in $t\in[0,T]$ for each positive T.

3. Singular coefficients

In equation (1) we let

$$g(s)=\alpha s^\beta, \tag{5}$$

where $\beta\in(-1,0)$ and α is an arbitrary real parameter. In this case the assumptions of Theorem 1 are not satisfied since the coefficient at the nonlinearity has a singularity at the origin. However, function (3) is still well-defined:

$$f(s)=\int\limits_0^s e^{\int\limits_0^x \alpha\tau^\beta d\tau}dx=\int\limits_0^s e^{\frac{\alpha}{\beta+1}x^{\beta+1}}dx.$$

This yields that $f'(s)=e^{\frac{\alpha}{\beta+1}s^{\beta+1}}>0$, and hence, f is a strictly monotone function. Then, $f''(s)=\alpha s^\beta e^{\frac{\alpha}{\beta+1}s^{\beta+1}}$, and therefore, $g(s)=\frac{f''(s)}{f'(s)}$.

Assume that a bounded positive function $u(x,t)$ solves equation (1) in the classical sense with the coefficient g defined by relation (5). Then, in the same way as in Section 2, we denote the function $f[u(x,t)]$ by $v(x,t)$ and by straightforward calculations we confirm that it satisfies equation (4). Moreover, due to the boundedness of the function $u(x,t)$ and a strict positivity of the exponent $\beta+1$, the function $v(x,t)$ is bounded:

$$|f[u(x,t)]|\leqslant\sup|u|e^{\frac{|\alpha|}{\beta+1}(\sup|u|)^{\beta+1}}.$$

Therefore, Theorem 1.1 from [1] can be applied in this case as well. This proves the following statement.

Theorem 2. *Let $u(x,t)$ be a classical positive bounded solution of the Cauchy problem for equation* (1), *where the coefficient g is defined by relation* (5), *$\beta\in(-1,0)$, $\alpha\in\mathbb{R}^n$, and the coefficient $\rho(x)$ and the initial function $u_0(x)$ satisfy the assumptions of Theorem 1. Then the statement of Theorem 1 holds true.*

4. Limiting case of singular coefficients

In equation (1) we let

$$g(s)=\frac{\alpha}{s}, \tag{6}$$

where $\alpha>-1$.

In this case substitution (3) is not applicable, but following [25], we can employ the substitution $f(s)=s^{\alpha+1}$. Assuming that a bounded positive function $u(x,t)$ solves equation (1) in the classical sense with the coefficient g defined by relation (6), we denote the function $u^{\alpha+1}(x,t)$

by $v(x,t)$; this function is well-defined and positive in the entire half-space $\mathbb{R}^n\times(0,+\infty)$ thanks to the positivity of the function u. We calculate:

$$\begin{aligned}\rho(x)\frac{\partial v}{\partial t}-\Delta v=&(\alpha+1)\rho(x)u^\alpha\frac{\partial u}{\partial t}-\sum_{j=1}^n\frac{\partial}{\partial x_j}\left[(\alpha+1)u^\alpha\frac{\partial u}{\partial x_j}\right]\\ =&(\alpha+1)\rho(x)u^\alpha\frac{\partial u}{\partial t}-\alpha(\alpha+1)u^{\alpha-1}\sum_{j=1}^n\left(\frac{\partial u}{\partial x_j}\right)^2-(\alpha+1)u^\alpha\Delta u\\ =&(\alpha+1)u^\alpha\left[\rho(x)\frac{\partial u}{\partial t}-\frac{\alpha}{u}\left|\nabla u\right|^2-\Delta u\right]=0;\end{aligned}$$

since u is a classical solution to equation (1) and is a positive function, all above differentiations and divisions are legible.

Thus, the function $v(x,t)$ is bounded, solves equation (4) in the classical sense and its trace on the hyperplane $\{t=0\}$ is equal to $u_0^{\alpha+1}(x)$.

Following [1], in $\mathbb{R}^n\times(0,+\infty)$ we define a function $V(x,t)=\int\limits_0^t v(x,\tau)d\tau$ and we suppose that the function ρ satisfies the assumptions of Theorem 1 and the function $u_0^{\alpha+1}$ belongs to $C^\alpha_{\text{loc}}(\mathbb{R}^n)$. Then, by [1, Th 1.1], there exists a function A Lipshitz on $[0,+\infty)$ such that the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}V(x,t)dx=\frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}A(t)$$

holds for each positive t and the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}[V(x,t)-A(t)]dx=0$$

is satisfied uniformly in $t\in[0,T]$ for each positive T.

Thus, the following statement is true.

Theorem 3. *Let $u(x,t)$ be a classical positive bounded solution to the Cauchy problem for the equation*

$$\rho(x)\frac{\partial u}{\partial t}=\Delta u+\frac{\alpha}{u}|\nabla u|^2,\qquad x\in\mathbb{R}^n,\quad t\in(0,+\infty),\tag{7}$$

where $\alpha>-1$, the coefficient $\rho(x)$ obeys the assumptions of Theorem 1, and the initial function $u_0(x)$ is such that $u_0^{\alpha+1}\in C^\varkappa_{\text{loc}}(\mathbb{R}^n)$. Then there exists a Lipschitz function A on $[0,+\infty)$ such that the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}U_{\alpha+1}(x,t)dx=\frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}A(t)$$

holds for each positive t and the relation

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}[U_{\alpha+1}(x,t)-A(t)]dx=0$$

is satisfied uniformly in $t\in[0,T]$ for each positive T, where

$$U_s(x,t)=\int\limits_0^t u^s(x,\tau)d\tau,\quad s>0.$$

4.1. Positive definite solutions. If we slightly strengthen the conditions on the solution assuming that it is not positive, but positive definite, namely, that it is lower bound is positive, then the condition for the coefficient α can be omitted.

For $\alpha < -1$, we apply the same power substitution as for $\alpha > -1$: assuming that a bounded positive definite function $u(x,t)$ satisfies equation (7) in the classical sense, by straightforward substitution, as in the case $\alpha > -1$, we confirm that the function $v(x,t) = u^{\alpha+1}(x,t)$ satisfies equation (4). In order to prove the boundedness of this function, we employ the positive definiteness of the function $u(x,t)$. Indeed, introducing a positive constant $\gamma \stackrel{\text{def}}{=} -1-\alpha$, we arrive at the inequality

$$v(x,t) \leqslant \frac{1}{(\inf u)^{\gamma}}.$$

Applying Theorem 1.1 from [1], we obtain the following statement.

Proposition 1. *Let $u(x,t)$ be a classical solution of the Cauchy problem for equation (7), where $\alpha \neq -1$, $\inf u \geqslant B > 0$, the coefficient $\rho(x)$ satisfies the assumptions of Theorem 1, and the initial function $u_0(x)$ is such that $u_0^{\alpha+1} \in C^{\varkappa}_{\text{loc}}\big(\mathbb{R}^n\big)$. Then the statement of Theorem 3 holds true.*

As $\alpha = -1$, we apply the change $v(x,t) = \ln\frac{u(x,t)}{B}$, where $B = \inf u > 0$ thanks to the positive definiteness of the solution u. Then $u(x,t) = Be^{v(x,t)}$,

$$\frac{\partial u}{\partial t} = Be^{v}\frac{\partial v}{\partial t}, \qquad \frac{\partial u}{\partial x_j} = Be^{v}\frac{\partial v}{\partial x_j}, \qquad \frac{\partial^2 u}{\partial x_j^2} = Be^{v}\Big(\frac{\partial v}{\partial x_j}\Big)^2 + Be^{v}\frac{\partial^2 v}{\partial x_j^2}, \quad j=\overline{1,n}.$$

This implies that

$$\Delta u = Be^{v}\big(\Delta v + |\nabla v|^2\big), \qquad |\nabla u|^2 = B^2e^{2v}|\nabla v|^2.$$

Now we take into consideration that u satisfies equation (7) as $\alpha = -1$; this yields the relation

$$\begin{aligned} 0 =& \rho(x)\frac{\partial u}{\partial t} - \Delta u + \frac{1}{u}|\nabla u|^2 = \rho(x)Be^{v}\frac{\partial v}{\partial t} - Be^{v}\big(\Delta v + |\nabla v|^2\big) + \frac{1}{Be^{v}}B^2e^{2v}|\nabla v|^2 \\ =& \rho(x)Be^{v}\frac{\partial v}{\partial t} - Be^{v}\Delta v = Be^{v}\left(\rho(x)\frac{\partial v}{\partial t} - \Delta v\right), \end{aligned}$$

and therefore, the function $v(x,t)$ is a classical solution to equation (4) bounded thanks to the boundedness of the function $u(x,t)$.

Applying [1, Th 1.1], we get a statement on the behavior of the mean:

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\int\limits_0^t \ln\frac{u(x,\tau)}{B}d\tau d\sigma_x = \lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\int\limits_0^t \ln u(x,\tau)d\tau d\sigma_x - \frac{2\pi^{\frac{n}{2}}\ln B}{\Gamma\left(\frac{n}{2}\right)}t.$$

Since the linear function $t\ln B$ is Lipschitz on $[0,+\infty)$, we can regard the function $t\ln B - A(t)$, where $A(t)$ is a function from [1, Th 1.1], as a new function $A(t)$. At the same time, we take into consideration that the function $\ln\frac{u_0}{B} = \ln u_0 - \ln B$ obviously belongs to each class of locally Hölder functions if and only if the function $\ln u_0$ belongs to the same class. This leads us to the following statement.

Proposition 2. *Let $u(x,t)$ be a classical bounded solution of the Cauchy problem for the equation*

$$\rho(x)\frac{\partial u}{\partial t} = \Delta u - \frac{1}{u}|\nabla u|^2,$$

where $\inf u \geqslant B > 0$, the coefficient $\rho(x)$ satisfies the assumptions of Theorem 1, and the initial function $u_0(x)$ is such that $\ln u_0 \in C^{\varkappa}_{\text{loc}}\big(\mathbb{R}^n\big)$. Then there exists a Lipschitz function A

on $[0,+\infty)$ *such that the relation*

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\int\limits_{0}^{t}\ln u(x,\tau)d\tau d\sigma_x=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}A(t)$$

holds for each positive t *and the relation*

$$\lim_{R\to\infty}\frac{1}{R^{n-1}}\int\limits_{|x|=R}\left(\int\limits_{0}^{t}\ln u(x,\tau)d\tau-A(t)\right)d\sigma_x=0$$

is satisfied uniformly in $t\in[0,T]$ *for each positive* T.

Remark. *It is known from* [26] *that Condition (i) in Theorem 1 (and, hence, in all five statements of the present work) can be replaced by the following equivalent condition*

$$\int\limits_{\mathbb{R}^n}\frac{\rho(\xi-x)d\xi}{|\xi|^{n-2}}\in L_\infty(\mathbb{R}^n).$$

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