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ON SOLVABILITY OF CLASS OF NONLINEAR INTEGRAL EQUATIONS IN p-ADIC STRING THEORY

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Abstract. In this paper, we study a class of integral equations with power nonlinearity on the entire real line. This class of equations arises in the p-adic theory of open-closed strings. Using the method of successive approximations and justifying their convergence, we prove the existence of a nontrivial continuous odd bounded solution on the entire line. The asymptotic behavior of the solution is studied as the argument increases unboundedly. We obtain integral estimates and some properties of approximations of the solution to the considered equation. Under some additional restrictions, we also establish the uniqueness of the constructed solution in a certain class of continuous functions. We provide examples of integral kernels of the equation satisfying all assumptions of the formulated theorems. As the nuclear function is a Gaussian distribution, from the proven results we obtain Vladimirov-Volovich theorem as a special case.

Keywords: successive approximations, limit of solution, pointwise convergence, continuity.

Mathematics Subject Classification: 47H10, 47H30

1. Introduction

The present work is devoted to studying the following nonlinear integral equation on the entire real line

$$\varphi^p(x) = \int_{-\infty}^{\infty} \lambda\left(|x|,|t|\right) K(x-t)\,\varphi(t)\,dt, \qquad x \in \mathbb{R} \tag{1.1}$$

for an unknown odd continuous on $\mathbb{R}$ function $\varphi(x)$. Here $p > 2$ is an arbitrary odd number and the functions λ and K possess the following properties:

(a) $\quad \lambda \in C(\mathbb{R}^+ \times \mathbb{R}^+); \qquad K \in C(\mathbb{R}), \qquad \mathbb{R}^+ \equiv [0,\infty);$

(b) $\quad K \in L_1(\mathbb{R}) \bigcap L_\infty(\mathbb{R}); \qquad K(\tau) \geqslant 0, \quad \tau \in \mathbb{R}; \quad \int\limits_{-\infty}^{\infty} K(\tau)\,d\tau = 1;$

(c) $\quad K(-\tau) = K(\tau), \quad \tau \in \mathbb{R}^+; \qquad \int\limits_{-\infty}^{\infty} t^2 K(t)\,dt < +\infty; \qquad K(\tau) \downarrow$ in τ on $\mathbb{R}^+$;

(d) $\quad 0 \leqslant \lambda(x,t) \leqslant 1, \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+; \qquad \int\limits_0^{\infty} t \sup\limits_{x\in\mathbb{R}^+} \left(1-\lambda(x,t)\right) dt < +\infty.$

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Equation (1.1) arises in p-adic string theory and describes the dynamics of tachyon of open-closed p-adic strings, see [1]-[3]. The considered equation is in some sense a further generalization of the equation with $\lambda(x,t)\equiv 1$ and with the kernel

$$K(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}$$

studied by academician V.S. Vladimirov and Ya.I. Volovich in work [2]. In work [4] by one of the authors, this equation (and the corresponding two-dimensional equation) was studied also in the case $\lambda(x,t)\equiv 1$, but with the kernel K satisfying only conditions (a)-(c).

In the present work we prove the existence of a nontrivial odd bounded solution on the entire real line. We calculate the limiting values of the constructed solution at $\pm\infty$. Under some additional restrictions we establish the uniqueness of the constructed solution in a certain class of continuous functions. In the end of the work, we provide examples of the functions $\lambda(x,t)$ and $K(x)$ obeying all assumptions of the formulated theorems.

2. Existence of a nontrivial continuous bounded solution on the entire real line

We begin with a key lemma, which will be used in proving the theorem on existence of a nontrivial continuous bounded solution to equation (1.1) on the entire real line. In order to do this, we consider the following auxiliary linear homogeneous Volterra equation:

$$\mathscr{B}(x)=\int_x^\infty \lambda(x,t)\left(V(t-x)-V(x+t)\right)\mathscr{B}(t)\,dt,\qquad x\in\mathbb{R}^+ \tag{2.1}$$

for a function $\mathscr{B}(x)$, where

$$V(\tau)=2K(\tau),\qquad \tau\in\mathbb{R}^+. \tag{2.2}$$

Lemma 2.1. *Under conditions (a)-(d), equation* (2.1) *possesses a non-negative nontrivial continuous bounded on the half-line* $\mathbb{R}^+$ *solution* $\mathscr{B}(x)$ *with the following asymptotics at infinity:* $\lim\limits_{x\to+\infty}\mathscr{B}(x)=1$.

Proof. We consider the following linear inhomogeneous Volterra equation:

$$\psi(x)=g(x)+\int_x^\infty \lambda(x,t)\left(V(t-x)-V(t+x)\right)\psi(t)\,dt,\qquad x\in\mathbb{R}^+, \tag{2.3}$$

with a special free term:

$$g(x)\equiv\int_x^\infty\left(1-\lambda(x,t)\right)V(t-x)\,dt+\int_x^\infty V(t+x)\,\lambda(x,t)\,dt,\qquad x\in\mathbb{R}^+. \tag{2.4}$$

It follows easily from properties (b) and (c) of the kernel K that

$$K(x-t)\geqslant K(x+t)\quad\text{for all}\quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+. \tag{2.5}$$

Due to (2.2), (2.5) and first property of the kernel K in (c) we have:

$$V(t-x)\geqslant V(x+t)\quad\text{for all}\quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+. \tag{2.6}$$

In view of properties (c) and (d) and (2.4), it is easy to confirm that the free term possesses the following properties:

$$g\in L_1(\mathbb{R}^+)\cap L_\infty(\mathbb{R}^+);\qquad m_1(g)\equiv\int_0^\infty xg(x)dx<+\infty. \tag{2.7}$$

By straightforward calculations we confirm that $\psi_0(x) \equiv 1$ is a solution to equation (2.3). Let us prove that apart of this solution, this equation possesses also a summable bounded on $\mathbb{R}^+$ solution $\psi_1(x)$. In order to do this, we consider the following successive approximations:

$$\psi^{(n+1)}(x) = g(x) + \int_x^\infty \lambda(x,t)\left(V(t-x) - V(t+x)\right)\psi^{(n)}(t)\,dt, \quad n = 0,1,2,\ldots \tag{2.8}$$

with the zero approximation $\psi^{(0)}(x) = 0$, $x \in \mathbb{R}^+$. By induction in n one can prove easily that the sequence of the functions $\{\psi^{(n)}(x)\}_{n=0}^\infty$ as $x \in \mathbb{R}^+$ increases in n:

$$\psi^{(n)}(x) \quad \uparrow \quad \text{in} \;\; n. \tag{2.9}$$

In what follows we shall show that this sequence of function is bounded from above by some summable on $\mathbb{R}^+$ function. Together with equation (2.3), we consider the following conservative recovering equation:

$$\tilde{\psi}(x) = g(x) + \int_x^\infty V(t-x)\,\tilde{\psi}(t)\,dt, \qquad x \in \mathbb{R}^+. \tag{2.10}$$

Taking into consideration the results of work [5], we conclude that equation (2.10) with the free term g of structure (2.4) and with properties (2.7) and with the kernel V of structure (2.2) and with properties (a)-(c) possesses a nonnegative summable bounded on $\mathbb{R}^+$ solution $\tilde{\psi}(x)$.

By induction in n, we are going to show that for all $x \in \mathbb{R}^+$, the estimate holds:

$$\psi^{(n)}(x) \leqslant \tilde{\psi}(x), \qquad n = 0,1,2,\ldots \tag{2.11}$$

Indeed, in the case $n = 0$, inequality (2.11) follows immediately (2.10) by the non-negativity of the functions V and $\tilde{\psi}$. Assume that (2.11) holds for some $n \in \mathbb{N}$. Then in view of property (d), inequality (2.6) and the non-negativity of the kernel function V, by (2.8) we get:

$$\psi^{(n+1)}(x) \leqslant g(x) + \int_x^\infty \lambda(x,t)\left(V(t-x) - V(t+x)\right)\tilde{\psi}(t)\,dt \leqslant g(x) + \int_x^\infty V(t-x)\,\tilde{\psi}(t)\,dt = \tilde{\psi}(x).$$

Applying again the induction in n, we easily confirm that

$$\psi^{(n)}(x) \leqslant 1, \quad x \in \mathbb{R}^+ \qquad \text{and} \qquad \psi^{(n)} \in C(\mathbb{R}^+), \quad n = 0,1,2,\ldots$$

Therefore, due to (2.9) and (2.11) we conclude that the sequence of continuous functions $\{\psi^{(n)}(x)\}_{n=0}^\infty$ has a pointwise limit as $n \to \infty$:

$$\lim_{n\to+\infty} \psi^{(n)}(x) = \psi(x).$$

Moreover, the limiting function satisfies the estimates

$$\psi(x) \leqslant 1, \qquad g(x) \leqslant \psi(x) \leqslant \tilde{\psi}(x), \quad x \in \mathbb{R}^+. \tag{2.12}$$

Since the pointwise limit of measurable function is measurable (see [6]), the limiting function ψ is also measurable. Since $\tilde{\psi} \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$, according inequality (2.12) we can state that

$$\psi \in L_1(\mathbb{R}^+) \,\cap\, L_\infty(\mathbb{R}^+). \tag{2.13}$$

It is obvious that the function

$$\mathscr{B}(x) = 1 - \psi(x) \geqslant 0, \qquad x \in \mathbb{R}^+, \tag{2.14}$$

is a non-trivial solution to homogeneous equation (2.1). Taking into consideration that $\lambda \in C(\mathbb{R}^+ \times \mathbb{R}^+)$, $V \in C(\mathbb{R}^+)$, by (2.1), (2.13) and (2.14) we infer that

$$\mathscr{B} \in C(\mathbb{R}^+), \tag{2.15}$$

and $1 - \mathscr{B} \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$.

Let us check that $\lim\limits_{x\to+\infty}\mathscr{B}(x)=1$. Indeed, due to formula (2.2) and properties (b), (d), it follows from equation (2.1) that

$$\begin{aligned}0\leqslant 1-\mathscr{B}(x)&=\int\limits_x^\infty\big(1-\lambda(x,t)\big)\,V(t-x)\,dt+\int\limits_{2x}^\infty V(\tau)\,d\tau\\&\leqslant\int\limits_x^\infty\sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,V(t-x)\,dt+\int\limits_{2x}^\infty V(\tau)\,d\tau\\&\leqslant C\int\limits_x^\infty\sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+\int\limits_{2x}^\infty V(\tau)\,d\tau\ \xrightarrow[x\to+\infty]{}0,\end{aligned}$$

where $C\equiv\sup\limits_{x\in\mathbb{R}^+}V(x)$. The proof is complete. □

We proceed to formulating and proving the first main result.

Theorem 2.1. *If conditions (a)-(d) hold, then equation* (1.1) *possesses a nontrivial odd continuous bounded solution on the entire real line* $\mathbb{R}$ *and*

$$\lim_{x\to\pm\infty}\varphi(x)=\pm1.$$

Moreover, if the function K *satisfies the strict inequality* $K(\tau)>0$, $\tau\in\mathbb{R}$, *then*

$$1-\varphi\in L_1(0,+\infty),\quad 1+\varphi\in L_1(-\infty,0).$$

Proof. Step I. By straightforward calculations one can easily check that if $f(x)$ is a continuous on $\mathbb{R}^+$ solution to the following nonlinear integral equation

$$f^p(x)=\int\limits_0^\infty\lambda(x,t)\big(K(x-t)-K(x+t)\big)\,f(t)\,dt,\qquad x\in\mathbb{R}^+,\tag{2.16}$$

then

$$\varphi(x)=\begin{cases}\ \ f(x)&\text{if}\quad x\geqslant0,\\-f(-x)&\text{if}\quad x<0,\end{cases}\tag{2.17}$$

is an odd continuous solution to equation (1.1). Letting

$$S(x)=f^p(x),\quad x\in\mathbb{R}^+,\tag{2.18}$$

we reduce the study of equation (2.16) to studying a nonlinear integral equation with a sum-difference kernel on the half-line for the function $S(x)$:

$$S(x)=\int\limits_0^\infty\lambda(x,t)\big(K(x-t)-K(x+t)\big)\,S^\alpha(t)\,dt,\quad x\in\mathbb{R}^+,\tag{2.19}$$

where

$$\alpha=\frac1p\in\Big(0,\,\frac12\Big).\tag{2.20}$$

Step II. For equation (2.19), we consider the following successive approximations:

$$\begin{aligned}&S_{n+1}(x)=\int\limits_0^\infty\lambda(x,t)\big(K(x-t)-K(x+t)\big)\,S_n^\alpha(t)\,dt,\\&S_0(x)\equiv1,\quad n=0,1,2,\dots,\quad x\in\mathbb{R}^+.\end{aligned}\tag{2.21}$$

Employing (2.5) and conditions (b)-(d), by induction in n we easily check that

$$S_n(x) \quad \downarrow \quad \text{in} \quad n. \tag{2.22}$$

Now we are going to prove that for each $n = 0, 1, 2, \ldots$ the following inequality holds:

$$S_n(x) \geqslant \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}} \mathscr{B}(x), \quad x \in \mathbb{R}^+. \tag{2.23}$$

In the case $n = 0$, the validity of inequality (2.23) is implied immediately by (2.14). Assume that (2.23) holds for some $n \in \mathbb{N}$. Then in view of (2.5), (2.2), (2.14) and (2.1), by (2.21) we get

$$\begin{aligned} S_{n+1}(x) &\geqslant \left(\frac{1}{2}\right)^{\frac{\alpha}{1-\alpha}} \int_0^\infty \lambda(x,t)\big(K(x-t) - K(x+t)\big)\mathscr{B}^\alpha(t)\,dt \\ &\geqslant \left(\frac{1}{2}\right)^{\frac{\alpha}{1-\alpha}} \int_x^\infty \lambda(x,t)\big(K(x-t) - K(x+t)\big)\mathscr{B}(t)\,dt \\ &\geqslant \left(\frac{1}{2}\right)^{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{2}\mathscr{B}(x) = \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}} \mathscr{B}(x). \end{aligned}$$

By induction in n one can check that

$$S_n \in C(\mathbb{R}^+), \quad n = 0, 1, 2, \ldots \tag{2.24}$$

Therefore, due to (2.22) and (2.23), the sequence of functions $\{S_n(x)\}_{n=0}^\infty$ possesses a pointwise limit as $n \to \infty$: $\lim\limits_{n\to\infty} S_n(x) = S(x)$ and according Levi theorem (see [6]), the limiting function solves equation (2.19) and satisfies the following two-sided inequality:

$$\left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}} \mathscr{B}(x) \leqslant S(x) \leqslant 1, \quad x \in \mathbb{R}^+. \tag{2.25}$$

Taking into consideration condition (a), by (2.19) and (2.24) we arrive at

$$S \in C(\mathbb{R}^+). \tag{2.26}$$

Then thanks to (2.24), (2.26) and Dini theorem, the convergence of the sequence of the functions $\{S_n(x)\}_{n=0}^\infty$ in each compact set in $\mathbb{R}^+$ is uniform.

Step III. Here the aim is to prove an additional property of the constructed solution S:

$$1 - S \in L_1(\mathbb{R}^+). \tag{2.27}$$

First, in view of inequality (2.23), we observe that

$$S_n^{\frac{1}{p}}(t) + S_n^{\frac{2}{p}}(t) + \cdots + S_n^{\frac{p-1}{p}}(t) \geqslant \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \mathscr{B}^{\frac{1}{p}}(t) + \left(\frac{1}{2}\right)^{\frac{2}{p-1}} \mathscr{B}^{\frac{2}{p}}(t) + \ldots + \frac{1}{2}\mathscr{B}^{\frac{p-1}{p}}(t). \tag{2.28}$$

Now we employ Lemma 2.1. Since $\lim\limits_{t\to+\infty} \mathscr{B}(t) = 1$, $\mathscr{B}(t) \leqslant 1$ and $\mathscr{B} \in C(\mathbb{R}^+)$, then for each $\varepsilon > 0$ there exists a number $r > 0$ such that the inequality $|1 - \mathscr{B}(t)| \leqslant \varepsilon$ is satisfied as $t > r$. Then, in particular, for $\varepsilon = \frac{1}{2}$ there exists a number $r_0 > 0$ such that

$$0 \leqslant 1 - \mathscr{B}(t) \leqslant \frac{1}{2} \qquad \text{and} \qquad t \geqslant r_0$$

or

$$\frac{1}{2} \leqslant \mathscr{B}(t) \leqslant 1 \quad \text{as} \quad t \geqslant r_0. \tag{2.29}$$

Taking into consideration inequality (2.29), by (2.28) we get that as $t \geqslant r_0$,

$$\begin{aligned} 1 + S_n^{\frac{1}{p}}(t) + S_n^{\frac{2}{p}}(t) + \cdots + S_n^{\frac{p-1}{p}}(t) & \\ \geqslant 1 + \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \left(\frac{1}{2}\right)^{\frac{1}{p}} + \left(\frac{1}{2}\right)^{\frac{2}{p-1}} \left(\frac{1}{2}\right)^{\frac{2}{p}} + \ldots + \frac{1}{2}\left(\frac{1}{2}\right)^{\frac{p-1}{p}} & \\ = \left(1 - \left(\frac{1}{2}\right)^{\frac{2p-1}{p-1}}\right)\left(1 - \left(\frac{1}{2}\right)^{\frac{2p-1}{p(p-1)}}\right)^{-1} \equiv \rho_1 > 1. & \end{aligned} \tag{2.30}$$

By induction in n we can check easily that

$$1 - S_n \in L_1(\mathbb{R}^+), \qquad n = 0, 1, 2, \ldots \tag{2.31}$$

In view of property (b) of the kernel K, we rewrite iterations (2.21) as follows:

$$\begin{aligned} 1 - S_{n+1}(x) = & \int_0^\infty \big(1 - \lambda(x,t)\, S_n^\alpha(t)\big)\, K(x-t)\, dt \\ & + \int_0^\infty \lambda(x,t)\, K(x+t)\, S_n^\alpha(t)\, dt + \int_x^\infty K(\tau)\, d\tau, \\ S_0(x) = & 1, \qquad n = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+. \end{aligned} \tag{2.32}$$

Since

$$\int_{-\infty}^\infty K(x) dx = 1 \quad \text{and} \quad K(x) > 0, \quad x \in \mathbb{R},$$

the inequality holds true:

$$\rho_0 \equiv \int_{-\infty}^{r_0} K(y) dy < 1. \tag{2.33}$$

Taking into consideration (2.30), (2.31), (2.33) and conditions (b)-(d), by (2.32) and Fubini theorem we have:

$$\begin{aligned} \int_0^\infty \big(1 - S_{n+1}(x)\big)\, dx = & \int_0^\infty \int_0^\infty \big(1 - \lambda(x,t)\, S_n^\alpha(t)\big)\, K(x-t)\, dt\, dx \\ & + \int_0^\infty \int_0^\infty \lambda(x,t)\, K(x+t)\, S_n^\alpha(t)\, dt\, dx + \int_0^\infty \int_x^\infty K(t)\, dt\, dx \\ \leqslant & \int_0^\infty \int_0^\infty \big(1 - \lambda(x,t)\big)\, K(x-t)\, dt\, dx + \int_0^\infty \int_0^\infty \lambda(x,t)\, K(x-t)\, \big(1 - S_n^\alpha(t)\big)\, dt\, dx \\ & + \int_0^\infty \int_0^\infty K(x+t)\, dt\, dx + \int_0^\infty t\, K(t)\, dt \\ \leqslant & \int_0^\infty \int_0^\infty \big(1 - \lambda(x,t)\big)\, K(x-t)\, dx\, dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_0^\infty K(x-t)\big(1-S_n^\alpha(t)\big)\,dx\,dt+2\int_0^\infty t\,K(t)\,dt\\
\leqslant&\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\int_0^\infty K(x-t)\,dx\,dt\\
&+\int_0^\infty\big(1-S_n^\alpha(t)\big)\int_{-\infty}^t K(u)\,du\,dt+2\int_0^\infty t\,K(t)\,dt\\
\leqslant&\int_0^\infty \frac{1-S_n(t)}{1+S_n^{\frac{1}{p}}(t)+S_n^{\frac{2}{p}}(t)+\cdots+S_n^{\frac{p-1}{p}}(t)}\int_{-\infty}^t K(u)\,du\,dt\\
&+\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+2\int_0^\infty t\,K(t)\,dt\\
=&\int_0^{r_0} \frac{1-S_n(t)}{1+S_n^{\frac{1}{p}}(t)+S_n^{\frac{2}{p}}(t)+\cdots+S_n^{\frac{p-1}{p}}(t)}\int_{-\infty}^t K(u)\,du\,dt\\
&+\int_{r_0}^\infty \frac{1-S_n(t)}{1+S_n^{\frac{1}{p}}(t)+S_n^{\frac{2}{p}}(t)+\cdots+S_n^{\frac{p-1}{p}}(t)}\int_{-\infty}^t K(u)\,du\,dt\\
&+\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+2\int_0^\infty t\,K(t)\,dt\\
\leqslant&\rho_0\int_0^{r_0}\big(1-S_n(t)\big)\,dt+\frac{1}{\rho_1}\int_{r_0}^\infty\big(1-S_n(t)\big)\,dt\\
&+\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+2\int_0^\infty t\,K(t)\,dt\\
\leqslant&\max\left(\rho_0,\frac{1}{\rho_1}\right)\int_0^\infty\big(1-S_{n+1}(t)\big)\,dt\\
&+\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+2\int_0^\infty t\,K(t)\,dt.
\end{aligned}
$$

This yields:

$$
\begin{aligned}
\int_0^\infty\big(1-S_{n+1}(x)\big)\,dx\leqslant&\left(1-\max\left(\rho_0,\frac{1}{\rho_1}\right)\right)^{-1}\\
&\cdot\left(\int_0^\infty \sup_{x\in\mathbb{R}^+}\big(1-\lambda(x,t)\big)\,dt+2\int_0^\infty t\,K\,(t)\,dt\right),\quad n=0,1,2,\ldots
\end{aligned}
\tag{2.34}
$$

Thus, in view of (2.22), (2.34) and Levi theorem, we confirm that (2.27) is true. We also obtain the following upper bound:

$$\int_0^\infty \big(1 - S(x)\big)\,dx \leqslant \left(1 - \max\left(\rho_0, \frac{1}{\rho_1}\right)\right)^{-1} \left(\int_0^\infty \sup_{x\in\mathbb{R}^+} \big(1-\lambda(x,t)\big)\,dt + 2\int_0^\infty t\,K(t)\,dt\right). \quad (2.35)$$

Step IV. At the last step we prove the last statement of the theorem:

$$\lim_{x\to\pm\infty} \varphi(x) = \pm 1, \qquad 1-\varphi \in L_1(0,+\infty), \qquad 1+\varphi \in L_1(-\infty, 0). \quad (2.36)$$

This is why we first show that $\lim\limits_{x\to+\infty} S(x) = 1$. Indeed, in view of (2.25), (2.27) and (b)-(d), by (2.19) we infer that

$$\begin{aligned} 0 \leqslant 1 - S(x) &\leqslant \int_0^\infty \big(1-\lambda(x,t)\big)\,K(x-t)\,dt + \int_0^\infty \lambda(x,t)\,K(x-t)\,\big(1-S^\alpha(t)\big)\,dt + 2\int_x^\infty K(u)\,du \\ &\leqslant \int_0^\infty \sup_{x\in\mathbb{R}^+} \big(1-\lambda(x,t)\big)\,K(x-t)\,dt + \int_0^\infty K(x-t)\,\big(1-S(t)\big)\,dt + 2\int_x^\infty K(u)\,du \xrightarrow[x\to+\infty]{} 0, \end{aligned}$$

since the convolution of summable and bounded functions tends to zero at infinity, see [7]. Therefore, $\lim\limits_{x\to+\infty} S(x) = 1$. Then by (2.18) we get that

$$\lim_{x\to+\infty} f(x) = 1. \quad (2.37)$$

Moreover, in view of (2.18), (2.27) and the following simple inequality

$$0 \leqslant 1 - f(x) \leqslant 1 - f^p(x), \qquad x\in\mathbb{R}^+,$$

we conclude that $1 - f \in L_1(\mathbb{R}^+)$.

Finally, taking into consideration the continuity of the function $f(x)$ on $\mathbb{R}^+$ as well as the formulae (2.17) and (2.37), we arrive at (2.36). The proof is complete. $\square$

3. Uniqueness of solution to equation (1.1) in a particular case

In what follows we prove three auxiliary lemmata employed in what follows.

We consider the following integral Hammerstein-Volterra equation with a power nonlinearity:

$$h(x) = \int_x^\infty \lambda(x,t)\big(K(t-x) - K(t+x)\big)h^\alpha(t)dt, \quad x\in\mathbb{R}^+, \quad (3.1)$$

for an unknown function $h(x)$. The following lemma holds.

Lemma 3.1. *Under assumptions of Theorem 2.1, equation* (3.1) *possesses a non-negative continuous bounded solution $h(x)$.*

Proof. We consider the following successive approximations:

$$\begin{aligned} h_{n+1}(x) &= \int_x^\infty \lambda(x,t)\big(K(t-x) - K(t+x)\big)\,h_n^\alpha(t)dt; \\ h_0(x) &= \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}}, \quad n = 0,1,2,\dots, \quad x\in\mathbb{R}^+. \end{aligned} \quad (3.2)$$

By induction one can show easily that the functions in the sequence $\{h_n(x)\}_{n=0}^{\infty}$ possess the following properties:

$$h_n(x) \downarrow \text{ in } n, \tag{3.3}$$

$$h_n(x) \geqslant \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}} \mathscr{B}(x), \quad n = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+, \tag{3.4}$$

$$h_n \in C(\mathbb{R}^+), \quad n = 0, 1, 2, \dots, \tag{3.5}$$

where $\mathscr{B}(x)$ is a solution to homogeneous equation (2.1) with the following asymptotics at infinity: $\lim\limits_{x\to+\infty} \mathscr{B}(x) = 1$. Therefore, the sequence of continuous functions $\{h_n(x)\}_{n=0}^{\infty}$ possesses a pointwise limit as $n \to +\infty$: $\lim\limits_{n\to+\infty} h_n(x) = h(x)$. According Levi theorem, the limiting function $h(x)$ satisfies equation (3.1) and the upper and lower bounds hold:

$$\left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}} \mathscr{B}(x) \leqslant h(x) \leqslant \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}}, \quad x \in \mathbb{R}^+. \tag{3.6}$$

Then, in view of (3.6) and the continuity of the functions λ and K, it follows from (3.1) that $h \in C(\mathbb{R}^+)$. The proof is complete. $\square$

Now let us confirm that if λ is *monotonically non-decreasing function in the variables* (x, t), that is, as $(x_i, t_i) \in \mathbb{R}^+ \times \mathbb{R}^+$, $i = 1, 2$, and $x_1 > x_2$, $t_1 > t_2$, the inequality

$$\lambda(x_1, t_1) \geqslant \lambda(x_2, t_2)$$

holds, then the constructed solution is also monotonically non-decreasing. Namely, the following lemma is true.

Lemma 3.2. *If the assumptions of Lemma 3.1 hold and the function λ is monotonically non-decreasing in its variables, then the constructed solution $h(x)$ of equation* (3.1) *is also monotonically non-decreasing and the following lower bound holds:*

$$h^{\frac{p-1}{p}}(x) \geqslant \lambda(x, x) \left(\frac{1}{2} - Q(2x)\right), \qquad x \in \mathbb{R}^+, \tag{3.7}$$

where

$$Q(\tau) \equiv \int\limits_{\tau}^{\infty} K(u)du, \quad \tau \in \mathbb{R}^+. \tag{3.8}$$

Proof. We rewrite iterations (3.2) as

$$\begin{gathered} h_{n+1}(x) = \int\limits_0^{\infty} \lambda(x, x+\tau)\big(K(\tau) - K(2x+\tau)\big)\, h_n^{\alpha}(x+\tau)\, d\tau, \\ h_0(x) = \left(\frac{1}{2}\right)^{\frac{1}{1-\alpha}}, \quad n = 0, 1, 2, \dots, \quad x \in \mathbb{R}^+. \end{gathered} \tag{3.9}$$

The monotonicity of the zero approximation follows immediately (3.9). Assume that $h_n(x) \uparrow$ in x for some fixed $n \in \mathbb{N}$. Then if x_1, $x_2 \in \mathbb{R}^+$ and $x_1 > x_2$, in view of the monotonicity of the function λ in its variables, properties (b), (c) of the kernel K and the induction assumption, by (3.9) we have:

$$h_{n+1}(x_1) - h_{n+1}(x_2) = \int\limits_0^{\infty} \lambda(x_1, x_1+\tau)\left(K(\tau) - K(2x_1+\tau)\right) h_n^{\alpha}(x_1+\tau)\, d\tau$$

$$-\int_0^\infty \lambda(x_2, x_2+\tau)\Big(K(\tau)-K(2x_2+\tau)\Big)\, h_n^\alpha(x_2+\tau)d\tau$$

$$\geqslant \int_0^\infty \Big(\lambda(x_1, x_1+\tau)-\lambda(x_2, x_2+\tau)\Big)\Big(K(\tau)-K(2x_2+\tau)\Big)\, h_n^\alpha(x_2+\tau)\, d\tau \geqslant 0,$$

that is, $h_{n+1}(x_1) \geqslant h_{n+1}(x_2)$. Thus, we have established the monotonicity in x of each function in the sequence $\{h_n(x)\}_{n=0}^\infty$. Then the limiting function h is also monotonically non-decreasing. In view of the monotonicity of the function λ in (3.1) we obtain:

$$h(x) \geqslant h^\alpha(x) \int_x^\infty \lambda(x,t)\left(K(t-x)-K(t+x)\right) dt \geqslant h^\alpha(x)\lambda(x,x)\left(\frac{1}{2}-Q(2x)\right)$$

since $\int\limits_0^\infty K(t)dt = \frac{1}{2}$. By the latter inequality we arrive at (3.7). The proof is complete. □

In the next lemma we prove that if the function λ is monotonically non-decreasing in its variables, then the solution $S(x)$ constructed by successive approximations (2.21) satisfies the lower bound:

$$S(x) \geqslant \left(\lambda(x,x)\left(\frac{1}{2}-Q(2x)\right)\right)^{\frac{p}{p-1}}, \quad x \in \mathbb{R}^+. \tag{3.10}$$

The following lemma holds.

Lemma 3.3. *Under assumptions of Lemma 3.2, the solution $S(x)$ to equation* (2.19) *constructed by means of iterations* (2.21) *satisfies estimate* (3.10).

Proof. We consider iterations (2.21). By induction in n we can show easily that

$$S_n(x) \geqslant h(x), \quad n = 0, 1, 2, \ldots, \quad x \in \mathbb{R}^+. \tag{3.11}$$

Therefore, the limiting function $S(x) = \lim\limits_{n\to\infty} S_n(x)$ satisfies the inequality:

$$S(x) \geqslant h(x), \quad x \in \mathbb{R}^+. \tag{3.12}$$

By (3.12) and (3.7) we now complete the proof. □

Now we can proceed to proving the uniqueness theorem, which is the second main result of the present work.

Theorem 3.1. *If under the assumptions of Lemma 3.2 the inequality*

$$\lambda(x,x)\Big(1-2Q(2x)\Big) > 1-2Q(x), \quad x > 0, \tag{3.13}$$

holds, then a solution to equation (2.16) *is unique in the following class of continuous on $\mathbb{R}^+$ functions:*

$$\mathfrak{M} \equiv \left\{ f(x) \,\middle|\, \left(\lambda(x,x)\left(\frac{1}{2}-Q(2x)\right)\right)^{\frac{1}{p-1}} \leqslant f(x) \leqslant 1, \quad x \in \mathbb{R}^+; \quad 1-f \in L_1(\mathbb{R}^+) \right\}. \tag{3.14}$$

Proof. Suppose that equation (2.16) possesses two solutions $f_1, f_2 \in \mathfrak{M}$. Then according Lagrangue mean value theorem and (3.14), by (2.16) we have:

$$p\,\lambda(x,x)\left(\frac{1}{2}-Q(2x)\right)\Big|f_1(x)-f_2(x)\Big| \leqslant \int_0^\infty \lambda(x,t)\left(K(x-t)-K(x+t)\right)\Big|f_1(t)-f_2(t)\Big|\, dt. \tag{3.15}$$

It follows directly from (3.14) that $f_1 - f_2 \in L_1(\mathbb{R}^+)$. Integrating both sides of (3.15) in x from 0 to $+\infty$ and taking into consideration conditions (b)-(d) and Fubini theorem, we obtain

$$p\int_0^\infty \lambda(x,x)\left(\frac{1}{2}-Q(2x)\right)\Big|f_1(x)-f_2(x)\Big|dx \leqslant \int_0^\infty \Big|f_1(t)-f_2(t)\Big|\Big(1-2Q(t)\Big)\,dt.$$

By condition $p > 2$ this leads us to the inequality:

$$\int_0^\infty \Big|f_1(x)-f_2(x)\Big|\Big(\lambda(x,x)\left(1-2Q(2x)\right)-\left(1-2Q(x)\right)\Big)\,dx \leqslant 0. \tag{3.16}$$

In view of (3.13), by (3.16) we obtain that $f_1(x) = f_2(x)$ almost everywhere in $\mathbb{R}^+$. The proof is complete. $\square$

Remark 3.1. *Since the sought function in equation* (1.1) *is odd and satsifies the boundary conditions* $\lim\limits_{x\to\pm\infty}\varphi(x) = \pm 1$, *by formula* (2.17) *and Theorem 3.1 we conclude that the boundary value problem for equation* (1.1) *for an odd on* $\mathbb{R}$ *function* φ

$$\begin{cases} \varphi^p(x) = \displaystyle\int_{-\infty}^{\infty}\lambda\big(|x|,|t|\big)\,K(x-t)\varphi(t)dt, \quad x\in\mathbb{R}, \\ \varphi(\pm\infty) = \pm 1 \end{cases}$$

possesses the unique solution in the following class of odd continuous functions:

$$\mathfrak{P} \equiv \left\{\varphi(x)\ \middle|\ \varphi\in C(\mathbb{R});\ \varphi(x) = \begin{cases} f(x) & if \quad x\geqslant 0, \\ -f(-x) & if \quad x<0; \end{cases} f\in\mathfrak{M}\right\}.$$

Remark 3.2. *We note that a solution to equation* (1.1) *is non-negative on the non-negative semi-axis.*

Remark 3.3. *Here we provide the examples of functions* $\lambda(x,t)$ *and* $K(x)$ *obeying all assumptions of the above theorems:*

$$\lambda(x,t) = 1-\varepsilon e^{-x}e^{-t}, \quad \varepsilon\in\left(0,\frac{1}{2}\right), \quad (x,t)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{3.17}$$

$$K(x) = \frac{1}{2}e^{-|x|}, \quad x\in\mathbb{R}. \tag{3.18}$$

It is obvious that the kernel $K(x)$ *satisfies conditions (a)-(c). By straightforward calculations one can confirm that* λ *satisfies conditions (a), (d) and is monotonous in its variables.*

Below we show that the functions λ and K given by (3.17) and (3.18) satisfy inequality (3.13). We observe that in this case estimate (3.13) casts into the form:

$$(1-e^{-2x})(1-\varepsilon e^{-2x}) > 1-e^{-x}, \quad x>0. \tag{3.19}$$

This inequality is equivalent to

$$e^x + \varepsilon e^{-2x} > \varepsilon + 1, \ x > 0. \tag{3.20}$$

We consider the function

$$G(x) = e^x + \varepsilon e^{-2x} - (\varepsilon+1), \quad x>0.$$

It is obvious that

$$G(+0) = 0, \qquad G'(x) = e^x - 2\varepsilon e^{-2x} = e^{-2x}(e^{3x}-2\varepsilon) \ > \ e^{-2x}(1-2\varepsilon) > 0$$

as $x>0$ and $\varepsilon\in\left(0,\frac{1}{2}\right)$. Thus, inequality (3.20) holds and hence, the same is true for (3.19).

Remark 3.4. *In the case* $K(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ *and* $\lambda(x, t) \equiv 1$ *we see easily that condition* (3.13) *holds as well.*

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