ISSN 2304-0122 Ufa Mathematical Journal. Vol. 10. No 3 (2018). P. 86-105.

doi:10.13108/2018-10-3-86 UDC 517.9

ALGEBRAIC PROPERTIES OF QUASILINEAR TWO-DIMENSIONAL LATTICES CONNECTED WITH INTEGRABILITY

I.T. HABIBULLIN, M.N. POPTSOVA

Abstract. In the paper we discuss a classification method for nonlinear integrable equations with three independent variables based on the notion of the integrable reductions. We call an equation integrable if it admits a large class of reductions being Darboux integrable systems of hyperbolic type equations with two independent variables. The most natural and convenient object to be studied in the framework of this scheme is the class of two dimensional lattices generalizing the well-known Toda lattice. In the present article we study the quasilinear lattices of the form

$$\begin{aligned}u_{n,xy} =&\alpha(u_{n+1},u_n,u_{n-1})u_{n,x}u_{n,y}+\beta(u_{n+1},u_n,u_{n-1})u_{n,x}\\&+\gamma(u_{n+1},u_n,u_{n-1})u_{n,y}+\delta(u_{n+1},u_n,u_{n-1}).\end{aligned}$$

We specify the coefficients of the lattice assuming that there exist cutting off conditions which reduce the lattice to a Darboux integrable hyperbolic type system of the arbitrarily high order. Under some extra assumption of nondegeneracy we describe the class of the lattices integrable in the above sense. There are new examples in the obtained list of chains.

Keywords: two-dimensional integrable lattice, x-integral, integrable reduction, cut-off condition, open chain, Darboux integrable system, characteristic Lie algebra

Mathematics Subject Classification: 37K10, 37K30, 37D99

1. Introduction

Integrable equations with three independent variables have a wide range of applications in physics. It suffices to recall such well-known nonlinear models as the KP equation, the Davey-Stewartson equation, the Toda lattice equation, and so on. From the point of view of integration and classification, multidimensional equations are the most complex. Different approaches to study the integrable multidimensional models were discussed, for example, in the papers [1]–[9]. It is known that the symmetry approach [10, 11], which proved to be a very effective method for classifying integrable equations in $1+1$ dimensions, is not so effective in the multidimensional case [12]. For studying multidimensional equations, the idea of the reduction is often used, and the matter is to replace an equation by a system of equations with less independent variables. The existence of a wide class of integrable reductions with two independent variables, as a rule, indicates the integrability of an equation with three independent variables. Among the specialists, the most popular method is the method of hydrodynamic reductions, when the presence of an infinite set of integrable systems of hydrodynamic type is taken as a sign of integrability of the equation, and the general solution of each such system generates some solution to the considered equation (see, for example, [13], [1], [2]). The history of the method and related references can be found in survey [3].

M.N. Poptsova, I.T. Habibullin, Algebraic properties of quasilinear two-dimensional lattices connected with integrability.

The authors gratefully acknowledge financial support from a Russian Science Foundation grant (project 15-11-20007).

In our works [14, 15] we use an alternative approach. We call a given equation integrable if it admits an infinite class of reductions in the form of Darboux-integrable systems of hyperbolic partial differential equations with two independent variables. In solving classification problems for multidimensional equations in this formulation, the theory of characteristic Lie algebras can be employed (a detailed exposition can be found in [17, 18]). This direction in the integrability theory seems to us promising.

Consider a nonlinear chain

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}) \tag{1.1}$$

with three independent variables, where the uknown function $u = u_n(x, y)$ depends on real x, y, and on integer n. For chain (1.1), the desired finite-field reductions are obtained in a natural way. It is sufficient to break off appropriately the chain at two integer points

$$u_{N_1} = \varphi_1(x, y, u_{N_1+1}, \ldots), \tag{1.2}$$

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \quad N_1 < n < N_2, \tag{1.3}$$

$$u_{N_2} = \varphi_2(x, y, u_{N_2-1}, \ldots). \tag{1.4}$$

Examples of such boundary conditions can be found below (see (4.28), (4.29)). The following two very significant circumstances should be noted:

i) for each known integrable chain of form (1.1), there are cut-off conditions reducing it to a Darboux-integrable system of form (1.2)-(1.4) of arbitrarily large order $N = N_2 - N_1 - 1$;
ii) specific form of the functions φ_1, φ_2, and f is constructively determined by the requirement of integrability of the system in the Darboux sense.

These two facts serve as motivation for the following definition (see also the work [14]):

Definintion 1. *A chain* (1.1) *is called integrable if there exist functions* φ_1 *and* φ_2 *such that for each choice of a pair of integers* N_1, N_2, *where* $N_1 < N_2 - 1$, *hyperbolic type system* (1.2)–(1.4) *is Darboux integrable.*

In the present paper we study quasilinear chains of the following form

$$u_{n,xy} = \alpha u_{n,x} u_{n,y} + \beta u_{n,x} + \gamma u_{n,y} + \delta, \tag{1.5}$$

assuming that the functions $\alpha = \alpha(u_{n+1}, u_n, u_{n-1})$, $\beta = \beta(u_{n+1}, u_n, u_{n-1})$, $\gamma = \gamma(u_{n+1}, u_n, u_{n-1})$, $\delta = \delta(u_{n+1}, u_n, u_{n-1})$ are analytic in a domain $D \subset \mathbb{C}^3$. We also assume that the derivatives

$$\frac{\partial \alpha(u_{n+1}, u_n, u_{n-1})}{\partial u_{n+1}} \quad \text{and} \quad \frac{\partial \alpha(u_{n+1}, u_n, u_{n-1})}{\partial u_{n-1}} \tag{1.6}$$

are non-zero.

The main result of this paper is the proof of the following assertion.

Theorem 1. *The quasilinear chain* (1.5), (1.6) *is integrable in the sense of Definition 1 if and only if it is reduced by point transformations to one of the following forms*

$$\begin{aligned}
&i) \quad u_{n,xy} = \alpha_n u_{n,x} u_{n,y},\\
&ii) \quad u_{n,xy} = \alpha_n (u_{n,x} u_{n,y} - u_n (u_{n,x} + u_{n,y}) + u_n^2) + u_{n,x} + u_{n,y} - u_n,\\
&iii) \quad u_{n,xy} = \alpha_n (u_{n,x} u_{n,y} - s_n (u_{n,x} + u_{n,y}) + s_n^2) + s_n' (u_{n,x} + u_{n,y} - s_n),
\end{aligned}$$

where

$$s_n = u_n^2 + C, \quad s_n' = 2u_n, \quad \alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n},$$

C *is an arbitrary constant.*

We note that equation i) was found earlier in the papers [27], [28] by Ferapontov, Shabat and Yamilov, equations ii) and iii) appear to be new. By applying additional conditions of the form $x = \pm y$ to the equations i)-iii), we obtain $1+1$-dimensional integrable chains. It can be shown that by point transformations they are reduced to the equations found earlier by Yamilov (see [29]).

Following Definition 1, we suppose that in the integrable case there are cut-off conditions at two integer points $n = N_1$, $n = N_2$ ($N_1 < N_2 - 1$), which reduce chain (1.5) to a finite system of hyperbolic equations

$$\begin{aligned} &u_{N_1} = \varphi_1, \\ &u_{n,xy} = \alpha_n u_{n,x} u_{n,y} + \beta_n u_{n,x} + \gamma_n u_{n,y} + \delta_n, \quad N_1 < n < N_2, && (1.7) \\ &u_{N_2} = \varphi_2, && (1.8) \end{aligned}$$

integrable in the Darboux sense.

We recall that the system of partial differential equations of the hyperbolic type (1.7) is Darboux integrable if it possesses a complete set of functionally independent $x-$ and $y-$integrals. A function I depending on a finite set of dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y, \dots$ is called a y-integral if it satisfies the equation $D_y I = 0$, where D_y is the operator of total differentiation with respect to the variable y, and the vector $\mathbf{u}$ has the coordinates $u_{N_1+1}, u_{N_1+2}, \dots, u_{N_2-1}$. Since system (1.7) is autonomous, we consider only autonomous nontrivial integrals. It can be shown that an $y-$integral is independent of $\mathbf{u}_y$, $\mathbf{u}_{yy}$, ... Therefore, we will consider only y-integrals depending on at least one dynamic variable $\mathbf{u}, \mathbf{u}_x, \dots$ We note that nowadays the Darboux integrable discrete and continuous models are intensively studied (see, [14], [17], [19]-[26]).

We give one more argument in favor of our Definition 1 concerning the integrability property of a two-dimensional chain. The problem on finding a general solution to a Darboux-integrable system reduces to solving a system of ordinary differential equations. Usually these ODEs are solved explicitly. On the other hand, any solution of the considered hyperbolic system (1.7) easily extends beyond the interval $[N_1, N_2]$ and generates the solution of the corresponding chain (1.5). Therefore, in this case chain (1.5) has a large set of exact solutions.

Let us briefly explain the structure of the paper. In Section 2 we recall the needed definitions and study the basic properties of the characteristic Lie algebra, which is the main tool in the theory of Darboux-integrable systems. In Section 3 we introduce the definition of test sequences, by means of which we obtain a system of differential equations for the refinement of the functions α, β, γ. Section 4 is devoted to the search for the function δ. Here we also give the final form of desired chain (4.27) integrable in the sense of Definition 1 and we provide the proof of Theorem 1.

2. Characteristic Lie algebras

Since chain (1.5) is invariant under the shift of the variable n, without loss of generality we can put $N_1 = -1$. In what follows we consider a system of hyperbolic equations

$$\begin{aligned} &u_{-1} = \varphi_1, \\ &u_{n,xy} = \alpha_n u_{n,x} u_{n,y} + \beta_n u_{n,x} + \gamma_n u_{n,y} + \delta_n, \quad 0 \leqslant n \leqslant N, && (2.1) \\ &u_{N+1} = \varphi_2. \end{aligned}$$

Recall that here $\alpha_n = \alpha(u_{n-1}, u_n, u_{n+1})$, $\beta_n = \beta(u_{n-1}, u_n, u_{n+1})$, $\gamma_n = \gamma(u_{n-1}, u_n, u_{n+1})$, $\delta_n = \delta(u_{n-1}, u_n, u_{n+1})$. Suppose that system (2.1) is Darboux integrable and that $I(\mathbf{u}, \mathbf{u}_x, \dots)$ is its nontrivial y-integral. The latter means that the function I must satisfy the equation $D_y I = 0$, where D_y is the operator of total derivative with respect to the variable y. The operator D_y acts on the class of functions of the form $I(\mathbf{u}, \mathbf{u}_x, \dots)$ by the rule $D_y I = YI$, where

$$Y = \sum_{i=0}^{N} \left(u_{i,y} \frac{\partial}{\partial u_i} + f_i \frac{\partial}{\partial u_{i,x}} + f_{i,x} \frac{\partial}{\partial u_{i,xx}} + \cdots \right). \tag{2.2}$$

Here $f_i = \alpha_i u_{i,x} u_{i,y} + \beta_i u_{i,x} + \gamma_i u_{i,y} + \delta_i$ is the right hand side of lattice (1.5). Hence, the function I solves the equation $YI = 0$. The coefficients of the equation $YI = 0$ depend on the variables $u_{i,y}$, while its solution I is independent of $u_{i,y}$, therefore the function I in fact satisfies the system of linear equations:

$$YI = 0, \quad X_j I = 0, \quad j = 1, \dots, N, \tag{2.3}$$

where $X_i = \frac{\partial}{\partial u_{i,y}}$. It follows from (2.3) that the commutator $Y_i = [X_i, Y]$ of the operators Y and X_i for $i = 0, 1, \dots N$ also annuls I. We use the explicit coordinate representation of the operator Y_i:

$$Y_i = \frac{\partial}{\partial u_i} + X_i(f_i)\frac{\partial}{\partial u_{i,x}} + X_i(D_x f_i)\frac{\partial}{\partial u_{i,xx}} + \cdots \tag{2.4}$$

By the special form of the function f_i, the operator Y can be represented as

$$Y = \sum_{i=0}^{N} u_{i,y} Y_i + R, \tag{2.5}$$

where

$$\begin{aligned} R &= \sum_{i=0}^{N} (f_i - u_{i,y} X_i(f_i))\frac{\partial}{\partial u_{i,x}} + (f_{i,x} - u_{i,y} X_i(D_x f_i))\frac{\partial}{\partial u_{i,xx}} + \cdots \\ &= \sum_{i=0}^{N} (\beta_i u_{i,x} + \delta_i)\frac{\partial}{\partial u_{i,x}} + \big((\alpha_i u_{i,x} + \gamma_i)(\beta_i u_{i,x} + \delta_i) + D_x(\beta_i u_{i,x} + \delta_i)\big)\frac{\partial}{\partial u_{i,xx}} + \cdots \end{aligned} \tag{2.6}$$

We denote by $\mathbf{F}$ the ring of locally analytic functions of the dynamical variables $\mathbf{u}$, $\mathbf{u}_x$, $\mathbf{u}_y$, ... We consider the Lie algebra $\mathcal{L}(y, N)$ over the ring $\mathbf{F}$ generated by the differential operators Y, Y_0, Y_1, ..., Y_N. It is clear that the commutator of two vector fields and the multiplication of a vector field by a function satisfy the following conditions:

$$[Z, gW] = Z(g)W + g[Z, W], \tag{2.7}$$

$$(gZ)h = gZ(h), \tag{2.8}$$

where $Z, W \in \mathcal{L}(y, N)$, $g, h \in \mathbf{F}$. Consequently, the pair $(\mathbf{F}, \mathcal{L}(y, N))$ has the structure of the Lie-Rinehart algebra[1] (see [30]). We call this algebra the characteristic Lie algebra of the system of equations (2.1) along the direction y. It is well known (see [20, 17]) that the function I is a y-integral of the system (2.1) if and only if it belongs to the kernel of each operator in $\mathcal{L}(y, N)$. Since the y-integral depends only on a finite number of dynamic variables, we can use the well-known Jacobi theorem on the existence of a nontrivial solution of a system to first-order linear differential equations with one unknown function. By this theorem, it is easy to confirm that in the Darboux integrable case in the algebra $\mathcal{L}(y, N)$ there exists a finite basis $Z_1, Z_2, \dots Z_k$, consisting of linearly independent operators such that each element Z of $\mathcal{L}(y, N)$ can be represented as a linear combination $Z = a_1 Z_1 + a_2 Z_2 + \dots a_k Z_k$, where the coefficients $a_1, a_2, \dots, a_k$ are analytic functions of dynamical variables defined in some open set. Moreover, the identity $a_1 Z_1 + a_2 Z_2 + \dots a_k Z_k = 0$ implies that $a_1 = a_2 = \dots = a_k = 0$. In this case, we call the algebra $\mathcal{L}(y, N)$ finite-dimensional. Similarly, we can define the characteristic algebra $\mathcal{L}(x, N)$ in the direction x. It is clear that the system (2.1) is Darboux integrable if and only if the characteristic algebras in both directions are finite-dimensional.

For the sake of convenience, we introduce the notation $\mathrm{ad}_X(Z) := [X, Z]$. We note that in our study the operator ad_{D_x} plays a key role. Below we shall apply D_x to smooth functions depending on dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots$. As it has been shown above, the operators D_y and Y coincide on this class of functions. Therefore, the identity $[D_x, D_y] = 0$ immediately implies $[D_x, Y] = 0$. Replacing Y by virtue of (2.5), and collecting in the resulting relation the coefficients of the independent variables $\{u_{i,y}\}_{i=0}^{N}$, we obtain

$$[D_x, Y_i] = -a_i Y_i, \quad \text{where} \quad a_i = \alpha_i u_{i,x} + \gamma_i. \tag{2.9}$$

[1]We thank D.V. Millionshchikov who drew our attention to this circumstance.

It is clear that the operator ad_{D_x} takes the characteristic Lie algebra into itself. The next lemma describes the kernel of this mapping.

Lemma 1. [16, 18, 17] *If the vector field of the form*

$$Z = \sum_i z_{1,i} \frac{\partial}{\partial u_{i,x}} + z_{2,i} \frac{\partial}{\partial u_{i,xx}} + \cdots \tag{2.10}$$

solves the equation $[D_x, Z] = 0$, *then* $Z = 0$.

3. Method of test sequences

We call the sequence of operators $W_0, W_1, W_2, \dots$ in the algebra $\mathcal{L}(y, N)$ a test sequence if

$$[D_x, W_m] = \sum_{j=0}^{m} w_{j,m} W_j \tag{3.1}$$

holds for all m. The test sequence allows us to derive the integrability conditions for a system of hyperbolic type (2.1) (see [19], [17], [20]). Indeed, assume that (2.1) is Darboux integrable. Then among the operators $W_0, W_1, W_2, \dots$ there is only a finite set of linearly independent elements in terms of which all the other operators can be expressed. In other words, there exists an integer k such that the operators $W_0, \dots W_k$ are linearly independent and W_{k+1} is expressed as follows:

$$W_{k+1} = \lambda_k W_k + \cdots + \lambda_0 W_0. \tag{3.2}$$

We apply the operator ad_{D_x} to both sides of identity (3.2). As a result, we obtain the relation

$$\begin{aligned} &\sum_{j=0}^{k} w_{j,k+1} W_j + w_{k+1,k+1} \sum_{j=0}^{k} \lambda_j W_j \\ &\qquad = \sum_{j=0}^{k} D_x(\lambda_j) W_j + \lambda_k \sum_{j=0}^{k} w_{j,k} W_j + \lambda_{k-1} \sum_{j=0}^{k-1} w_{j,k-1} W_j + \cdots + \lambda_0 w_{0,0} W_0. \end{aligned} \tag{3.3}$$

Collecting coefficients at independent operators, we obtain a system of differential equations for the coefficients $\lambda_0, \lambda_1, \dots \lambda_k$. The resulting system is overdetermined, since λ_j is a function of a finite number of dynamical variables $\mathbf{u}, \mathbf{u}_x, \dots$. The compatibility conditions for this system define the integrability conditions for (2.1). For example, collecting the coefficients at W_k, we get the first equation of the mentioned system:

$$D_x(\lambda_k) = \lambda_k (w_{k+1,k+1} - w_{k,k}) + w_{k,k+1}, \tag{3.4}$$

which is also overdetermined.

Below in this section, we use two test sequences to refine the form of the functions α_n, β_n, γ_n.

3.1. First test sequence.

Let us define a sequence of operators in the characteristic algebra $\mathcal{L}(y, N)$ by the following recurrence formula:

$$Y_0, \quad Y_1, \quad W_1 = [Y_0, Y_1], \quad W_2 = [Y_0, W_1], \dots W_{k+1} = [Y_0, W_k], \dots \tag{3.5}$$

Above (see (2.9)), the commutation relations for the first two terms of this sequence were derived:

$$[D_x, Y_0] = -a_0 Y_0 = -(\alpha_0 u_{0,x} + \gamma_0) Y_0, \quad [D_x, Y_1] = -a_1 Y_1 = -(\alpha_1 u_{1,x} + \gamma_1) Y_1. \tag{3.6}$$

Applying the Jacobi identity and using the last formulae, we derive:

$$[D_x, W_1] = -(a_0 + a_1) W_1 - Y_0(a_1) Y_1 + Y_1(a_0) Y_0. \tag{3.7}$$

We can prove by induction that (3.5) is a test sequence. Moreover, for each $k \geqslant 2$ the formula

$$[D_x, W_k] = p_k W_k + q_k W_{k-1} + \cdots, \tag{3.8}$$

holds true, where the functions p_k, q_k are found by the rule

$$p_k = -(a_1 + ka_0), \quad q_k = \frac{k-k^2}{2} Y_0(a_0) - Y_0(a_1)k. \tag{3.9}$$

By assumption,there exists only a finite set of linearly independent elements of the sequence (3.5) in the algebra $\mathcal{L}(y, N)$. Hence, there exists a natural M such that

$$W_M = \lambda W_{M-1} + \cdots, \tag{3.10}$$

the operators $Y_0, Y_1, W_1, \ldots, W_{M-1}$ are linearly independent, and the dots stand for a linear combination of the operators Y_0, Y_1, W_1, $\ldots$, W_{M-2}.

Lemma 2. *The operators Y_0, Y_1, W_1 are linearly independent.*

Proof. Let us assume the contrary, namely, that the identity holds:

$$\lambda_1 W_1 + \mu_1 Y_1 + \mu_0 Y_0 = 0. \tag{3.11}$$

The operators Y_0, Y_1 are of the form $Y_0 = \frac{\partial}{\partial u_0} + \cdots$, $Y_1 = \frac{\partial}{\partial u_1} + \cdots$, while W_1 contain no terms of the form $\frac{\partial}{\partial u_0}$ and $\frac{\partial}{\partial u_1}$. Hence, the coefficients μ_1, μ_0 are zero. If, in addition, $\lambda_1 \neq 0$, then $W_1 = 0$. We apply the operator ad_{D_x} to both sides of the last identity, then by (3.7) we obtain the equation

$$Y_0(a_1)Y_1 - Y_1(a_0)Y_0 = 0.$$

It implies:

$$Y_0(a_1) = \alpha_{1,u_0} u_{1,x} + \gamma_{1,u_0} = 0, \qquad Y_1(a_0) = \alpha_{0,u_1} u_{0,x} + \gamma_{0,u_1} = 0.$$

Due to the independence of the variables $u_{0,x}$ and $u_{1,x}$, we obtain $\alpha_{1,u_0} = \alpha_{0,u_1} = 0$. But this contradicts the assumption of (1.6) that $\frac{\partial \alpha(u_{n+1}, u_n, u_{n-1})}{\partial u_{n\pm1}} \neq 0$. The proof is complete. ☐

Lemma 3. *If the expansion of form* (3.10) *holds, then*

$$\alpha(u_1, u_0, u_{-1}) = \frac{P'(u_0)}{P(u_0) + Q(u_{-1})} + \frac{1}{M-1} \frac{Q'(u_0)}{P(u_1) + Q(u_0)} - c_1(u_0). \tag{3.12}$$

Proof. It is not difficult to show that equation (3.4) for the sequence (3.5) is of the form:

$$D_x(\lambda) = -a_0 \lambda - \frac{M(M-1)}{2} Y_0(a_0) - M Y_0(a_1). \tag{3.13}$$

We simplify relation (3.13) using the formulae

$$\begin{aligned} Y_0(a_0) &= \left(\frac{\partial}{\partial u_0} + (\alpha_0 u_{0,x} + \gamma_0) \frac{\partial}{\partial u_{0,x}} \right) (\alpha_0 u_{0,x} + \gamma_0) = \left(\alpha_{0,u_0} + \alpha_0^2 \right) u_{0x} + \gamma_{0,u_0} + \alpha_0 \gamma_0, \\ Y_0(a_1) &= \alpha_{1,u_0} u_{1,x} + \gamma_{1,u_0}. \end{aligned} \tag{3.14}$$

A simple analysis of the equation (3.13) shows that $\lambda = \lambda(u_0, u_1)$. Therefore, (3.13) is rewritten as

$$\begin{aligned} \lambda_{u_0} u_{0,x} + \lambda_{u_1} u_{1,x} = &- \left((\alpha_0 \lambda + \frac{M(M-1)}{2} (\alpha_{0,u_0} + \alpha_0^2) \right) u_{0,x} - M \alpha_{1,u_0} u_{1,x} \\ &- \left(\gamma_0 \lambda + \frac{M(M-1)}{2} (\gamma_{0,u_0} + \alpha_0 \gamma_0) + M \gamma_{1,u_0} \right). \end{aligned}$$

Collecting the coefficients at the independent variables $u_{0,x}$, $u_{1,x}$, we get an overdetermined system of differential equations for λ:

$$\lambda_{u_0} = -\alpha_0 \lambda - \frac{M(M-1)}{2} (\alpha_{0,u_0} + \alpha_0^2), \quad \lambda_{u_1} = -M \alpha_{1,u_0}, \tag{3.15}$$

$$\gamma_0 \lambda + \frac{M(M-1)}{2} (\gamma_{0,u_0} + \alpha_0 \gamma_0) + M \gamma_{1,u_0} = 0. \tag{3.16}$$

Note that equations (3.15) do not involve the function γ and coincide completely with the equations studied in our paper [15]. Lemma 3 is implied immediately by Lemma 3.2 in [15]. In what follows we use equation (3.16) to refine the function γ. $\square$

3.2. Second test sequence. We construct a test sequence containing the operators Y_0, Y_1, Y_2 and their multiple commutators:

$$\begin{aligned} &Z_0 = Y_0, && Z_1 = Y_1, && Z_2 = Y_2, && Z_3 = [Y_1, Y_0]\,, && Z_4 = [Y_2, Y_1]\,, \\ &Z_5 = [Y_2, Z_3]\,, && Z_6 = [Y_1, Z_3]\,, && Z_7 = [Y_1, Z_4]\,, && Z_8 = [Y_1, Z_5]\,. \end{aligned} \tag{3.17}$$

Elements of the sequence Z_m for $m > 8$ are determined by the recurrence formula $Z_m = [Y_1, Z_{m-3}]$. Note that this is the simplest test sequence generated by iterations of the map $Z \to [Y_1, Z]$, which contains the operator $[Y_2, [Y_1, Y_0]] = Z_5$.

Lemma 4. *The operators Z_0, $Z_1, \dots Z_5$ are linearly independent.*

Proof. Arguing as in the proof of Lemma 1, we confirm that the operators Z_0, Z_1, $\dots$, Z_4 are linearly independent. Let us prove Lemma 4 by contradiction. Assume that

$$Z_5 = \sum_{j=0}^{4} \lambda_j Z_j. \tag{3.18}$$

First we derive the formulae describing the action of the operator ad_{D_x} on the operators Z_i. For $i = 0, 1, 2$, the needed formulae are immediately obtained from the relation

$$[D_x, Y_i] = -a_i Y_i.$$

Recall that $a_i = \alpha_i u_{i,x} + \gamma_i = \alpha(u_{i-1}, u_i, u_{i+1}) u_{i,x} + \gamma(u_{i-1}, u_i, u_{i+1})$. For $i = 3, 4, 5$ we have

$$\begin{aligned} &[D_x, Z_3] = -(a_1 + a_0) Z_3 + \cdots\,, \\ &[D_x, Z_4] = -(a_2 + a_1) Z_4 + \cdots\,, \\ &[D_x, Z_5] = -(a_0 + a_1 + a_2) Z_5 + Y_0(a_1) Z_4 - Y_2(a_1) Z_3 + \cdots \end{aligned}$$

By applying the operator ad_{D_x} to both sides of (3.18), we obtain

$$\begin{aligned} -(a_0 + a_1 + a_2)(\lambda_4 Z_4 + \lambda_3 Z_3 + \cdots) + Y_0(a_1) Z_4 - Y_2(a_1) Z_3 + \cdots \\ = \lambda_{4,x} Z_4 + \lambda_{3,x} Z_3 - \lambda_4 (a_1 + a_2) Z_4 - \lambda_3 (a_0 + a_1) Z_3 + \cdots \end{aligned} \tag{3.19}$$

Collecting the coefficients at Z_4 in identity (3.19), we obtain the following equation:

$$\lambda_{4,x} = -(\alpha_0 u_{0,x} + \gamma_0)\lambda_4 - (\alpha_{1,u_0} u_{1,x} + \gamma_{1,u_0}). \tag{3.20}$$

A simple analysis of equation (3.20) shows that $\lambda = \lambda(u_0, u_1)$. Consequently,

$$\lambda_{4,x} = \lambda_{4,u_0} u_{0,x} + \lambda_{4,u_1} u_{1,x}$$

and equation (3.20) reduces to a system of three equations

$$\gamma_0 \lambda_4 + \gamma_{1,u_0} = 0, \qquad \lambda_{4,u_0} = -\alpha_0 \lambda_4, \qquad \lambda_{4,u_1} = -\alpha_{1,u_0}.$$

It follows from these equations that $\lambda_4 = 0$. Otherwise, if $\lambda_4 \neq 0$, then $\alpha_0 = -\left(\log \lambda_4\right)_{u_0}$, which implies that $(\alpha_0)_{u_{-1}} = 0$ and this contradicts the requirement that $\alpha(u_1, u_0, u_{-1})$ essentially depends on u_1 and u_{-l}. Hence, $\lambda_4 = 0$. Then from (3.20) we have $\alpha_{1,u_0} = 0$, which again leads to a contradiction. $\square$

We return back to sequence (3.17). For further work, we need to describe the action of the operator ad_{D_x} on all elements of this sequence. It is convenient to separate the sequence (3.17) into three subsequences $\{Z_{3m}\}$, $\{Z_{3m+1}\}$ and $\{Z_{3m+2}\}$.

Lemma 5. *The action of the operator* ad_{D_x} *on the sequence* (3.17) *is given by the following formulae:*

$$
\begin{aligned}
[D_x, Z_{3m}] &= -(a_0 + ma_1)Z_{3m} + \left(\frac{m-m^2}{2}Y_1(a_1) - mY_1(a_0)\right) Z_{3m-3} + \cdots ,\\
[D_x, Z_{3m+1}] &= -(a_2 + ma_1)Z_{3m+1} + \left(\frac{m-m^2}{2}Y_1(a_1) - mY_1(a_2)\right) Z_{3m-2} + \cdots ,\\
[D_x, Z_{3m+2}] &= -(a_0 + ma_1 + a_2)Z_{3m+2} + Y_0(a_1)Z_{3m+1} + Y_2(a_1)Z_{3m} -\\
&\quad - (m-1)\left(\frac{m}{2}Y_1(a_1) + Y_1(a_0 + a_2)\right) Z_{3m-1} + \cdots
\end{aligned}
$$

Lemma 5 is easily proved by an induction.

Theorem 2. *Assume that the operator* Z_{3k+2} *is represented as a linear combination*

$$Z_{3k+2} = \lambda_k Z_{3k+1} + \mu_k Z_{3k} + \nu_k Z_{3k-1} + \cdots \tag{3.21}$$

of the previous terms in sequence (3.17) *and none of the operators* Z_{3j+2} *for* $j < k$ *is a linear combination of the operators* Z_s *with* $s < 3j + 2$. *Then the coefficient* ν_k *satisfies the equation*

$$D_x(\nu_k) = -a_1\nu_k - \frac{k(k-1)}{2}Y_1(a_1) - (k-1)Y_1(a_0 + a_2). \tag{3.22}$$

Lemma 6. *Assume that all the conditions of Theorem 2 are satisfied. Suppose that the operator* Z_{3k} *(the operator* Z_{3k+1}*) is linearly expressed in terms of the operators* Z_i, $i < 3k$. *Then in this expansion the coefficient at* Z_{3k-1} *is zero.*

Proof. We prove the lemma by contradiction assuming that in the formula

$$Z_{3k} = \lambda Z_{3k-1} + \cdots \tag{3.23}$$

the coefficient λ is nonzero. We apply the operator ad_{D_x} to both sides of equation (3.23). As a result, according to Lemma 5, we get:

$$-(a_0 + ka_1)\lambda Z_{3k-1} + \cdots = D_x(\lambda)Z_{3k-1} - \lambda(a_0 + (k-1)a_1 + a_2)Z_{3k-1} + \cdots \tag{3.24}$$

Collecting the coefficients at Z_{3k-1}, we obtain that the coefficient λ should satisfy the equation

$$D_x(\lambda) = \lambda(a_2 - a_1).$$

According to our assumption above, λ does not vanish and, therefore,

$$D_x(\log\lambda) = a_2 - a_1. \tag{3.25}$$

Since λ depends on finitely many dynamical variables, according to equation (3.25), λ can depend only on u_1 and u_2. Therefore, from (3.24) we get that

$$(\log\lambda)_{u_1}u_{1,x} + (\log\lambda)_{u_2}u_{2,x} = \alpha_2 u_{2,x} + \gamma_2 - \alpha_1 u_{1,x} - \gamma_1.$$

The variables $u_{1,x}$, $u_{2,x}$ are independent, so the last equation is equivalent to the system of equations $\alpha_1 = -(\log\lambda)_{u_1}$, $\alpha_2 = (\log\lambda)_{u_2}$, $\gamma_2 - \gamma_1 = 0$. Consequently, $\alpha_1 = \alpha_1(u_1, u_2)$ depends only on u_1, u_2. The latter contradicts the assumption that α_1 essentially depends on u_0. The contradiction shows that the assumption $\lambda \neq 0$ is false. The proof is complete. □

In order to prove Theorem 2, we apply the operator ad_{D_x} to both sides of identity (3.21) and simplify by means of the formulae in Lemma 5. Collecting the coefficients at Z_{3k-1}, we obtain equation (3.22).

We find the exact values of the coefficients in equation (3.22):

$$
\begin{aligned}
Y_1(a_0) &= Y_1(\alpha_0 u_{0,x} + \gamma_0) = \alpha_{0,u_1}u_{0,x} + \gamma_{0,u_1},\\
Y_1(a_2) &= Y_1(\alpha_2 u_{2,x} + \gamma_2) = \alpha_{2,u_1}u_{2,x} + \gamma_{2,u_1},\\
Y_1(a_1) &= Y_1(\alpha_1 u_{1,x} + \gamma_1) = (\alpha_{1,u_1} + \alpha_1^2)u_{1,x} + \gamma_{1,u_1} + \gamma_1\alpha_1.
\end{aligned}
$$

and substitute them into (3.22):

$$\begin{aligned} D_x(\nu_k) = &-(\alpha_1 u_{1,x} + \gamma_1)\nu_k - \frac{k(k-1)}{2}\big((\alpha_{1,u_1} + \alpha_1^2)u_{1,x} + \gamma_{1,u_1} + \gamma_1\alpha_1\big) \\ &-(k-1)(\alpha_{0,u_1}u_{0,x} + \alpha_{2,u_1}u_{2,x} + \gamma_{0,u_1} + \gamma_{2,u_1}). \end{aligned} \tag{3.26}$$

A simple analysis of equation (3.26) shows that ν_k can depend only on the variables u_0, u_1, u_2. Consequently,

$$D_x(\nu_k) = \nu_{k,u_0}u_{0,x} + \nu_{k,u_1}u_{1,x} + \nu_{k,u_2}u_{2,x}. \tag{3.27}$$

Substituting (3.27) in (3.26) and collecting coefficients at independent variables, we obtain a system of equations for the coefficient ν_k:

$$\nu_{k,u_0} = -(k-1)\alpha_{0,u_1}, \tag{3.28}$$

$$\nu_{k,u_1} = -\alpha_1\nu_k - \frac{k(k-1)}{2}(\alpha_{1,u_1} + \alpha_1^2), \tag{3.29}$$

$$\nu_{k,u_2} = -(k-1)\alpha_{2,u_1}, \tag{3.30}$$

$$0 = \gamma_1\nu_k + \frac{k(k-1)}{2}(\gamma_{1,u_1} + \gamma_1\alpha_1) + (k-1)(\gamma_{0,u_1} + \gamma_{2,u_1}). \tag{3.31}$$

Substituting the expression for the function α given by formula (3.12) into equation (3.28), we get

$$\nu_{k,u_0} = \frac{k-1}{M-1}\frac{P'(u_1)Q'(u_0)}{(P(u_1) + Q(u_0))^2}.$$

We integrate the last equation with respect to the variable u_0

$$\nu_k = -\frac{k-1}{M-1}\frac{P'(u_1)}{P(u_1) + Q(u_0)} + H(u_1, u_2). \tag{3.32}$$

Since $\nu_{k,u_2} = H_{u_2}$, equation (3.30) is rewritten as

$$H_{u_2} = (k-1)\frac{P'(u_2)Q'(u_1)}{(P(u_2) + Q(u_1))^2}.$$

Integrating the latter, we obtain an exact expression for the function H:

$$H = -(k-1)\left(\frac{Q'(u_1)}{P(u_2) + Q(u_1)} + A(u_1)\right),$$

which gives

$$\nu_k = -(k-1)\left(\frac{1}{M-1}\frac{P'(u_1)}{P(u_1) + Q(u_0)} + \frac{Q'(u_1)}{P(u_2) + Q(u_1)} + A(u_1)\right). \tag{3.33}$$

We substitute the found expressions for the functions α and ν_k into the equation (3.29)

$$\begin{aligned}
&-\frac{(k-1)}{M-1}\left(\frac{P''(u_1)}{P(u_1)+Q(u_0)}-\frac{P'^2(u_1)}{(P(u_1)+Q(u_0))^2}\right)\\
&-(k-1)\left(\frac{Q''(u_1)}{P(u_2)+Q(u_1)}-\frac{Q'^2(u_1)}{(P(u_2)+Q(u_1))^2}+A'(u_1)\right)\\
=&(k-1)\left(\frac{P'(u_1)}{P(u_1)+Q(u_0)}+\frac{1}{M-1}\frac{Q'(u_1)}{P(u_2)+Q(u_1)}-c_1(u_1)\right)\\
&\cdot\left(\frac{1}{M-1}\frac{P'(u_1)}{P(u_1)+Q(u_0)}+\frac{Q'(u_1)}{P(u_2)+Q(u_1)}+A(u_1)\right)\\
&-\frac{k(k-1)}{2}\left(\frac{P''(u_1)}{P(u_1)+Q(u_0)}+\frac{1}{M-1}\frac{Q''(u_1)}{P(u_2)+Q(u_1)}\right.\\
&-\frac{1}{M-1}\frac{Q'^2(u_1)}{(P(u_2)+Q(u_1))^2}+\frac{1}{M-1}\frac{2Q'(u_1)P'(u_1)}{(P(u_1)+Q(u_0))(P(u_2)+Q(u_1))}\\
&+\frac{1}{(M-1)^2}\frac{Q'^2(u_1)}{(P(u_2)+Q(u_1))^2}\\
&\left.-c_1'(u_1)-2c_1(u_1)\left(\frac{P'(u_1)}{P(u_1)+Q(u_0)}+\frac{1}{M-1}\frac{Q'(u_1)}{P(u_2)+Q(u_1)}\right)+c_1^2(u_1)\right).
\end{aligned}\tag{3.34}$$

Obviously, according to the assumption

$$\frac{\partial}{\partial u_1}\alpha(u_1,u_0,u_{-1})\neq 0,\qquad \frac{\partial}{\partial u_{-l}}\alpha(u_1,u_0,u_{-1})\neq 0,$$

the functions $P'(u_2)$ and $Q'(u_0)$ do not vanish. Consequently, the variables

$$\frac{Q'^2(u_1)}{(P(u_2)+Q(u_1))^2},\quad \frac{P'^2(u_1)}{(P(u_1)+Q(u_0))^2},\quad \frac{P'(u_1)Q'(u_1)}{(P(u_1)+Q(u_0))(P(u_2)+Q(u_1))}$$

are independent. Collecting the coefficients of these variables in (3.34), we obtain a system of two equations

$$\left(1-\frac{1}{M-1}\right)\left(1-\frac{k}{2(M-1)}\right)=0,\qquad 1+\frac{1}{(M-1)^2}=\frac{k}{M-1}.\tag{3.35}$$

The system (3.35) has two solutions: $M=0$, $k=-2$ and $M=2$, $k=2$. Since k must be greater than zero, we have $M=2$, $k=2$. The last argument completes the proof of Theorem 2.

Thus, we have proved that $M=2$, $k=2$. Expansions (3.10) and (3.21) take the form

$$W_2=\lambda W_1+\sigma Y_1+\delta Y_0,\tag{3.36}$$

$$Z_8=\lambda Z_7+\mu Z_6+\nu Z_5+\rho Z_4+\kappa Z_3+\sigma Z_2+\delta Z_1+\eta Z_0.\tag{3.37}$$

The following theorem is true.

Theorem 3. *Expansions (3.36), (3.37) hold if and only if the functions α, γ in equation (1.5) are of the form:*

$$\alpha(u_{n+1},u_n,u_{n-1})=\frac{1}{u_n-u_{n-1}}-\frac{1}{u_{n+1}-u_n},\tag{3.38}$$

$$\gamma(u_{n+1},u_n,u_{n-1})=r'(u_n)-r(u_n)\alpha(u_{n+1},u_n,u_{n-1}),\tag{3.39}$$

where $r(u_n)=\frac{k_1}{2}u_n^2+k_2u_n+k_3$ and the factors k_i are arbitrary constants.

Proof. Consider relation (3.36). Using relations (3.6), (3.7) and applying the Jacobi identity, we get

$$[D_x,W_2]=-(2a_0+a_1)W_2-Y_0(a_0+2a_1)W_1+(2Y_0Y_1(a_0)-Y_1Y_0(a_0))Y_0-Y_0Y_0(a_1)Y_1.\tag{3.40}$$

It is obvious that only one term in formula (3.36) contains the operator of differentiation $\frac{\partial}{\partial u_1}$, namely σY_1, and only one term contains $\frac{\partial}{\partial u_0}$, namely, σY_0. Consequently, $\sigma = 0$, $\delta = 0$ and expansion (3.36) casts into the form

$$W_2 = \lambda W_1.$$

Applying the operator ad_{D_x} to both sides of the last relation, we obtain

$$\begin{aligned} -(2a_0+a_1)W_2 - Y_0(a_0+2a_1)W_1 + (2Y_0Y_1(a_0) - Y_1Y_0(a_0))Y_0 - Y_0Y_0(a_1)Y_1 \\ = D_x(\lambda)W_1 + \lambda\left(-(a_0+a_1)W_1 + Y_1(a_0)Y_0 - Y_0(a_1)Y_1\right). \end{aligned}$$

Collecting the coefficients for the operators W_2, W_1, Y_1, Y_0, we obtain the following system:

$$D_x(\lambda) = -a_0\lambda - Y_0(a_0+2a_1), \tag{3.41}$$

$$-Y_0Y_0(a_1) = -\lambda Y_0(a_1), \tag{3.42}$$

$$2Y_0Y_1(a_0) - Y_1Y_0(a_0) = \lambda Y_1(a_0). \tag{3.43}$$

Examining the first equation of the obtained system, we observe that $\lambda = \lambda(u_0, u_1)$ and then simplifying all the equations, we arrive at the following system:

$$\lambda_{u_0} = -\alpha_0\lambda - (\alpha_{0,u_0} + \alpha_0^2), \tag{3.44}$$

$$\lambda_{u_1} = -2\alpha_{1,u_0}, \tag{3.45}$$

$$\alpha_{1,u_0u_0} = \lambda\alpha_{1,u_0}, \quad \alpha_{0,u_0u_1} = \lambda\alpha_{0,u_1}. \tag{3.46}$$

$$\gamma_0\lambda + \gamma_{0,u_0} + \gamma_0\alpha_0 + 2\gamma_{1,u_0} = 0, \tag{3.47}$$

$$\gamma_{1,u_0u_0} = \lambda\gamma_{1,u_0}, \tag{3.48}$$

$$\gamma_{0,u_0u_1} + \gamma_0\alpha_{0,u_1} - \gamma_{0,u_1}\alpha_0 = \lambda\gamma_{0,u_1}. \tag{3.49}$$

We note that equations (3.44)-(3.46) will be employed to refine the functions α and λ, and the equations (3.47)–(3.49) will be used to specify the function γ by substituting the already found expression for α.

We proceed to decomposition (3.37). Letting $k = 2$ in the formulae in Lemma 5, we obtain

$$[D_x, Z_6] = -(\alpha_0u_{0,x} + 2\alpha_1u_{1,x})Z_6 + \cdots, \tag{3.50}$$

$$[D_x, Z_7] = -(\alpha_2u_{2,x} + 2\alpha_1u_{1,x})Z_7 - (Y_1(\alpha_1u_{1,x}) + 2Y_1(\alpha_2u_{2,x}))Z_4 + \cdots, \tag{3.51}$$

$$\begin{aligned} [D_x, Z_8] = &- (\alpha_0u_{0,x} + 2\alpha_1u_{1,x} + \alpha_2u_{2,x})Z_8 + Y_0(\alpha_1u_{1,x})Z_7 + Y_2(\alpha_1u_{1,x})Z_6 \\ &- \left(Y_1(\alpha_1u_{1,x}) + Y_1(\alpha_0u_{0,x} + \alpha_2u_{2,x})\right)Z_5 + \cdots \end{aligned} \tag{3.52}$$

Then we apply the operator ad_{D_x} to both parts of relation (3.37) and simplify the resulting equation by using (3.50), (3.51), (3.52). Comparing the coefficients at Z_7 and Z_6, we get $\lambda = 0$ and $\mu = 0$. Thus, formula (3.37) is simplified:

$$Z_8 = \nu Z_5 + \rho Z_4 + \kappa Z_3 + \sigma Z_2 + \delta Z_1 + \eta Z_0. \tag{3.53}$$

The following commutation relations hold:

$$\begin{aligned} [D_x, Z_8] = &- (a_2+2a_1+a_0)Z_8 + Y_0(a_1)Z_7 - Y_2(a_1)Z_6 - Y_1(a_2+a_1+a_0)Z_5 \\ &+ Y_1Y_0(a_1)Z_4 - Y_1Y_2(a_1)Z_3 + (Y_1Y_2Y_0(a_1) + Z_5(a_1))Z_1, \end{aligned} \tag{3.54}$$

$$[D_x, Z_5] = -(a_0+a_1+a_2)Z_5 + Y_0(a_1)Z_4 - Y_2(a_1)Z_3 + Y_2Y_0(a_1)Z_1. \tag{3.55}$$

We apply ad_{D_x} to (3.53), then simplify according to (3.54), (3.55), (3.53) and collect the coefficients at Z_5:

$$-(a_2+2a_1+a_0)\nu - Y_1(a_2+a_1+a_0) = D_x(\nu) - (a_2+a_1+a_0)\nu$$

or the same

$$D_x(\nu) = -a_1\nu - Y_1(a_2+a_1+a_0). \tag{3.56}$$

It follows from equation (3.56) that ν depends on three variables $\nu = \nu(u, u_1, u_2)$. Thus, equation (3.56) reduces to a system of equations:

$$\nu_u = -\alpha_{0,u_1}, \tag{3.57}$$

$$\nu_{u_1} = -\alpha_1\nu - \alpha_{1,u_1} - \alpha_1^2, \tag{3.58}$$

$$\nu_{u_2} = -\alpha_{2,u_1}, \tag{3.59}$$

$$\gamma_1\nu + \gamma_{2,u_1} + \gamma_1\alpha_1 + \gamma_{1,u_1} + \gamma_{0,u_1} = 0. \tag{3.60}$$

So, as a result of studying relations (3.36), (3.37), we arrive at equations (3.44)–(3.46) and (3.57)-(3.59), which exactly coincide with the corresponding systems of equations from the work [15] and, therefore, we obtain that

$$\alpha(u_{n+1}, u_n, u_{n-1}) = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n}.$$

Using the remaining equations (3.47)–(3.49) and (3.60), we find γ:

$$\gamma(u_{n+1}, u_n, u_{n-1}) = r'(u_n) - r(u_n)\alpha(u_{n+1}, u_n, u_{n-1}).$$

It is easy to show that relations (3.36), (3.37) become

$$W_2 = \lambda W_1, \qquad \lambda = \frac{2}{u_1 - u_0}, \qquad Z_8 = \nu Z_5, \qquad \nu = -\frac{u_2 - 2u_1 + u_0}{(u_1 - u_0)(u_2 - u_1)}.$$

□

Similarly, we have

$$\beta(u_{n+1}, u_n, u_{n-1}) = \tilde{r}'(u_n) - \tilde{r}(u_n)\alpha(u_{n+1}, u_n, u_{n-1}), \tag{3.61}$$

where $\tilde{r}(u_n) = \frac{\tilde{k}_1}{2}u_n^2 + \tilde{k}_2 u_n + \tilde{k}_3$, and the coefficients $\tilde{k}_i$ are arbitrary constants.

The next step of our study is to refine the function δ. To do this, we construct a new sequence in a set of multiple commutators.

4. Refining of function δ

We recall that since the right-hand side f_i of the system (1.5) is of a special form, the operator Y can be represented as follows (see (2.5)):

$$Y = \sum_{i=0}^{N} u_{i,y} Y_i + R,$$

Here the operator R is defined by formula (2.6). Consider the following sequence of the operators in the characteristic algebra $\mathcal{L}(y, N)$:

$$\begin{gathered} Y_{-1}, \quad Y_0, \quad Y_1, \quad Y_{0,-1} = [Y_0, Y_{-1}], \quad Y_{1,0} = [Y_1, Y_0], \\ R_0 = [Y_0, R], \quad R_1 = [Y_0, R_0], \quad R_2 = [Y_0, R_1], \quad \ldots, \quad R_{k+1} = [Y_0, R_k]. \end{gathered} \tag{4.1}$$

The following commutation relations hold true:

$$[D_x, Y_{-1}] = -a_{-1}Y_{-1}, \quad [D_x, Y_0] = -a_0 Y_0, \quad [D_x, Y_1] = -a_1 Y_1, \tag{4.2}$$

$$[D_x, Y_{1,0}] = -(a_0 + a_1)Y_{1,0} - Y_1(a_0)Y_0 + Y_0(a_1)Y_1, \tag{4.3}$$

$$[D_x, Y_{0,-1}] = -(a_{-1} + a_0)Y_{0,-1} - Y_0(a_{-1})Y_{-1} + Y_{-1}(a_0)Y_0, \tag{4.4}$$

$$[D_x, R] = -\sum_i h_i Y_i, \tag{4.5}$$

where $a_i = \alpha_i u_{i,x} + \gamma_i$, $h_i = \beta_i u_{i,x} + \delta_i$. Using the Jacobi identity and formulae (4.2)–(4.5), we get the formulae:

$$\begin{aligned} [D_x, R_0] &= [D_x, [Y_0, R]] = -[Y_0, [R, D_x]] - [R, [D_x, Y_0]] \\ &= -a_0 R_0 + h_1 Y_{1,0} - h_{-1}Y_{0,-1} - Y_0(h_1)Y_1 - Y_0(h_{-1})Y_{-1} + (R(a_0) - Y_0(h_0))Y_0, \end{aligned} \tag{4.6}$$

$$[D_x, R_1] = -2a_0R_1 - Y_0(a_0)R_0 + \cdots , \tag{4.7}$$
$$[D_x, R_2] = -3a_0R_2 - 3Y_0(a_0)R_1 - Y_0^2(a_0)R_0 + \cdots , \tag{4.8}$$
$$[D_x, R_3] = -4a_0R_2 - 6Y_0(a_0)R_1 - 4Y_0^2(a_0)R_1 - Y_0^3(a_0)R_0 + \cdots , \tag{4.9}$$

where the dots stand for a linear combination of the operators $Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$. It can be proved by an induction that

$$[D_x, R_n] = p_nR_n + q_nR_{n-1} + \cdots , \tag{4.10}$$

where

$$p_n = -(n+1)a_0, \quad q_n = -\frac{n^2+n}{2}Y_0(a_0), \tag{4.11}$$

and three dots stand for a linear combination of the operators R_k, $k < n-1$ and $Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$.

We consider two different cases:

i) The operator R_0 is expressed linearly in terms of operators (4.1).

ii) The operator R_0 is not expressed linearly in terms of operators (4.1).

Let us focus on the case *i*). It follows from formula (4.6) that this linear expansion must be of the form

$$R_0 = \lambda R + \mu Y_{1,0} + \tilde{\mu} Y_{0,-1} + \nu Y_1 + \eta Y_0 + \epsilon Y_{-1}. \tag{4.12}$$

The operators in the right hand side of this formula are linearly independent.

Applying the operator ad_{D_x} to both sides of (4.12), we obtain

$$\begin{aligned}-a_0(\lambda R + \mu Y_{1,0} + \tilde{\mu} Y_{0,-1} + \cdots) + h_1Y_{1,0} - h_{-1}Y_{0,-1} + \cdots&\\ = D_x(\lambda)R + D_x(\mu)Y_{1,0} + \mu(-(a_0+a_1)Y_{1,0} + \cdots)&\\ + D_x(\tilde{\mu})Y_{0,-1} + \tilde{\mu}(-(a_{-1}+a_0)Y_{0,-1} + \cdots).&\end{aligned} \tag{4.13}$$

Three dots stand for a linear combination of the operators Y_{-1}, Y_0, Y_1. Collecting coefficients for the independent operators R, $Y_{1,0}$, $Y_{0,-1}$, we obtain the system of differential equations for the coefficients λ, μ, $\tilde{\mu}$

$$D_x(\lambda) = -a_0\lambda, \tag{4.14}$$
$$D_x(\mu) = a_1\mu + h_1, \quad D_x(\tilde{\mu}) = a_{-1}\tilde{\mu} - h_{-1}. \tag{4.15}$$

The equation (4.14) reads as $D_x(\lambda) = -(\alpha_0 u_{0,x} + \gamma_0)\lambda$. It is easy to see that $\lambda = \lambda(u_0)$ and hence $\lambda'(u_0) = -\alpha_0\lambda(u_0)$, $0 = \gamma_0\lambda$. If $\lambda \neq 0$ then $\alpha_0 = -(\log\lambda(u_0))'$. But this contradicts assumption (1.6) that $\frac{\partial\alpha_0(u_1,u_0,u_{-1})}{\partial u_{\pm1}} \neq 0$. Hence, we have $\lambda = 0$.

Consider equations (4.15):

$$D_x(\mu) = (\alpha_1 u_{1,x} + \gamma_1)\mu + \beta_1 u_{1,x} + \delta_1, \tag{4.16}$$
$$D_x(\tilde{\mu}) = (\alpha_{-1} u_{-1,x} + \gamma_{-1})\tilde{\mu} - \beta_{-1} u_{-1,x} - \delta_{-1}. \tag{4.17}$$

From (4.16) we obtain that μ depends only on u_1 and from (4.17) we obtain that $\tilde{\mu}$ depends only on u_{-1}. Hence, equations (4.16) and (4.17) are reduced to the following system:

$$\mu'(u_1) = \alpha_1\mu(u_1) + \beta_1, \quad 0 = \gamma_1\mu(u_1) + \delta_1, \tag{4.18}$$
$$\tilde{\mu}(u_{-1}) = \alpha_{-1}\tilde{\mu}(u_{-1}) - \beta_{-1}, \quad 0 = \gamma_{-1}\tilde{\mu}(u_{-1}) - \delta_{-1}. \tag{4.19}$$

By shifting the argument n backwards and forwards by one in the equation (4.18) and, respectively, in (4.19) we obtain:

$$\mu'(u_0) = \alpha_0\mu(u_0) + \beta_0, \qquad 0 = \gamma_0\mu(u_0) + \delta_0, \tag{4.20}$$
$$\tilde{\mu}(u_0) = \alpha_0\tilde{\mu}(u_0) - \beta_0, \qquad 0 = \gamma_0\tilde{\mu}(u_0) - \delta_0. \tag{4.21}$$

We exclude μ and $\tilde{\mu}$ from these equations and arrive at the differential equation for the function δ_0:

$$\delta_{0,u_0} = \left(\frac{\gamma_{0,u_0}}{\gamma_0} + \alpha_0\right)\delta_0 - \beta_0\gamma_0. \tag{4.22}$$

Equation (4.22) is solved easily:

$$\begin{aligned}\delta_0(u_{-1},u_0,u_1) =& \frac{1}{4}\frac{1}{(u_0-u_{-1})(u_0-u_1)}\Big(k_1(u_0^2u_1 - 2u_0u_{-1}u_1 + u_0^2u_{-1})\\ &\qquad + 2k_2(u_0^2 - u_{-1}u_1) + 2k_3(-u_1 + 2u_0 - u_{-1})\Big)\\ &\cdot\Big(\tilde{k}_1(u_0u_1 - u_{-1}u_1 + u_{-1}u_0) + 2\tilde{k}_2u_0 + 2\tilde{k}_3\\ &\quad + 4F_1(u_{-1},u_1)(u_0-u_{-1})(u_0-u_1)\Big).\end{aligned} \tag{4.23}$$

Here k_1, k_2, k_3 and $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ are constants, which appear in the description of functions (3.39), (3.61) and $F_1(u_{-1},u_1)$ is a function to be found.

Substituting (4.23) into the second equation of (4.20), we obtain that $F_1(u_{-1},u_1) = \frac{1}{2}\tilde{k}_1$ and

$$\mu(u_0) = \frac{1}{2}\tilde{k}_1u_0^2 + \tilde{k}_2u_0 + \tilde{k}_3$$

From the second equation of (4.21) we get

$$\tilde{\mu}(u_0) = -\frac{1}{2}\tilde{k}_1u_0^2 - \tilde{k}_2u_0 - \tilde{k}_3.$$

Further, we collect the coefficients at Y_1, Y_{-1} in identity (4.14). Substituting the above functions α, β, γ, δ, μ, $\tilde{\mu}$ into the obtained equations, we get identities that do not give any additional condition on the unknown functions.

Let us collect the coefficients at Y_0 in (4.14). We find:

$$D_x(\eta) = R(a_0) - Y_0(h_0) + \mu Y_1(a_0) - \tilde{\mu}Y_{-1}(a_0).$$

Calculating each term and simplifying the last equation, we obtain

$$D_x(\eta) = \big(-\beta_{0,u_0} + \mu\alpha_{0,u_1} - \tilde{\mu}\alpha_{0,u_{-1}}\big)u_{0,x} + \delta_0\alpha_0 - \delta_{0,u_0} - \gamma_0\beta_0 + \mu\gamma_{0,u_1} - \tilde{\mu}\gamma_{0,u_{-1}}.$$

A simple analysis of the last equation shows that η can depend only on u_0. Therefore, the last equation reduces to a system of two equations:

$$\eta'(u_0) = -\beta_{0,u_0} + \mu\alpha_{0,u_1} - \tilde{\mu}\alpha_{0,u_{-1}}, \tag{4.24}$$

$$0 = \delta_0\alpha_0 - \delta_{0,u_0} - \gamma_0\beta_0 + \mu\gamma_{0,u_1} - \tilde{\mu}\gamma_{0,u_{-1}}. \tag{4.25}$$

By straightforward calculations we obtain that the right hand side of (4.24) vanishes identically.

Studying equation (4.25), we get some additional relations between the constants k_i and $\tilde{k}_i$:

$$k_1 = \frac{\tilde{k}_1}{\tilde{k}_2}k_2, \qquad k_3 = \frac{\tilde{k}_3}{\tilde{k}_2}k_2.$$

Thus, we have proved that if decomposition (4.12) holds, then it should be of the form

$$R_0 = \mu Y_{1,0} + \tilde{\mu}Y_{0,-1}. \tag{4.26}$$

Herewith we completely determine the desired coefficients of quasilinear chain (1.5)

$$u_{n,xy} = \alpha_n u_{n,x}u_{n,y} + \beta_n u_{n,x} + \gamma_n u_{n,y} + \delta_n. \tag{4.27}$$

We obtain explicit expressions for these coefficients

$$\begin{aligned}\alpha_n &= \alpha(u_{n+1},u_n,u_{n-1}) = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1}-u_n},\\ \beta_n &= \beta(u_{n+1},u_n,u_{n-1}) = r'(u_n) - r(u_n)\alpha(u_{n+1},u_n,u_{n-1}),\end{aligned}$$

$$\gamma_n = \gamma(u_{n+1}, u_n, u_{n-1}) = \varepsilon(r'(u_n) - r(u_n)\alpha(u_{n+1}, u_n, u_{n-1})),$$
$$\delta_n = \delta(u_{n+1}, u_n, u_{n-1}) = -\varepsilon r(u_n)(r'(u_n) - r(u_n)\alpha(u_{n+1}, u_n, u_{n-1})),$$

where $r(u_n) = \frac{c_1}{2}u_n^2 + c_2 u_n + c_3$ is a polynomial of the degree at most two with arbitrary coefficients and $c_i = \tilde{k}_i$, $\varepsilon = k_2/\tilde{k}_2$. The boundary conditions reducing the chain to an integrable system of hyperbolic equations are given in the form

$$u_{-1} = \lambda, \qquad u_{N+1} = \lambda \tag{4.28}$$

where λ is a root of the polynomial $r(\lambda)$, that is, $r(\lambda) = 0$. In the degenerate case when $r(u_n) = c_3$ the boundary conditions are taken as

$$u_{-1} = c_3(\varepsilon x + y) + c_4, \qquad u_{N+1} = c_3(\varepsilon x + y) + c_5, \tag{4.29}$$

where c_4, c_5 are arbitrary constants.

Let us study case *ii*). Assume that some element R_n, $n > 0$ of sequence (4.1) is linearly expressed in terms of the previous elements:

$$R_n = \lambda R_{n-1} + \cdots, \tag{4.30}$$

but elements R_k, $k < n$ are not expressed linearly in terms of the previous elements R_j, $j < k$ and $Y_{1,0}, Y_{0,-1}, Y_1, Y_0, Y_{-1}$. We apply the operator ad_{D_x} to both sides of (4.30) and obtain

$$p_n(\lambda R_{n-1} + \cdots) + q_n R_{n-1} + \cdots = D_x(\lambda)R_{n-1} + \lambda(p_{n-1}R_{n-1} + \cdots).$$

We collect the coefficients at the operator R_{n-1} in the resulting identity and we find:

$$D_x(\lambda) = \lambda(p_n - p_{n-1}) + q_n.$$

We substitute explicit expressions for p_n, p_{n-1}, q_n into the last equation to get

$$D_x(\lambda) = -a_0\lambda - \frac{n^2+n}{2}Y_0(a_0).$$

We substitute the explicit expression for a_0 and evaluate $Y_0(a_0)$. We obtain

$$D_x(\lambda) = -(\alpha_0 u_{0,x} + \gamma_0)\lambda - \frac{n^2+n}{2}\left((\alpha_{0,u_0} + \alpha_0^2)u_{0,x} + \gamma_{0,u_0} + \gamma_0\alpha_0\right) \tag{4.31}$$

It follows from the last identity that λ depends only on u_0. Then the equation reduces to a system of two equations

$$\lambda'(u_0) = -\alpha_0\lambda - \frac{n^2+n}{2}(\alpha_{0,u_0} + \alpha_0^2), \tag{4.32}$$

$$\gamma_0\lambda + \frac{n^2+n}{2}(\gamma_{0,u_0} + \gamma_0\alpha_0) = 0. \tag{4.33}$$

We rewrite equation (4.32) as

$$\lambda'(u_0) = \frac{-\lambda(u_0)(2u_0 - u_1 - u_{-1}) - (n^2+n)}{(u_0 - u_{-1})(u_0 - u_1)} \tag{4.34}$$

or

$$\lambda'(u_0)(u_0^2 - u_0u_1 - u_{-1}u_0 + u_1u_{-1}) = -\lambda(u_0)(2u_0 - u_1 - u_{-1}) - (n^2+n). \tag{4.35}$$

Since the variables u_{-1}, u_0, u_1 are independent then the last equation implies immediately that $\lambda = 0$ and $n^2 + n = 0$. Thus, we have $n = 0$ or $n = -1$. Both solutions contradict the assumption $n > 0$. Therefore, case *ii*) is never realized.

Up to point transformations, there are three essentially different versions of chain (4.27):

1) If $c_1 = c_2 = 0$, then by the shift transformation $u \to u - c_3(\varepsilon x + y)$ chain (4.27) reduces to the known Ferapontov-Shabat-Yamilov chain (see [27, 28])

$$u_{n,xy} = \alpha_n u_{n,x} u_{n,y}, \tag{4.36}$$

2) If $c_1=0$, $c_2\neq 0$, then by shifting $u\to u-\frac{c_3}{c_2}$ and scaling $x\to\frac{x}{\varepsilon c_2}$, $y\to\frac{y}{c_2}$ we obtain the chain

$$u_{n,xy}=\alpha_n(u_{n,x}u_{n,y}-u_n(u_{n,x}+u_{n,y})+u_n^2)+u_{n,x}+u_{n,y}-u_n, \tag{4.37}$$

3) For $c_1\neq 0$ by the shift transformation $u\to u-\frac{c_2}{c_1}$ and by the scaling $x\to\frac{2}{\varepsilon c_1}x$, $y\to\frac{2}{c_1}y$ chain (4.27) can be reduced to

$$u_{n,xy}=\alpha_n(u_{n,x}u_{n,y}-s_n(u_{n,x}+u_{n,y})+s_n^2)+s_n'(u_{n,x}+u_{n,y}-s_n), \tag{4.38}$$

where $s_n=u_n^2+C$ and $C=\frac{c_3}{c_1}-\left(\frac{c_2}{c_1}\right)^2$ is an arbitrary constant.

Thus we have proved that each chain integrable in the sense of Definition 1 is of form (4.27). In order to complete the proof of Theorem 1, we have to prove the opposite statement. This is done in the following theorem.

Theorem 4. *Chain* (4.27) *found as a result of the classification is integrable in the sense of Definition 1 formulated in Introduction.*

We introduce special notations for multiple commutators of the operators $\{Y_i\}$:

$$Y_{i_k,\dots,i_0}=[Y_{i_k},Y_{i_{k-1},\dots,i_0}]. \tag{4.39}$$

The structure of the Lie algebra generated by the operators $\{Y_i\}$ can be studied by a method developed in our previous paper [15]. One can prove that each element in this algebra can be represented as a linear combination of the following operators

$$Y_i,\ Y_{i+1,i},\ Y_{i+2,i+1,i},\ \dots. \tag{4.40}$$

Formula (2.5) implies that the algebra $\mathcal{L}(y,N)$ corresponding to system (2.1) is an extension of this algebra obtained by adding one more generator, namely, the operator R.

We recall that in paper [15], a particular case of a chain (4.27) was studied in detail. Namely, the following theorem was proved.

Theorem 5. *The chain*

$$u_{n,xy}=\left(\frac{1}{u_n-u_{n-1}}-\frac{1}{u_{n+1}-u_n}\right)u_{n,x}u_{n,y} \tag{4.41}$$

is integrable in the sense of Definition 1 formulated in the Introduction.

We recall briefly the scheme of the proof of Theorem 5. The basis

$$\{Y_i\}_{i=0}^N,\quad \{Y_{i+1,i}\}_{i=0}^{N-1},\quad \{Y_{i+2,i+1,i}\}_{i=0}^{N-2},\quad \dots,Y_{N,N-1,\dots,0}. \tag{4.42}$$

was constructed in the set of multiple commutators of the operators $Y_0,\dots,Y_N$ corresponding to chain (4.41).

In order to prove that there is basis (4.42) in the set of multiple commutators of the operators $Y_0,\dots,Y_N$ corresponding to chain (4.27), we can repeat the proof of Theorem 5.2 from paper [15] (see Appendix) letting $a_i=\alpha_i u_{i,x}+\gamma_i$. This proof is cumbersome and we do not give it here.

In order to prove the main Theorem 4 we consider the algebra Lie $\mathcal{L}(y,N)$ generated by the operators $Y_0,\dots,Y_N,R$ and we prove that a finite basis exists in this algebra

$$R,\quad \{Y_i\}_{i=0}^N,\quad \{Y_{i+1,i}\}_{i=0}^{N-1},\quad \{Y_{i+2,i+1,i}\}_{i=0}^{N-2},\quad \dots,\quad Y_{N,N-1,\dots,0}. \tag{4.43}$$

It remains to check that each multiple commutator of the operator R with the operators (4.42) is linearly expressed in terms of the operators in set (4.43).

We proceed to proving Theorem 4.

Proof of Theorem 4. Here we consider truncated chains, that is, finite systems of hyperbolic equations (2.1) obtained by imposing cut-off conditions to the initial chain. We note that while passing from an infinite chain to a truncated one, the commutation relations near the cut-off points change.

We prove Theorem 4 by an induction. Let us prove the induction base. The first step of the proof requires the following formulae:

$$\big[D_x, \bar{R}_0\big] = -a_0\bar{R}_0 + h_1 Y_{1,0} - Y_0(h_1)Y_1 + \big(R(a_0) - Y_0(h_0)\big)Y_0, \tag{4.44}$$

$$\begin{aligned}\big[D_x, \bar{R}_N\big] = &- a_N\bar{R}_N - h_{N-1}\,[Y_N, Y_{N-1}]\\ &- Y_N(h_{N-1})Y_{N-1} + \big(R(a_N) - Y_N(h_N)\big)Y_N,\end{aligned} \tag{4.45}$$

$$\begin{aligned}\big[D_x, \bar{R}_k\big] = &- a_k\bar{R}_k - h_{k-1}\,[Y_k, Y_{k-1}] + h_{k+1}\,[Y_{k+1}, Y_k]\\ &- Y_k(h_{k-1})Y_{k-1} + \big(R(a_k) - Y_k(h_k)\big)Y_k - Y_k(h_{k+1})Y_{k+1}.\end{aligned} \tag{4.46}$$

Here $\bar{R}_j = [Y_j, R]$, $j = 0, 1 \ldots, N$.

First we study the end points $k = 0$ and $k = N$. Let us prove the following identity:

$$\bar{R}_0 = \lambda^{(0)}R + \mu^{(0)}Y_{1,0} + \nu^{(0)}Y_1 + \eta^{(0)}Y_0. \tag{4.47}$$

We apply the operator ad_{D_x} to both sides of identity (4.47) and simplify using (4.2), (4.3), (4.5), (4.44). As a result we obtain

$$\begin{aligned}&-a_0(\lambda^{(0)}R + \mu^{(0)}Y_{1,0} + \cdots) + h_1Y_{1,0} + \cdots\\ &\qquad = D_x(\lambda^{(0)})R + D_x(\mu^{(0)})Y_{1,0} + \mu^{(0)}(-(a_1 + a_0)Y_{1,0} + \cdots).\end{aligned} \tag{4.48}$$

Here the dots stand for a linear combination of the operators Y_1, Y_0. Collecting the coefficients at the operators R and $Y_{1,0}$ in (4.48), we obtain a system of the equations

$$D_x(\lambda^{(0)}) = -a_0\lambda^{(0)}, \tag{4.49}$$

$$D_x(\mu^{(0)}) = a_1\mu^{(0)} + h_1. \tag{4.50}$$

Equation (4.49) coincides with equation (4.14) for $i = 0$, hence, $\lambda^{(0)} = 0$. Equation (4.50) coincides with the first equation in (4.15) for $i = 0$, hence, $\mu^{(0)} = \mu$. It is easy to show that $\nu^{(0)} = \eta^{(0)} = 0$. Thus, we have proved that decomposition (4.47) is of the form

$$\bar{R}_0 = \mu^{(0)}Y_{1,0}. \tag{4.51}$$

Let us prove the identity:

$$\bar{R}_N = \lambda^{(N)}R + \tilde{\mu}^{(N)}Y_{N,N-1} + \eta^{(N)}Y_N + \epsilon^{(N)}Y_{N-1}. \tag{4.52}$$

We apply ad_{D_x} to both sides of relation (4.52):

$$\begin{aligned}&-a_N(\lambda^{(N)}R + \tilde{\mu}^{(N)}Y_{N,N-1} + \cdots) - h_{N-1}Y_{N,N-1} + \cdots\\ &\qquad = D_x(\lambda^{(N)})R + D_x(\tilde{\mu}^{(N)})Y_{N,N-1} + \tilde{\mu}^{(N)}(-(a_N + a_{N-1})Y_{N,N-1} + \cdots).\end{aligned} \tag{4.53}$$

Here the dots stand for a linear combination of the operators Y_N, Y_{N-1}. Collecting the coefficients for R and $Y_{N,N-1}$, we get the system:

$$D_x(\lambda^{(N)}) = -a_N\lambda^{(N)}, \tag{4.54}$$

$$D_x(\tilde{\mu}^{(N)}) = a_{N-1}\tilde{\mu}^{(N)} - h_{N-1}. \tag{4.55}$$

Equation (4.54) coincides with equation (4.14) for $i = N$, hence $\lambda^{(N)} = 0$. Equation (4.55) coincides with the second equation in (4.15) for $i = N$ and hence, we have $\tilde{\mu}^{(N)} = D_n^N\tilde{\mu}(u_{-1}) = \tilde{\mu}(u_{N-1})$. It is easy to show that $\eta^{(N)} = \epsilon^{(N)} = 0$. Thus, we have proved that decomposition (4.52) reads as

$$\bar{R}_N = \tilde{\mu}^{(N)}Y_{N,N-1}. \tag{4.56}$$

Let $0 < k < N$. We are going to show that

$$\bar{R}_k = \lambda^{(k)}R + \mu^{(k)}Y_{k+1,k} + \tilde{\mu}^{(k)}Y_{k,k-1} + \nu^{(k)}Y_{k+1} + \eta^{(k)}Y_k + \epsilon^{(k)}Y_{k-1}. \tag{4.57}$$

We apply the operator ad_{D_x} to both sides of relation (4.57):

$$\begin{aligned}-a_k(\lambda^{(k)}R+\mu^{(k)}Y_{k+1,k}+\tilde{\mu}^{(k)}Y_{k,k-1}+\cdots)-h_{k-1}Y_{k,k-1}+h_{k+1}Y_{k+1,k}+\cdots\\ =D_x(\lambda^{(k)})R+D_x(\mu^{(k)})Y_{k+1,k}+D_x(\tilde{\mu}^{(k)})Y_{k,k-1}\\ +\mu^{(k)}(-(a_{k+1}+a_k)Y_{k+1,k}+\cdots)+\tilde{\mu}^{(k)}(-(a_k+a_{k-1})Y_{k,k-1}+\cdots).\end{aligned} \tag{4.58}$$

Here the dots stand for a linear combination of the operators $Y_0, Y_1, \ldots, Y_{N-1}, Y_N$. Collecting the coefficients at R, $Y_{k+1,k}$, $Y_{k,k-1}$ in (4.58), we obtain the system

$$D_x(\lambda^{(k)})=-a_k\lambda^{(k)}, \tag{4.59}$$

$$D_x(\mu^{(k)})=a_{k+1}\mu^{(k)}+h_{k+1}, \tag{4.60}$$

$$D_x(\tilde{\mu}^{(k)})=a_{k-1}\tilde{\mu}^{(k)}-h_{k-1}. \tag{4.61}$$

Equation (4.59) coincides with (4.14) if $i=k$. That is why we obtain $\lambda^{(k)}=0$. Equation (4.60) coincides with the first equation in (4.15) if $i=k$, and equation (4.61) coincides with the second equation in (4.15) if $i=k$. Hence, $\mu^{(k)}=D_n^k(\mu(u_1))=\mu(u_{k+1})$, $\tilde{\mu}^{(k)}=D_n^k(\tilde{\mu}(u_{-1}))=\tilde{\mu}(u_{k-1})$. It is easy to show that $\nu^{(k)}=\eta^{(k)}=\epsilon^{(k)}=0$. Thus, we have proved that decomposition (4.57) reads as

$$\bar{R}_k=\mu^{(k)}Y_{k+1,k}+\tilde{\mu}^{(k)}Y_{k,k-1}. \tag{4.62}$$

We calculate the commutator $[Y_{i+1,i},R]$ for some i, $0\leqslant i\leqslant N-1$. Using the Jacobi identity, we obtain

$$\begin{aligned}[Y_{i+1,i},R]=&-[R,Y_{i+1,i}]=-[R,[Y_{i+1},Y_i]]=[Y_{i+1},[Y_i,R]]+[Y_i,[R,Y_{i+1}]]\\ =&\left[Y_{i+1},\mu^{(i)}Y_{i+1,i}+\tilde{\mu}^{(i)}Y_{i,i-1}\right]-\left[Y_i,\mu^{(i+1)}Y_{i+2,i+1}+\tilde{\mu}^{(i+1)}Y_{i+1,i}\right]\\ =&\Lambda^{(i)}Y_{i+2,i+1,i}+M^{(i)}Y_{i+1,i,i-1}+\kappa^{(i)}Y_{i+2,i+1}+\eta^{(i)}Y_{i+1,i}+\zeta^{(i)}Y_{i,i-1},\end{aligned}$$

where $\Lambda^{(i)}, M^{(i)}, \kappa^{(i)}, \eta^{(i)}, \zeta^{(i)}$ are some functions depending on dynamic variables. Herewith $\zeta^{(0)}=0$, $M^{(0)}=0$, $\Lambda^{N-1}=0$, $\kappa^{N-1}=0$.

Let us justify the inductive step. Given M, $0\leqslant k<M\leqslant N-1$, we assume that

$$\begin{aligned}[Y_{M,M-1,\ldots,k},R]=&\Lambda Y_{M+1,M,M-1,\ldots,k}+MY_{M,M-1,\ldots,k,k-1}+\nu Y_{M,M-1,\ldots,k}\\ &+\varepsilon Y_{M+1,M,M-1,..,k+1}+\eta Y_{M-1,\ldots,k,k-1}+\zeta Y_{M-1,M-2,\ldots,k}\\ &+\theta Y_{M,M-1,\ldots,k+1}+\xi Y_{M-2,M-2,\ldots,k-1}+\cdots\\ &+\cdots+\kappa Y_{M+1,M}+\varphi Y_{M,M-1}+\cdots+\chi Y_{k,k-1}.\end{aligned} \tag{4.63}$$

Let us prove a similar representation holds for $M+1$. Using the Jacobi identity, we obtain:

$$\begin{aligned}[Y_{M+1,M,M-1,\ldots,k},R]=&-[R,[Y_{M+1},Y_{M,M-1,\ldots,k}]]\\ =&[Y_{M+1},[Y_{M,M-1,\ldots,k},R]]+[Y_{M,M-1,..,k},[R,Y_{M+1}]]\\ =&[Y_{M+1},[Y_{M,M-1,\ldots,k},R]]-[Y_{M,M-1,..,k},R_{M+1}].\end{aligned}$$

We substitute the decomposition (4.63) and one of equations (4.62), (4.51) or (4.56) (this depends on a particular value of M: $M=0$, $M=N$ or $0<M<N$) into the last formula. Then we expand the commutators using the linearity property. The latter completes the proof of Theorem 4. □

Conclusion

In this paper the problem of the integrable classification of two-dimensional chains of type (1.1) is studied. For chains of special type (1.5), (1.6), a complete description of the integrable cases is obtained. By integrability of the chain we mean here the existence of reductions in the form of arbitrarily high order systems of hyperbolic type equations that are Darboux integrable. Apart of the known chains, the obtained list contains new chains (see Chains ii) and iii) in Theorem 1).

The algorithm used for classification is relatively new and testing this algorithm is one of the goals of the work. It is based on the concept of the characteristic Lie algebra applied earlier to the systems of hyperbolic type equations with two independent variables (see, for instance, [17], [20] and the references therein). It is well known that for a Darboux integrable system, the characteristic algebras in both directions have finite dimensions. In the present article we adapted this concept to the classification of $1+2$-dimensional lattices.

As the examples show (see [31] - [34]), the characteristic algebras of the hyperbolic systems with two independent variables integrable by means of the inverse scattering method are slow growth algebras.

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Mariya Nikolaevna Poptsova,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: mnpoptsova@gmail.com

Ismagil Talgatovich Habibullin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia,
Bashkir State University,
Validy str. 32,
450077, Ufa, Russia
E-mail: habibullinismagil@gmail.com