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NEW CHARACTERIZATIONS OF BLOCH SPACES, BERS-TYPE AND ZYGMUND-TYPE SPACES AND RELATED QUESTIONS

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Abstract. In terms of Berezin symbols, we give new characterizations of the Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$, Bers-type and the Zygmund-type spaces of analytic functions on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. We discuss some properties of Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$. We provide a new characterization of certain function space with variable exponents. Namely, given a function $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ with a bounded sequence $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ of Taylor coefficients $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0, 1, 2, \ldots)$, we have $f \in H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ if and only if

$$\int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\frac{\gamma(t)p(t)-q(t)}{p(t)}} dr < +\infty.$$

Here $D_{(a_n)}$ denotes the associate diagonal operator on the Hardy-Hilbert space H^2 defined by the formula $D_{(a_n)} z^n = a_n z^n$ $(n = 0, 1, 2, \ldots)$.

Keywords: Bers-type space, Zygmund-type space, Bloch spaces, Berezin symbol.

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1. Introduction

Let $\mathbb{D}$ be the unit disc in complex plane $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathrm{Hol}(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. We denote by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$. Recall that a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to the Bloch space $\mathcal{B} = \mathcal{B}(\mathbb{D})$ if

$$\|f\|_b := \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < +\infty.$$

Being equipped with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_b$, $\mathcal{B}$ is a Banach space. Let $\mathcal{B}_0 = \mathcal{B}_0(\mathbb{D})$ be the space consisting of all $f \in \mathcal{B}$ satisfying

$$\lim_{|z|\to 1} (1 - |z|^2) |f'(z)| = 0.$$

This space is called a little Bloch space.

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Let $\alpha \geqslant 0$. The Bers-type space, denoted by $H_\alpha^\infty = H_\alpha^\infty(\mathbb{D})$, is a Banach space consisting of all $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f(z)| < +\infty.$$

Clearly, $H_0^\infty = H^\infty$.

Let $0 < p,\ q < +\infty$, $\gamma > -1$. If a function $f \in \mathrm{Hol}(\mathbb{D})$ is such that

$$\|f\|_{Hp,q,\gamma}^q := \int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \right)^{\frac{q}{p}} (1-r)^\gamma dr < +\infty,$$

we say that f belongs to a mixed norm space denoted by $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$.

Let $\beta > 0$. The Zygmund-type space denoted by $\mathcal{Z}^\beta$ consist of all $f \in \mathrm{Hol}(\mathbb{D})$ obeying

$$\|f\|_{\mathcal{Z}^\beta} := |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left|f''(z)\right| < +\infty.$$

The space $\mathcal{Z}^\beta$ becomes a Banach space with the above norm $\|\cdot\|_{\mathcal{Z}^\beta}$. Let $\beta = 1$. Then $\mathcal{Z}^1 = \mathcal{Z}$ is the classical Zygmund space. For more information on the Bers-type, Zygmund-type and Bloch spaces on the unit disc $\mathbb{D}$, see, e.g., K. Zhu [14, 15], X. Zhu [16], S. Stević [11], [12], Y. Ren [9], P. Duren [2], J. Shi [10].

Recall that the following old problem for most of functional spaces X of analytic functions in $\mathbb{D}$, including the Hardy space $H^p(\mathbb{D})$, is open (see Privalov [8] and Duren [2]):

How a function in X can be recovered by the behavior of its Taylor coefficients?

Ideally, one would like to find a condition on the $\widehat{f}(n) := \frac{f^n(0)}{n!}$ (Taylor coefficient), which is both necessary and sufficient for f to be in X Of course, for $X = H^p$, $p = 2$, the problem is completely solved: $f \in H^2$ if and only if $\sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 < +\infty$. For $p = \infty$, the problem of coefficients was solved by I. Schur in 1919 (see, Privalov [8, Ch. 2]). Some classical results on the Taylor coefficients of functions in Hardy and Bergman spaces are also known (see, for instance, [2, 8, 14]). Some recent results about Taylor coefficients of H^1 functions and entire functions in the Fock spaces F_α^p have been obtained, respectively, by Pavlović [6] and Tung [13]. But the general situation is much more complicated, and no complete answer is available (more informations about this are contained in [1] and references therein.)

Note that in his book [6], Pavlović proved the following characterization of functions belonging to the Hardy space H^1 :

$$H^1 = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right| dt < +\infty \right\}.$$

Theorem 1.1. *For a function f analytic in $\mathbb{D}$, the following statements are equivalent:*
(a) $f \in H^1$;
(b) $\sup_{n\geqslant 0} \frac{1}{a_n}\sum_{j=0}^{n} \frac{1}{j+1} \|S_j(f)\|_{H^1} < +\infty$;
(c) $\sup_{n\geqslant 0} \|P_n f\|_{H^1} < +\infty$.

Here $P_n f := \frac{1}{a_n}\sum_{j=0}^{n} \frac{1}{j+1} S_j(f)$, where $a_n = \sum_{j=0}^{n} \frac{1}{j+1}$, $n = 0, 1, 2, \ldots$, and $S_j(f)$ are the partial sums of the Taylor series for f.

Popa [7] gave some interesting generalization of this result of Pavlović by proving a similar characterization of upper triangular trace class matrices. Recently, Karaev [3] gave some characterizations of Hardy and Besov classes of analytic functions on $\mathbb{D}$ with the variable exponents.

In this paper, we use the so-called Berezin symbols technique to characterize the Bers-type, Zygmund-type spaces, and the Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$. We also consider some spaces defined by the variable exponent. We discuss Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$

2. Berezin symbols and characterizations of the Bers-type, Zygmund-type and Bloch spaces

In this section, we characterize the spaces H_α^∞, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$ in terms of Berezin symbols of diagonal operators (associated with the Taylor coefficients of the functions from the spaces H_α^∞, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$) acting on the Hardy space $H^2 = H^2(\mathbb{D})$.

Recall that Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ is the collection of analytic functions in $\mathbb{D}$ which satisfy the inequality

$$\|f\|_{H^2} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^2 d\theta \right)^{\frac{1}{2}} < +\infty.$$

It is well-known that H^2 is a reproducing kernel Hilbert space with the orthonormal basis $e_n(z) = z^n$ $(n = 0, 1, 2, \ldots)$, and consequently with the reproducing kernel

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z) = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n = \frac{1}{1 - \overline{\lambda} z},\ \lambda \in \mathbb{D}.$$

The reproducing property means that $\langle f, k_\lambda \rangle_{H^2} = f(\lambda)$ for all $f \in H^2$ and $\lambda \in \mathbb{D}$. For any bounded sequence $(a_n)_{n \geqslant 0}$ of complex numbers a_n, we denote by $D_{(a_n)}$ the associate diagonal operator on the Hardy-Hilbert space H^2 which is defined by the formula

$$D_{(a_n)} z^n = a_n z^n, \qquad n = 0, 1, 2, \ldots \tag{1}$$

For any bounded linear operator T on H^2, its Berezin symbol $\widetilde{T}$ is the following bounded complex-valued function in $\mathbb{D}$:

$$\widetilde{T}(\lambda) := \langle T\mathcal{K}_\lambda, \mathcal{K}_\lambda \rangle \ \ (\lambda \in \mathbb{D}),$$

where

$$\mathcal{K}_\lambda(z) := \frac{k_\lambda(z)}{\|k_\lambda(z)\|_{H^2}} = (1 - |\lambda|^2)^{\frac{1}{2}} (1 - \overline{\lambda} z)^{-1}$$

is the normalized reproducing kernel of H^2, and $ber(T) := \sup_{\lambda \in \mathbb{D}} \left| \widetilde{T}(\lambda) \right|$ is the Berezin number of T.

The following lemma is well-known.

Lemma 2.1. *The Berezin symbol of the diagonal operator $D_{(a_n)}$ on the Hardy space H^2 is the following radial function:*

$$\widetilde{D}_{(a_n)}(|\lambda|) = (1 - |\lambda|^2) \sum_{k=0}^{\infty} a_k |\lambda|^{2k} \ \ (\lambda \in \mathbb{D}) \tag{2}$$

Proof. Indeed, by using that

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z} = \sum_{k=0}^{\infty} \overline{\lambda}^k z^k$$

is the reproducing kernel of H^2, we have:

$$\begin{aligned}\widetilde{D}_{(a_n)}(\lambda) &= \left\langle D_{(a_n)}\mathcal{K}_\lambda(z), \mathcal{K}_\lambda(z)\right\rangle = \left\langle D_{(a_n)}\frac{k_\lambda(z)}{\|k_\lambda(z)\|_2}, \frac{k_\lambda(z)}{\|k_\lambda(z)\|_2}\right\rangle \\ &= \left(1-|\lambda|^2\right)\left\langle D_{(a_n)}\sum_{k=0}^{\infty}\overline{\lambda}^k z^k, \sum_{k=0}^{\infty}\overline{\lambda}^k z^k\right\rangle = \left(1-|\lambda|^2\right)\left\langle \sum_{k=0}^{\infty}\overline{\lambda}^k a_k z^k, \sum_{k=0}^{\infty}\overline{\lambda}^k z^k\right\rangle \\ &= \left(1-|\lambda|^2\right)\sum_{k=0}^{\infty} a_k\,|\lambda|^{2k}.\end{aligned}$$

Hence,

$$\widetilde{D}_{(a_n)}(\lambda) = \left(1-|\lambda|^2\right)\sum_{k=0}^{\infty} a_k\,|\lambda|^{2k}, \qquad \lambda\in\mathbb{D},$$

which proves formula (2). □

Our next result characterizes the spaces H_α^∞, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$ in terms of behavior of the Berezin symbols of the corresponding diagonal operators mentioned above.

Theorem 2.1. *Let a function* $f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ *have a bounded sequence* $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ *of Taylor coefficients* $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0,1,2,\ldots)$. *Then* $f\in H_\alpha^\infty$ $(\alpha\geqslant 0)$ *if and only if*

$$\sup_{\substack{0\leqslant r<1\\ 0\leqslant\theta<2\pi}} (1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right| < +\infty.$$

Proof. Indeed, let a function f be as in the statement of the theorem. Then, rewriting f and using Lemma 2.1, we obtain:

$$\begin{aligned}f(z) &= f(|z|\,e^{i\arg(z)}) = f(re^{i\theta}) = \sum_{n=0}^{\infty}\widehat{f}(n)(re^{i\theta})^n \\ &= \sum_{n=0}^{\infty}\widehat{f}(n)e^{in\theta}r^n = \frac{(1-r)\sum_{n=0}^{\infty}\widehat{f}(n)e^{in\theta}r^n}{1-r} = \frac{\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})}{1-r}.\end{aligned}$$

Hence, for any $z = re^{i\theta}$, with $r = |z|$ and $\theta = \arg(z)$, $0\leqslant r<1$, $0\leqslant\theta<2\pi$, we have

$$f(z) = \frac{\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})}{1-r}. \tag{3}$$

Using (3), we get

$$\left(1-|z|^2\right)^\alpha|f(z)| = (1-r^2)^\alpha\frac{\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})}{1-r} = (1-r)^{\alpha-1}(1+r)^\alpha\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right| \tag{4}$$

for all r, $0\leqslant r<1$ and all $\theta\in[0,2\pi)$. Hence, we obtain

$$(1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right| \leqslant \left(1-|z|^2\right)^\alpha|f(z)| \leqslant 2^\alpha(1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right|$$

for all $z = re^{i\theta}\in\mathbb{D}$. In particular, by (5) we conclude that that $f\in H_\alpha^\infty$ if and only if

$$\sup_{\substack{0\leqslant r<1\\ \theta\in[0,2\pi)}} (1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right| < +\infty.$$

This completes the proof. □

An immediate corollary of inequalities (5) is as follows.

Corollary 1. *If $f \in H^p_\alpha$ and $\alpha > 0$, then*

$$\sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} (1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right| \leqslant \|f\|_{p,\alpha} \leqslant 2^\alpha \sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} (1-r)^{\alpha-1}\left|\widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r})\right|.$$

Theorem 2.2. *Let a function $f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ have the sequence $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ of Taylor coefficients such that $\widehat{f}(n) = O\left(\frac{1}{n}\right)$ as $n\to\infty$. Then*

(*a*) *f belongs to Bloch space $\mathcal{B}$ if and only if*

$$\sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} \left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right| < +\infty;$$

(*b*) *$f\in\mathcal{B}_0$ if and only if*

$$\lim_{r\to 1}\left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right| = 0$$

for all $\theta\in[0,2\pi)$.

Proof. It follows from the condition $\widehat{f}(n) = O\left(\frac{1}{n}\right)$ $(n\to\infty)$ that $(n\widehat{f}(n))_n$ is bounded, and hence, the diagonal operator $D_{(n\widehat{f}(n))}$ is bounded on the Hardy-Hilbert space H^2. This implies immediately that $D_{((n+1)\widehat{f}(n+1)e^{in\theta})}$ is also bounded in H^2 for every fixed $\theta\in[0,2\pi)$. Therefore, by formula (2) in Lemma 2.1, we have

$$\begin{aligned}
\left(1-|z|^2\right)f'(z) &= (1-r^2)\sum_{n=0}^{\infty}\widehat{f}(n)z^n = (1-r^2)\sum_{n=1}^{\infty}n\widehat{f}(n)z^{n-1}\\
&= (1-r^2)\sum_{n=0}^{\infty}(n+1)\widehat{f}(n+1)z^n\\
&= (1-r)(1+r)\sum_{n=0}^{\infty}(n+1)\widehat{f}(n+1)e^{in\theta}r^n\\
&= (1+r)\left[(1-r)\sum_{n=0}^{\infty}\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)r^n\right]\\
&= (1+r)\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}).
\end{aligned}$$

Hence,

$$\left(1-|z|^2\right)f'(z) = (1+r)\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \tag{6}$$

for all $z = re^{i\theta}\in\mathbb{D}$, where $r\in[0,1)$ and $\theta\in[0,2\pi)$. Now formula (6) shows that $f\in\mathcal{B}$ if and only if

$$\sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} \left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right| < +\infty$$

and this proves (a).

On the other hand, it follows also immediately from (6) that $f\in\mathcal{B}_0$ if and only if

$$\lim_{r\to 1}\left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right| = 0$$

for all θ, $0\leqslant\theta<2\pi$. This proves (b). □

Formula (6) implies also the following results.

Corollary 2. *Assume that* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ *satisfies the condition of Theorem 2.2. If* $f \in \mathcal{B}$*, then*

$$\inf_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} \left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right| + |f(0)| \leqslant \|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} \left|\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r})\right|.$$

Corollary 3. *If* f *is the same as in Corollary* 2*, then we have:*
(*a*) $\|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{0\leqslant\theta<2\pi} ber(D_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)})$.
(*b*) $\|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{n\geqslant 0}(n+1)\left|\widehat{f}(n+1)\right|$.

Clearly, Statement (a) implies Statement (b) since for each arbitrary fixed $\theta \in [0, 2\pi)$

$$ber(D_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)}) \leqslant \left\|D_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)}\right\| = \sup_{n\geqslant 0}(n+1)\left|\widehat{f}(n+1)\right|.$$

Corollary 4. *Each function* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ *obeying* $\widehat{f}(n) = o(\frac{1}{n})$ *as* $n \to \infty$ *belongs to the little Bloch space* $\mathcal{B}_0$*.*

Proof. Indeed, since by assumption $\lim_{n\to\infty} n\left|\widehat{f}(n)\right| = 0$, we see that $D_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)}$ is a compact diagonal operator, and since H^2 is the standard reproducing kernel Hilbert space (which means that the normalized reproducing kernel $\mathcal{K}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}z}$ weakly tends to zero as λ approaches any boundary point of the unit disc $\mathbb{D}$), we obtain that $\lim_{r\to 1}\left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)}(\sqrt{r})\right| = 0$ for all $\theta \in [0, 2\pi)$, which implies that $f \in \mathcal{B}_0$. The proof is complete. $\square$

Theorem 2.3. *Let* $\beta > 0$ *and* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function such that* $\widehat{f}(n) = O\left(\frac{1}{(n-1)n}\right)$ *as* $n \to \infty$*. Then* $f \in \mathcal{Z}^\beta$ *if and only if*

$$\sup_{\substack{0\leqslant r<1\\0\leqslant\theta<2\pi}} (1-r)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right)}(\sqrt{r})\right| < +\infty.$$

Proof. Since by assumption $((n-1)n\widehat{f}(n))_{n\geqslant 2}$ is a bounded sequence, the diagonal operator $D_{((n-1)n\widehat{f}(n)e^{in\theta})}$ is a bounded operator on the Hardy space H^2 for all $\theta \in [0, 2\pi)$. Then, by Lemma 2.1 we have:

$$\begin{aligned}
(1-|z|^2)^\beta |f''(z)| &= (1-r^2)\left|\sum_{n=2}^{\infty}(n-1)n\widehat{f}(n)z^{n-2}\right| \\
&= (1-r^2)^\beta\left|\sum_{n=0}^{\infty}(n+1)(n+2)\widehat{f}(n+2)z^n\right| \\
&= (1-r^2)^\beta\left|\sum_{n=0}^{\infty}(n+1)(n+2)\widehat{f}(n+2)e^{in\theta}r^n\right| \\
&= (1+r)^\beta(1-r)^{\beta-1}\left|(1-r)\sum_{n=0}^{\infty}\left[(n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right]r^n\right|
\end{aligned}$$

$$= (1+r)^{\beta}(1-r)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right)}\sqrt{r}\right|$$

and hence,

$$(1-|z|^2)^{\beta}\,|f''(z)| = (1+r)^{\beta}(1-r)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right)}\left(\sqrt{r}\right)\right| \tag{7}$$

for all $z = re^{i\theta} \in \mathbb{D}$. This equality shows that $f \in \mathcal{Z}^{\beta}$ if and only if

$$\sup_{\substack{0\leqslant r<1\\ 0\leqslant\theta<2\pi}} (1-r)^{\beta-1}\left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)}(\sqrt{r})\right| < +\infty,$$

which proves the theorem. □

3. Generalized subharmonicity and Toeplitz operators on Bergman space

In this section we apply representations (4), (6) and (7) for studying the boundedness, compactness and belonging to Schatten-Neumann class for Toeplitz operators acting in the Bergman space $L_a^2 = L_a^2(\mathbb{D})$.

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In terms of Cartesian and polar coordinates is reads as

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$$

For$1 \leqslant p < +\infty$, the usual Lebesgue space $L^p(\mathbb{D},dA)$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with the norm

$$\|f\|_p = \left[\int_{\mathbb{D}} |f(z)|^p\, dA(z)\right]^{\frac{1}{p}} < +\infty.$$

The Bergman space $L_a^p = L_a^p(\mathbb{D})$ is defined to be the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions. For $p = 2$, L_a^2 is a reproducing kernel Hilbert space with the reproducing kernel

$$k(z,w) = \frac{1}{(1-z\overline{w})^2}$$

Recall that $P : L^2(\mathbb{D}, dA) \to L_a^2$ is the Bergman projection and this is an integral operator given by the formula

$$Pf(z) = \int_{\mathbb{D}} k(z,w)f(w)dA(w) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^2}dA(w).$$

Given a function $\varphi \in L^{\infty}(\mathbb{D})$, we define the Toeplitz operator T_{φ} on L_a^2 by $T\varphi f = P(\varphi f)$, $f \in L_a^2$. Since the Bergman projection has the unit norm, we clearly get $\|T_{\varphi}\| \leqslant \|\varphi\|_{\infty}$. More information about Bergman space Toeplitz operators can be found, for example, in the book by K.Zhu [14].

Definition 3.1 ([14]). *Suppose f is a nonnegative function on $\mathbb{D}$. We say that f has a generalized subharmonic property if there exists a constant $C > 0$ such that*

$$f(z) \leqslant \frac{C}{|D(z,r)|}\int_{D(z,r)} f(w)dA(w)$$

for all $z \in \mathbb{D}$. Here for each $r > 0$ and $a \in \mathbb{D}$

$$D(a,r) := \{z \in \mathbb{D} : \beta(z,a) < r\}$$

is the Bergman disc with the Bergman metric

$$\beta(z,w)=\frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where $\rho(z,w)=\left|\frac{z-w}{1-\bar{z}w}\right|$ $(z,w\in\mathbb{D})$ *is the pseudo-hyperbolic distance on* $\mathbb{D}$*) and* $|D(a,r)|$ *is the normalized area of* $D(a,r)$*.*

Before proving a next theorem, we mention the following known result [14].

Lemma 3.1. *If* $\varphi\geqslant 0$ *possesses the generalized subharmonic property, then*
(1) T_φ *is bounded on* L_a^2 *if and only if* φ *is bounded as* $|z|\to 1^-$;
(2) T_φ *is compact on* L_a^2 *if and only if* $\varphi(z)\to 0$ *as* $|z|\to 1^-$;
(3) T_φ *is in* $S_p(L_a^2)$ *if and only if* $\varphi\in L^p(\mathbb{D},d\lambda)$*, where* $d\lambda(z)=\frac{dA(z)}{(1-|z|^2)^2}$ *is the Möbius invariant measure on* $\mathbb{D}$*.*

Theorem 3.1. *Let a function* $f(z)=\sum\limits_{n=0}^{\infty}\widehat{f}(n)z^n\in\mathrm{Hol}(\mathbb{D})$ *have a bounded sequence* $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ *of Taylor coefficients. Let* $\alpha\geqslant 0$*. Then*
(*a*) $T_{(1+|z|)^\alpha(1-|z|)^{\alpha-1}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|}$ *is bounded on* L_a^2 *if and only if*

$$(1-|z|)^{\alpha-1}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|$$

is bounded as $|z|\to 1^-$;
(*b*) $T_{(1+|z|)^\alpha(1-|z|)^{\alpha-1}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|}$ *is compact on* L_a^2 *if and only if*

$$(1-|z|)^{\alpha-1}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|\to 0$$

as $|z|\to 1^-$;
(*c*) T *belongs to the Schatten-Neumann class* $S_p:=S_p\left(L_a^2\right)$ *if and only if*

$$\int\limits_{\mathbb{D}}(1+|z|)^{\alpha p}(1-|z|)^{p(\alpha-1)}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|^p d_\lambda(z)<+\infty,$$

where $d_\lambda(z)=\frac{dA(z)}{(1-|z|)^2}$ *is the Möbius invariant measure on* $\mathbb{D}$*.*

Proof. Indeed, one can prove that if f is analytic, $p>0$, and α is real, then the function $(1-|z|)^\alpha|f(z)|^p$ possesses the generalized subharmonic property (see K.Zhu [14]). In particular, the function $(1-|z|^2)^\alpha|f(z)|$ possesses the same property. On the other hand, by formula (4),

$$(1-|z|^2)^\alpha|f(z)|=(1+|z|)^\alpha(1-|z|)^{\alpha-1}\left|\widetilde{D}_{\left(\widehat{f}(n)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|$$

for all $z\in\mathbb{D}$. Now the statement of theorem is immediately implied by Lemma 3.1. □

By using formulae (6) and (7), the nextg two results can be proved by the same method as in the above proof, and therefore the proof is omitted.

Theorem 3.2. *Let a function* $f(z)=\sum\limits_{n=0}^{\infty}\widehat{f}(n)z^n\in\mathrm{Hol}(\mathbb{D})$ *have the sequence* $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ *of Taylor coefficients such that* $\widehat{f}(n)=O\left(\frac{1}{n}\right)$ *as* $n\to\infty$*. Then*
(*a*) $T_{\left(1-|z|^2\right)|f'(z)|}$ *is bounded on* L_a^2 *if and only if* $\left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|$ *is bounded as* $|z|\to 1^-$;

(b) $T_{(1-|z|^2)|f'(z)|}$ *is compact on* L_a^2 *if and only if*

$$\left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\arg(z)}\right)}(\sqrt{|z|})\right| \to 0$$

as $|z| \to 1^-$;

(c) $T_{(1-|z|^2)|f'(z)|}$ *is in* $S_p(L_a^2)$ *if and only if*

$$\int\limits_{\mathbb{D}} (1+|z|)^p \left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|^p d_\lambda(z) < +\infty.$$

Theorem 3.3. *Let* $\beta > 0$ *and* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function such that* $\widehat{f}(n) = O(((n-1)n)^{-1})$ *as* $n \to \infty$. *Then*

(a) $T_{(1-|z|^2)^\beta |f''(z)|}$ *is bounded on* L_a^2 *if and only if* $(1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\arg(z)}\right)}(\sqrt{|z|})\right|$ *is bounded as* $|z| \to 1^-$;

(b) $T_{(1-|z|^2)^\beta |f''(z)|}$ *is compact on* L_a^2 *if and only if*

$$(1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\arg(z)}\right)}(\sqrt{|z|})\right| \to 0$$

as $|z| \to 1^-$;

(c) $T_{(1-|z|^2)^\beta |f''(z)|}$ *is in* $S_p(L_a^2)$ *if and only if*

$$(1+|z|)^\beta (1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\arg(z)}\right)}(\sqrt{|z|})\right| \in L^p(\mathbb{D}, d\lambda).$$

4. Characterization of mixed norm space $H_{p(t),q(t),\gamma(t)}$ with variable exponents

Let $\mathbb{T} = \partial\mathbb{D}$ and let $p = p(t)$ and $q(t)$, $t \in \mathbb{T}$, be bounded positive measurable functions defined on $\mathbb{T}$, and let $\gamma(t) > -1$ on $\mathbb{T}$. Following by Kokilashvili and Paatashvili [4, 5] (see also Karaev [2]), we say that the analytic function f in $\mathbb{D}$ belongs to the Hardy class $H^{p(t)}$ if

$$\sup_{0<r<1} \frac{1}{2\pi} \int\limits_0^{2\pi} \left|f(re^{it})\right|^{p(t)} dt < +\infty,$$

where $p(t) = p(e^{it})$, $t \in [0, 2\pi)$.

Similarly, we say that a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to the mixed norm space $H_{p(t),q(t),\gamma(t)}$ with the variable exponents if

$$\|f\|^{q(t)}_{H_{p(t),q(t),\gamma(t)}} := \int\limits_0^1 \left(\frac{1}{2\pi}\int\limits_0^{2\pi} \left|f(re^{it})\right|^{p(t)} dt\right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr < +\infty,$$

where $p(t) = p(e^{it})$, $q(t) = q(e^{it})$ and $\gamma(t) = \gamma(e^{it})$, $t \in [0, 2\pi)$.

For $p(t) = p = const > 0$, $q(t) = q = const > 0$ and $\gamma(t) = \gamma = const > -1$, the class $H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ coincides with the class $H_{p,q,\gamma}$.

The following theorem characterize the spaces $H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ in terms of Berezin symbols and Taylor coefficients.

Theorem 4.1. *Let* $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function with the bounded sequence* $\left\{\widehat{f}(n)\right\}_{n\geqslant 0}$ *of Taylor coefficients* $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0, 1, 2, \ldots)$. *Then* $f \in H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ *if and only if*

$$\int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\frac{\gamma(t)p(t)-q(t)}{p(t)}} dr < +\infty.$$

Proof. Using formula (2) in Lemma 2.1, we rewrite the function $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n$ as follows:

$$f(z) = f(re^{it}) = \frac{\widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r})}{1-r} \tag{8}$$

for each $z = re^{it} \in \mathbb{D}$. Now the statement of the theorem follows the definition of the space $H_{p(\cdot),q(\cdot),\gamma(\cdot)}$. Namely, in view of (8), we obtain

$$\int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| f(re^{it}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr = \int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{\widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r})}{1-r} \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr$$

$$= \int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)-\frac{q(t)}{p(t)}} dr.$$

This yields that $f \in H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ if and only if

$$\int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)-\frac{q(t)}{p(t)}} dr < +\infty.$$

The proof is complete. □

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