FOURIER METHOD FOR FIRST ORDER DIFFERENTIAL EQUATIONS WITH INVOLUTION AND GROUPS OF OPERATORS

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Abstract. In the paper we study a mixed problem for a first-order differential equation with an involution. It is written by means of a differential operator with an involution acting in the space functions square integrable on a finite interval. We construct a similarity transform of this operator in an operator being an orthogonal direct sum of an operator of finite rank and operators of rank 1. The method of our study is the method of similar operators. Theorem on similarity serves as the basis for constructing groups of operators, whose generator is the original operator. We write out asymptotic formulae for groups of operators. The constructed group allows us to introduce the notion of a mild solution, and also to describe the mild solutions to the considered problem.

This serves to justify the Fourier method. Almost periodicity of bounded mild solutions is established. The proof of almost periodicity is based on the asymptotic representation of the spectrum of a differential operator with an involution.

Keywords: method of similar operator, spectrum, mixed problem, group of operators, differential operator with involution.

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1. INTRODUCTION AND MAIN RESULTS

We study the mixed problem for a hyperbolic equation with an involution:

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} = \dfrac{\partial u(t,s)}{\partial s} - v(s)u(t,\omega - s), \\ u(t,0) = u(t,\omega), \\ u(0,s) = \varphi(s), \\ t \in \mathcal{J}, \quad s \in [0,\omega], \end{cases} \tag{1}$$

where the symbol $\mathcal{J}$ denotes of the intervals $(-\infty, \infty)$, $(-\infty, \beta]$, $[\alpha, \beta]$, $[\alpha, \infty)$. Hereinafter we assume that the considered interval contains zero.

Together with homogeneous problem (1) we consider the corresponding inhomogeneous problem

$$\begin{cases} \dfrac{\partial u(t,s)}{\partial t} = \dfrac{\partial u(t,s)}{\partial s} - v(s)u(t,\omega - s) + f(t,s), \\ u(t,0) = u(t,\omega), \\ u(0,s) = \varphi(s), \\ t \in \mathcal{J}, \quad s \in [0,\omega]. \end{cases} \tag{2}$$

To formulate the problem, we introduce the following functional spaces. By $L_2 = L_2[0,\omega]$ we denote the Hilbert space of Lebesgue measurable square integrable functions (equivalence classes) on $[0,\omega]$ with values in $\mathbb{C}$. The scalar product in L_2 is defined by the formula

$$(x,y) = \frac{1}{\omega} \int_0^\omega x(s)\overline{y(s)}\,ds, \quad x,y \in L_2,$$

and the norm is generated by this scalar product.

The space L_2 is isometrically isomorphic to the Hilbert space $L_{2,\omega} = L_{2,\omega}(\mathbb{R},\mathbb{C})$ of complex ω-periodic functions defined on the entire axis $\mathbb{R}$ and square integrable on $[0,\omega]$. In what follows, each function $x \in L_2$ will be identified with its ω-periodic continuation on $\mathbb{R}$.

By $W_2^1 = W_2^1[0,\omega]$ we denote the Sobolev space of absolutely continuous functions in L_2 with derivatives in L_2; this space is equipped with the scalar product $(x,y)_{W_2^1} = (x,y) + (x',y')$, x, $y \in W_2^1$.

By symbol $\operatorname{End}\mathcal{H}$ we denote the Banach algebra of bounded linear operators acting in an abstract Hilbert space $\mathcal{H}$ with the norm

$$\|X\|_\infty = \sup_{\|x\|\leqslant 1} \|Xx\|, \qquad x \in \mathcal{H}, \quad X \in \operatorname{End}\mathcal{H}.$$

We introduce a two-sided ideal of Hilbert-Schmidt operators $\mathfrak{S}_2(\mathcal{H})$ in the algebra $\operatorname{End}\mathcal{H}$. By $\|X\|_2$ we denote the norm of Hilbert-Schmidt operators $X \in \mathfrak{S}_2(\mathcal{H})$, that is, $\|X\|_2 = (\operatorname{tr}(XX^*))^{1/2}$. Here $\operatorname{tr}(XX^*)$ is the trace of the operator XX^* belonging to a two-sided ideal $\mathfrak{S}_1(\mathcal{H})$ of nuclear operators in $\operatorname{End}\mathcal{H}$ (see [1]) with the norm $\|X\|_1 = \operatorname{tr}(XX^*) = \sum_{n\in\mathbb{Z}} s_n$, where (s_n) is the sequence of s-numbers of an operator X. The formula $(X,Y) = \operatorname{tr}(XY^*)$, X, $Y \in \mathfrak{S}_2(\mathcal{H})$, defines a scalar product in $\mathfrak{S}_2(\mathcal{H})$.

In the paper we assume that the potential v belongs to the space L_2. In what follows, by the symbol $C(\mathcal{J},L_2)$ we denote a linear space of functions $u : \mathcal{J} \times [0,\omega] \to \mathbb{C}$ possessing the following properties. For a fixed $t \in \mathcal{J}$, the function $s \mapsto u(t,s)$ belongs L_2. Moreover, the function

$$\widetilde{u} : \mathcal{J} \to L_2, \quad \widetilde{u}(t)(s) = u(t,s), \quad t \in \mathcal{J}, \quad s \in [0,\omega],$$

is assumed to be continuous. If $\mathcal{J}$ is a finite interval, then the space $C(\mathcal{J},L_2)$ is Banach. In this case as the norm, the quantity $\|u\|_\infty = \max_{t\in\mathcal{J}} \|\widetilde{u}(t)\|_2$ serves. The function $\widetilde{u}$ is called associated with the function u and they will be identified.

The function $f : \mathcal{J} \times [0,\omega] \to \mathbb{C}$ in inhomogeneous problem (2) is assumed to belong to the space $C(\mathcal{J},L_2)$.

We mention a series of works [2]–[5], where mixed problem (1) was considered with a smooth potential $v : [0,\omega] \to \mathbb{C}$. In these works, the problem on justification of Fourier method for homogeneous problem (1) was studied by a resolvent method (by means of contour integration). There was also studied the asymptotics of eigenvalues and equiconvergence of spectral resolutions for the L defined by problem (1). We shall define this operator a little bit later.

Second order differential operators with involution were studied in papers [6]–[8]. In work [8] there was given a complete bibliographic survey of results on equiconvergence of spectral resolutions for first and second order operators with involution.

A qualitative analysis of solutions to boundary value problems for first and second order equations with involutions was made in [9]–[13]. The basicity property of root functions were considered in [14]. The operator generated by a second order differential expression with an involution at a derivative was studied in [15], [16].

The spectral properties of the operator L (the asymptotics of eigenvalues and estimates for the rate of equiconvergence of spectral resolutions) for a system of differential equations with involution were obtained by the method of similar operators in [17]. In the present work, the main approach is also the method of similar operators. We also mention papers [18]–[22], in which the method of similar operators was developed.

Operators with involution arise in the filtration theory, theory of prediction and in studying subharmonic oscillations [23]–[26].

Problems (1) and (2) in Hilbert space L_2 are written respectively as

$$\widetilde{u}'_t = L\widetilde{u}, \quad \widetilde{u}(0) = \varphi, \tag{3}$$

$$\widetilde{u}'_t = L\widetilde{u} + \widetilde{f}, \quad \widetilde{u}(0) = \varphi. \tag{4}$$

The operator $L : D(L) \subset L_2 \to L_2$ in equations (3), (4) is defined by the formula

$$(Ly)(s) = \frac{dy(s)}{ds} - v(s)y(\omega - s), \quad s \in [0, \omega]. \tag{5}$$

Its domain $D(L)$ is defined by periodic boundary conditions

$$D(L) = \{y \in W_2^1 : y(0) = y(\omega)\}. \tag{6}$$

Let us formulate definitions related to the notion of a solution to considered Cauchy problems (1), (2).

Definition 1. *([27]) A classical solution to problem (2) is a continuously differentiable function $u : \mathcal{J} \times [0, \omega] \to \mathbb{C}$ belonging to the space $C(\mathcal{J}, L_2)$ such that the associated function $\widetilde{u} : \mathcal{J} \to L_2$ is continuously differentiable and for each $t \in \mathcal{J}$ it satisfies the condition $\widetilde{u}(t) \in D(L)$ and equation (4).*

Definition 2. *([27, § 3.1]) A function $u \in C(\mathcal{J}, L_2)$ is called a mild solution to problem (4) if $\int_0^t \widetilde{u}(s)\, ds \in D(L)$ for each $t \in \mathcal{J}$ and*

$$\widetilde{u}(t) = \varphi + L\int_0^t \widetilde{u}(s)\, ds + \int_0^t \widetilde{f}(s)\, ds, \quad t \in \mathcal{J},$$

where the Riemann integrals are considered for continuous functions defined on $\mathcal{J}$ with values in Hilbert space L_2.

The next definition follows naturally Definition 1.

Definition 3. *A classical solution to problem (1), where $\varphi \in W_2^1$, is a function $u : \mathcal{J} \times [0, \omega] \to \mathbb{C}$ belonging to the space $C(\mathcal{J}, L_2)$ such that the associated function $\widetilde{u} : \mathcal{J} \to L_2$ is continuously differentiable and solves problem (1).*

Definition 4. *A function $\widetilde{u} : \mathcal{J} \to L_2$ is called a mild solution to problem (1) if it is a uniform limit on each bounded interval in $\mathcal{J}$ of classical solutions $(\widetilde{u}_n)$, $n \geqslant 1$, obeying $\widetilde{u}_n(0) = \varphi_n \in W_2^1$, $n \geqslant 1$, where $\lim_{n \to \infty} \varphi_n = \varphi$.*

Definition 5. *Problem (1) is said to be uniformly well-defined if for each initial condition $\varphi \in L_2$ there exists a unique mild solution $x \in C(\mathcal{J}, L_2)$ obeying the condition $\widetilde{x}(0) = \varphi$.*

A close definition of a well-defined problem for an abstract differential operator with a constant operator coefficient was given in monograph [28, Ch. II, Sect. 3]. We observe that the definition of a well-definite problem (1) is equivalent to the fact the operator L is a generator of a strongly continuous group of operators.

In this paper, using the theory of semi-groups of operators will play a key role in justification of Fourier method for the equation with involution.

Theorem 1. *Problem (1) is uniformly well-defined. The differential operator L is a generator of some strongly continuous group of operators*

$$T : \mathbb{R} \to \operatorname{End} L_2.$$

Each classical solution $u \in C(\mathcal{J}, L_2)$ to problem (1) is defined by the formula

$$u(t, s) = (T(t)\varphi)(s), \quad s \in [0, \omega], \quad t \in \mathcal{J}, \tag{7}$$

where $\varphi \in W_2^1$ and $\varphi(0) = \varphi(\omega)$. Each mild solution is given by (7), where $\varphi \in L_2$.

Owing to application of the method of similar operators, further Theorem 1 will be significantly strengthened, see Theorems 6–8. It should be noted that at least for a bounded function v, Theorem 1 can be obtained on the base of general theorems on perturbed semi-groups (groups) of operators (see [29], [30]).

Remark 1. *It follows from [27, Prop 3.1.16] that each mild solution $u \in C(\mathcal{J}, L_2)$ to problem (2) written as (4) can be represented as*

$$\widetilde{u}(t) = T(t - t_0)\widetilde{u}(t_0) - \int_{t_0}^{t} T(t - \tau)\widetilde{f}(\tau) \, d\tau, \quad t, t_0 \in \mathcal{J}, \tag{8}$$

where $T : \mathbb{R} \to \operatorname{End} L_2$ is the group of operators in Theorem 1, whose generator is the operator L.

In view of Remark 1, the following definition of classical solution is equivalent to Definition 1.

Definition 6. *A classical solution to problem (2) is a function $u \in C(\mathcal{J}, L_2)$ satisfying identity (8) (integral equation) such that $\widetilde{u}(t) \in D(L) \subset W_2^1$.*

Thus, classical solution satisfies identity (4).

Theorem 2. *Cauchy problem (2) has a unique mild solution $x \in C(\mathcal{J}, L_2)$ such that the associated function $\widetilde{x}$ can be represented as*

$$\widetilde{x}(t) = T(t)\varphi + \int_{0}^{t} T(t - \tau)\widetilde{f}(\tau) \, d\tau, \quad t \in \mathcal{J}.$$

The statement of Theorem 2 follows Theorem 1 and Definition 6.

Throughout the rest of the paper are regularly employed the following definition and properties of similar operators.

Definition 7. *Two linear operators $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1, 2$, are called similar if there exists a continuously invertible operator $U \in \operatorname{End} \mathcal{H}$ such that*

$$A_1 U x = U A_2 x, \quad x \in D(A_2), \quad U D(A_2) = D(A_1).$$

The operator U is called the operator of transformation operator A_1 into A_2.

Such operators possess a series of coinciding spectral properties. It is convenient to formulate an appropriate statement as the following lemma.

Lemma 1. *Let $A_i : D(A_i) \subset \mathcal{H} \to \mathcal{H}$, $i = 1, 2$, be two similar operators and $U \in \operatorname{End} \mathcal{H}$ is the operator of transformation operator A_1 into operator A_2. Then the following statements are true:*

1) $\sigma(A_1) = \sigma(A_2)$, $\sigma_d(A_1) = \sigma_d(A_2)$, $\sigma_c(A_1) = \sigma_c(A_2)$, $\sigma_r(A_1) = \sigma_r(A_2)$, where $\sigma(A_i)$, $\sigma_d(A_i)$, $\sigma_c(A_i)$, $\sigma_r(A_i)$, $i = 1, 2$, are the spectrum, the discrete spectrum. the continuous spectrum and the residual spectrum of the operators A_i, $i = 1, 2$, respectively;

2) if the operator A_2 admits the expansion $A_2 = A_{21} \oplus A_{22}$, where $A_{2k} = A_2|\mathcal{H}_k$, $k = 1, 2$, is the restriction of A_2 on $\mathcal{H}_k$ with respect to the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ invariant with respect to A_2, then subspaces $\widetilde{\mathcal{H}}_k = U(\mathcal{H}_k)$, $k = 1, 2$, are invariant with respect to the operator A_1 and $A_1 = A_{11} \oplus A_{12}$, where $A_{1k} = A_1|\widetilde{\mathcal{H}}_k$, $k = 1, 2$, at that, $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$. Moreover, if P is a projector corresponding to the expansion $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, that is, $\mathcal{H}_1 = \operatorname{Im} P$ is the image of the projector P, $\mathcal{H}_2 = \operatorname{Im}(I - P)$ is the image of the additional projector $I - P$, then the projector $\widetilde{P} \in \operatorname{End} \mathcal{H}$ corresponding to the expansion $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ is defined by the formula

$$\widetilde{P} = UPU^{-1}.$$

3) if a is an eigenvector of the operator A_2 associated with the eigenvalue λ_0, then $b = Ua$ is an eigenvector of the operator A_1 associated with the same eigenvalue λ_0.

4) if the operator A_2 is a generator of a strongly continuous semi-group (group) of the operators $T_2 : \mathbb{J} \to \operatorname{End} \mathcal{H}$ (of class C_0), then the operator A_1 is a generator a strongly continuous semi-group (group) of operators

$$T_1(t) = UT_2(t)U^{-1}, \quad t \in \mathbb{J}, \quad T_1 : \mathbb{J} \to \operatorname{End} \mathcal{H},$$

where $\mathbb{J}$ coincides with one of the sets $\mathbb{R}_+$, $\mathbb{R}$.

Let $\mathcal{H}$ be an abstract Hilbert space represented as the direct sum of orthogonal non-zero closed subspaces $\mathcal{H}_n$, $n \in \mathbb{Z}$, that is,

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n, \tag{9}$$

where $\mathcal{H}_i$ is orthogonal to $\mathcal{H}_j$ as $i \neq j$, $i, j \in \mathbb{Z}$, and $x = \sum_{n \in \mathbb{Z}} x_n$, $x_n \in \mathcal{H}_n$, $\|x\|^2 = \sum_{n \in \mathbb{Z}} \|x_n\|^2$. Such representation of the space $\mathcal{H}$ gives rise the partition of the unity of the system of ortho-projectors $\{\mathcal{P}_n, n \in \mathbb{Z}\}$. The projectors $\mathcal{P}_n$, $n \in \mathbb{Z}$, possess the following properties:

1) $\mathcal{P}_n^* = \mathcal{P}_n$, $n \in \mathbb{Z}$;

2) $\mathcal{P}_i \mathcal{P}_j = 0$ as $i \neq j$, $i, j \in \mathbb{Z}$;

3) the series $\sum_{n \in \mathbb{Z}} \mathcal{P}_n x$ unconditionally converges to $x \in \mathcal{H}$ and $\|x\|^2 = \sum_{n \in \mathbb{Z}} \|\mathcal{P}_n x\|^2$;

3') the identities $\mathcal{P}_k x = 0$, $k \in \mathbb{Z}$ imply that the vector x is zero;

4) $\mathcal{H}_k = \operatorname{Im} \mathcal{P}_k$, $x_k = \mathcal{P}_k x$, $k \in \mathbb{Z}$.

We observe that the properties 3) and 3') are equivalent.

Definition 8. *A linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is called an orthogonal direct sum of bounded operators $\mathcal{A}_n \in \operatorname{End} \mathcal{H}_n$, $n \in \mathbb{Z}$, with respect to resolution (9) and it is written as*

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n, \tag{10}$$

if

1) $\mathcal{H}_n \subset D(\mathcal{A}) = \{x \in \mathcal{H} : \sum_{k \in \mathbb{Z}} \|\mathcal{A}_k x_k\|^2 < \infty, x_k = \mathcal{P}_k x, k \in \mathbb{Z}\}$ for all $n \in \mathbb{Z}$;

2) each subspace $\mathcal{H}_n$, $n \in \mathbb{Z}$, is invariant with respect to the operator $\mathcal{A}$ and $\mathcal{A}_n$, $n \in \mathbb{Z}$, is the restriction of the operator $\mathcal{A}$ on $\mathcal{H}_n$, $n \in \mathbb{Z}$.

3) $\mathcal{A}x = \sum_{k \in \mathbb{Z}} \mathcal{A}_k x_k$, $x \in D(\mathcal{A})$, where $x_k = \mathcal{P}_k x$, $k \in \mathbb{Z}$.

We observe the inclusion of the spectra: $\sigma(\mathcal{A}_k) \subset \sigma(\mathcal{A})$, $k \in \mathbb{Z}$.

Without additional restrictions, the spectrum $\sigma(\mathcal{A})$ does not necessarily coincide with the union of the spectra $\sigma(\mathcal{A}_k)$, $k \in \mathbb{Z}$, and even with the closure of the union.

Definition 9. *The expansion of the Hilbert space $\mathcal{H}$ of form*

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} U\mathcal{H}_k, \tag{11}$$

where U is the invertible operator in $\mathrm{End}\,\mathcal{H}$ and $\mathcal{H}$ is the orthogonal direct sum of subspaces $\mathcal{H}_k$, $k \in \mathbb{Z}$, of form (9) is called quasi-orthogonal or U-orthogonal. If the operator U is represented as $U = I + W$, where $W \in \mathfrak{S}_2(\mathcal{H})$, then the quasi-orthogonal expansion of the space $\mathcal{H}$ is called Riesz expansion.

We observe that U-orthogonal expansion (11) of the space $\mathcal{H}$ is called orthogonal if in $\mathcal{H}$ we introduce an equivalent scalar product of form

$$(x, y)_* = (Ux, Uy), \quad x, y \in \mathcal{H}.$$

Definition 10. *We shall say that a linear closed operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a quasi-orthogonal (U-orthogonal) direct sum of bounded operators $\widetilde{A}_k$, $k \in \mathbb{Z}$, with respect to the quasi-orthogonal expansion of space $\mathcal{H}$ of form (11) if $\widetilde{A}_k = UA_kU^{-1}$, $k \in \mathbb{Z}$, for some invertible operator $U \in \mathrm{End}\,\mathcal{H}$. This is written as*

$$A = \bigoplus_{k \in \mathbb{Z}} \widetilde{A}_k.$$

We return back to the operator L defined by formulae (5), (6). We represent it as $L = L_0 - V$, where $L_0 : D(L_0) = D(L) \subset L_2 \to L_2$ is the operator of differentiating $L_0 = \frac{d}{ds}$ and

$$(Vy)(s) = v(s)y(\omega - s). \tag{12}$$

The operator V is well-defined thanks to the inclusion $D(L_0) \subset D(V)$. In what follows, the operator L_0 serves as an unperturbed operator, while the operator V is regarded as a perturbation.

Let us describe the spectral property of the unperturbed operator L_0. The spectrum $\sigma(L_0)$ of the operator L_0 is represented as

$$\sigma(L_0) = \cup_{k \in \mathbb{Z}} \sigma_k,$$

where $\sigma_k = \{\lambda_k\}$, $\lambda_k = \frac{i2\pi k}{\omega}$, $k \in \mathbb{Z}$, are the isolated eigenvalues. The corresponding eigenvectors are the functions $e_k(s) = e^{i\frac{2\pi k}{\omega}s}$, $s \in [0, \omega]$, $k \in \mathbb{Z}$, forming an orthonormalized basis in the spaces L_2 with respect to the introduced scalar product. The corresponding spectral projector $\mathcal{P}_n = P(\sigma_n, L_0)$, $n \in \mathbb{Z}$, (Riesz projector) is defined by the formula

$$(\mathcal{P}_n x)(s) = \frac{1}{\omega}\left(\int_0^\omega x(\tau)e^{-i\frac{2\pi k}{\omega}\tau}\,d\tau\right)e^{i\frac{2\pi k}{\omega}s} = \widehat{x}(n)e^{i\frac{2\pi n}{\omega}s}, \ x \in L_2, \ s \in [0, \omega]. \tag{13}$$

Here $\widehat{x}(n)$, $n \in \mathbb{Z}$, is the Fourier coefficient of a periodic function $x \in L_2$ defined by the formula

$$\widehat{x}(n) = \frac{1}{\omega}\int_0^\omega x(\tau)e^{-i\frac{2\pi n}{\omega}\tau}\,d\tau.$$

We observe that an unperturbed operator $L_0 = \frac{d}{ds}$ is an orthogonal sum of the operators $(L_0)_k = L_0|\mathcal{H}_k = \frac{i2\pi k}{\omega}I_k$, where I_k stands for the identity operator in the one-dimensional space $\mathcal{H}_k = \mathrm{Im}\,\mathcal{P}_k$, that is, $L_0 = \oplus_{k \in \mathbb{Z}}(L_0)_k$. At that, all operators $(L_0)_k$, $k \in \mathbb{Z}$, have rank one 1 and are represented as $(L_0)_k = \lambda_k I_k$, $k \in \mathbb{Z}$.

Let $\mathcal{P}_{(k)} = \sum\limits_{|i| \leqslant k} \mathcal{P}_i$, $k \geqslant 0$. Then the operator L_0 is also an orthogonal direct sum of the operators

$$L_0 = (L_0)_{(k)} \oplus \left(\bigoplus_{|j|>k} (L_0)_j \right) = (L_0)_{(k)} \oplus \left(\bigoplus_{|j|>k} i\frac{2\pi j}{\omega} I_j \right), \quad k \in \mathbb{Z}_+, \quad j \in \mathbb{Z},$$

where $(L_0)_{(k)}$ is the restriction of the operator $L_0 = \frac{d}{ds}$ on the subspace $\mathcal{H}_{(k)} = \operatorname{Im} \mathcal{P}_{(k)}$ with respect of the representation of the space L_2 as

$$L_2 = \mathcal{H}_{(k)} \oplus \left(\bigoplus_{|j|>k} \mathcal{H}_j \right). \tag{14}$$

We note that $(L_0)_{(k)} = \bigoplus\limits_{|j| \leqslant k} \frac{i2\pi j}{\omega} I_j$ with respect to the orthogonal expansion $\mathcal{H}_{(k)} = \bigoplus\limits_{|j| \leqslant k} \mathcal{H}_j$.

The next theorem is a base of all results and estimates in the paper.

Theorem 3. *There exists a number $k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ such that the operator $L = L_0 - V$ is similar to the operator $L_0 - V_0$, where the operator V_0 belongs to the space $\mathfrak{S}_2(\mathcal{H})$, the identity*

$$LU = U(L_0 - V_0)$$

holds true and the subspaces $\mathcal{H}_{(k)} = \operatorname{Im} \mathcal{P}_{(k)}$ and $\mathcal{H}_j = \operatorname{Im} \mathcal{P}_j$, $|j| > k$, are invariant with respect to the operators V_0, L_0. Moreover, the operator L is an U-orthogonal direct sum of form

$$L = U(L_0 - \left(V_{0(k)} \oplus \bigoplus_{|j|>k} V_{0j} \right))U^{-1}$$

with respect to the Riesz expansion (U-orthogonal expansion) of the space $\mathcal{H} = U\mathcal{H}_{(k)} \oplus \bigoplus\limits_{|j|>k} U\mathcal{H}_j$. The invertible operator of transformation U in $\operatorname{End} L_2$ is represented as $U = I + W$, where $W \in \mathfrak{S}_2(\mathcal{H})$.

A specific form of the operators U and V_0 will be written out in Sections 3, 4, 5.

Theorem 4. *The spectrum of the operator L can be represented as*

$$\sigma(L) = \widetilde{\sigma}_{(k)} \cup \left(\bigcup_{|j|>k} \widetilde{\sigma}_j \right),$$

where the set $\widetilde{\sigma}_{(k)}$ contains at most $2k+1$ eigenvalues, the sets $\widetilde{\sigma}_j$, $|j| > k$, consist of a single point and $\widetilde{\sigma}_j = \{\widetilde{\lambda}_j\}$, $\widetilde{\lambda}_j = i\frac{2\pi j}{\omega} - b_j$, and $\sum\limits_{|j|>k} |b_j|^2 < \infty$.

The associated eigenvectors $\widetilde{e}_n$ form a Bari basis in the space L_2 (in particular, Riesz basis) and the estimate holds:

$$\|\widetilde{e}_n - e_n\| = b_{1n}, \quad |n| > m,$$

where $\sum\limits_{|n|>m} b_{1n}^2 < \infty$.

In work [5], there were given more specified formulae for the eigenvalues and eigenvectors of the operator L in the case of a smooth on the segment $[0,1]$ potential v such that $v(0) = v(1)$.

In paper [17] there were obtained the estimates for the weighted means of the eigenvalues in the sets σ_n, $|n| > m$, and the estimates for the deviations $\operatorname{dist}(i\frac{2\pi j}{\omega}, \widetilde{\sigma}_j)$.

We note that the properties of differential operators with involution discussed in the present work differs significantly from the spectral properties of differential operators [31], [32].

The next theorem is formulated under the assumptions of Theorems 3 and 4 in terms of notations introduced in Theorem 4.

Let $\widetilde{\mathcal{P}}_{(k)} = P(\widetilde{\sigma}_{(k)}, L)$, $k \geqslant 0$, $\widetilde{\mathcal{P}}_n = P(\widetilde{\lambda}_n, L)$, $|n| > m$, be the spectral projectors constructed by the spectral sets $\widetilde{\sigma}_{(k)}$ and $\widetilde{\sigma}_j$, $|j| > k$, respectively.

Theorem 5. *The limiting relation*

$$\lim_{n \to \infty} \|\widetilde{\mathcal{P}}_{(k)} + \sum_{|j|=k+1}^{n} \widetilde{\mathcal{P}}_j - \mathcal{P}_{(k)} - \sum_{|j|=k+1}^{n} \mathcal{P}_j\|_2 = 0$$

holds true.

The next theorem strengthens essentially the result of Theorem 1.

Theorem 6. *The differential operator L is a generator of a strongly continuous group of operators*

$$T : \mathbb{R} \to \operatorname{End} L_2.$$

The group $T : \mathbb{R} \to \operatorname{End} L_2$ is similar to the group $\widetilde{T} : \mathbb{R} \to \operatorname{End} L_2$, which admits an orthogonal expansion of form

$$\widetilde{T}(t) = \bigoplus_{j=-\infty}^{-k-1} e^{(\frac{i2\pi j}{\omega} - b_j)I_j t} \oplus e^{B_{(k)}t} \bigoplus_{j=k+1}^{\infty} e^{(\frac{i2\pi j}{\omega} - b_j)I_j t}, \quad t \in \mathbb{R}, \tag{15}$$

with respect to the expansion of the space L_2 of form (14) for some integer $k \geqslant 0$, and $T(t) = U\widetilde{T}(t)U^{-1}$, $t \in \mathbb{R}$. Moreover, the operator of transformation $U \in \operatorname{End} L_2$ of the group $\widetilde{T}$ into the group T can be represented as $U = I + W$, where $W \in \mathfrak{S}_2(\mathcal{H})$. The operator $B_{(k)}$ belongs to $\operatorname{End} \mathcal{H}_{(k)}$ and $\sum_{|j|>k} |b_j|^2 < \infty$.

Thus, the group of operators $T(t)$, $t \in \mathbb{R}$, admits an U-orthogonal expansion with respect to an U-orthogonal expansion of the space L_2 of form (11). The number k in Theorem 6 will be found later in Section 4, see Theorem 16.

Let $\psi = U^{-1}\varphi$ and the function ψ is expanded into the Fourier series

$$\psi(s) = \sum_{n \in \mathbb{Z}} \widehat{\psi}(n)e^{i\frac{2\pi n}{\omega}s}, \qquad s \in [0, \omega].$$

It is obvious that $\mathcal{P}_j\psi = \widehat{\psi}(j)e_j$, $j \in \mathbb{Z}$.

Definition 11. *The representation of the group of operators (mild solutions) of form*

$$T(t)\varphi = U\widetilde{T}(t)U^{-1}\varphi = \sum_{j=-\infty}^{-k-1} e^{(\frac{i2\pi j}{\omega} - b_j)t}U\mathcal{P}_j\psi + Ue^{B_{(k)}t}\mathcal{P}_{(k)}\psi + \sum_{j=k+1}^{\infty} e^{(\frac{i2\pi j}{\omega} - b_j)t}U\mathcal{P}_j\psi \tag{16}$$

is called the Fourier series of the mild solution $u(t, s) = (T(t)\varphi)(s)$, $t \in \mathbb{R}$, $s \in [0, \omega]$, $\varphi \in L_2$, to problem (1).

Representation (15) of Theorem 6 implies immediately the next theorem.

Theorem 7. *The group $\widetilde{T}$ can be represented as*

$$\widetilde{T}(t) = e^{B_{(k)}t}\mathcal{P}_{(k)} + \sum_{|j| \geqslant k+1} e^{(i\frac{2\pi j}{\omega} - b_j)t}\mathcal{P}_j.$$

We consider sequences of the numbers $\beta_l = \|\mathcal{P}_l W\|_2$, $\alpha_l = \sup_{|j| \geqslant l} |b_j|$, $l \in \mathbb{Z}$. They possess the properties $\lim_{l \to \infty} a_l = 0$ and (β_l) is a square integrable sequence.

The representation of solution as (16) implies the following theorem.

Theorem 8. *For each function $\varphi \in L_2$ the estimates hold:*

$$\|T(t)\varphi - Ue^{B_{(k)}t}\mathcal{P}_{(k)}\psi - \sum_{k \leqslant |j| \leqslant N} e^{(\frac{i2\pi j}{\omega} - b_j)t}U\mathcal{P}_j\psi(j)\|_2$$

$$\leqslant C\left(\sum_{|j| \geqslant N+1} e^{2b_j|t|}(|\widehat{\psi}(j)|^2 + \beta_j^2\|\psi\|^2)\right)^{\frac{1}{2}} \leqslant Ce^{\alpha_{N+1}|t|}\left(\sum_{|j| \geqslant N+1} (|\widehat{\psi}(j)|^2 + \beta_j^2\|\psi\|^2)\right)^{\frac{1}{2}}, \quad t \in \mathbb{R},$$

for $N > k$ and some constant $C > 0$.

The presence of the group of operators T allows us to well define the operator $\frac{d}{dt} - L$ in a series of functional spaces. B the symbol $C_b = C_b(\mathcal{J}, L_2)$ we denote a Banach space of continuous bounded on $\mathcal{J}$ functions with values in L_2 and the norm $\|x\|_\infty = \sup_{t \in \mathcal{J}} \|x(t)\|_2$, $x \in C_b$. It follows from Theorem 4 that $\sigma(L) \cap i\mathbb{R}$ is at most countable set having no finite accumulation points. This is [33] implies immediately the next theorem.

Theorem 9. *Each mild bounded solution $\widetilde{u} \in C_b$ of problem (1) is a Bohr almost periodic function (see [34]).*

Hereinafter till the end of the section we assume that $\mathcal{J}$ coincides either with $\mathbb{R}_+$ or with $\mathbb{R}$.

Definition 12 ([35], [36]). *A function $x_0 \in C_b$ is called slowly varying at infinity if $\lim_{|s| \to \infty} \|x_0(t+s) - x_0(s)\|_2 = 0$.*

Definition 13 ([35], [36]). *A uniformly continuous function $x \in C_b$ is called almost periodic at infinity if for each $\varepsilon > 0$ there exists a generalized trigonometric polynomial of form $\varphi_n(t) = \sum_{k=1}^{n} x_k(t)e^{i\lambda_k t}$, $t \in \mathcal{J}$, where x_k, $1 \leqslant k \leqslant n$, are slowly varying at infinity functions and λ_k, $1 \leqslant k \leqslant n$, are real numbers such that*

$$\sup_{t \in \mathcal{J}} \|x(t) - \sum_{k=1}^{n} x_k(t)e^{i\lambda_k t}\| < \varepsilon.$$

By $C_0 = C_0(\mathcal{J}, L_2)$ we denote the space of functions vanishing at infinity, that is, $\lim_{|t| \to \infty} \|x(t)\|_2 = 0$, $x \in C_0$. Theorem 6.3 in [35] implies immediately the following theorem.

Theorem 10. *Let $\widetilde{u} \in C_0$ be a mild solution to inhomogeneous problem (2) and $\widetilde{f} \in C_0$. Then $\widetilde{u} : \mathcal{J} \to L_2$ is almost periodic at infinity.*

2. Method of similar operators

We observe that the method of similar operator is usually employed in the spectral analysis of various classes of differential (see [37] – [40]) and difference (see [41], [42]) operators. While presenting the method, we shall base on works [21], [22].

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a linear closed operator with a dense domain acting in a complex (abstract) Hilbert space $\mathcal{H}$. We introduce a linear space $\mathfrak{L}_A(\mathcal{H})$ of operators acting in $\mathcal{H}$ and relatively bounded with respect to the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$. We shall say that an operator $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ is relatively bounded with respect to the operator A if $D(B) \supset D(A)$ and the quantity $\|B\|_A = \inf\{C > 0 : \|Bx\| \leqslant C(\|x\| + \|Ax\|), x \in D(A)\}$ is finite. This quantity is chosen to be the norm of the operator $B \in \mathfrak{L}_A(\mathcal{H})$ in $\mathfrak{L}_A(\mathcal{H})$. Thus, $\mathfrak{L}_A(\mathcal{H})$ is a Banach space.

We consider a perturbed abstract operator $A - B : D(A) \subset \mathcal{H} \to \mathcal{H}$, where $B \in \mathfrak{L}_A(\mathcal{H})$, and the spectral properties of a linear closed operator A are known.

We define a transformer (that is, a linear operator in spaces of operators; a terminology due to M.G. Krein) $\mathrm{ad}_A : D(\mathrm{ad}_A) \subset \mathrm{End}\,\mathcal{H} \to \mathrm{End}\,\mathcal{H}$ by the formula

$$\mathrm{ad}_A X = AX - XA, \quad X \in D(\mathrm{ad}_A),$$

with domain $D(\mathrm{ad}_A)$ formed by operators $X \in \mathrm{End}\,\mathcal{H}$ possessing the properties:

1) $XD(A) \subset D(A)$;

2) the operator $\mathrm{ad}_A X : D(A) \to \mathcal{H}$ admits a bounded extension Y on $\mathcal{H}$ and we let $\mathrm{ad}_A X = Y$ (such extension is unique).

The most important notion of the method of similar operator is the notion of an admissible triple.

Definition 14 ([21], [22])**.** *Let $\mathcal{M}$ be a linear subspace in $\mathfrak{L}_A(\mathcal{H})$ and $J : \mathcal{M} \to \mathcal{M}$, $\Gamma : \mathcal{M} \to \mathrm{End}\,\mathcal{H}$ be transformers. A triple $(\mathcal{M}, J, \Gamma)$ is called an admissible triple for the operator A, and $\mathcal{M}$ is called a space of admissible perturbations if*

1) $\mathcal{M}$ is a Banach space with its norm $\|\cdot\|_*$ continuously embedded into $\mathfrak{L}_A(\mathcal{H})$, that is, there exists a constant $C > 0$ such that $\|X\|_A \leqslant C\|X\|_*$ for each $X \in \mathcal{M}$;

2) J and Γ are bounded linear operator and J is a projector;

3) $(\Gamma X)D(A) \subset D(A)$ and the identities

$$(\mathrm{ad}_A \Gamma X)x = (X - JX)x, \quad x \in D(A), \quad X \in \mathcal{M},$$

hold and $\Gamma X \in \mathrm{End}\,\mathcal{H}$ is the unique solution to the equation

$$\mathrm{ad}_A Y = AY - YA = X - JX,$$

satisfying the condition $JY = 0$;

4) $X\Gamma Y$, $(\Gamma X)Y \in \mathcal{M}$ for all $X, Y \in \mathcal{M}$ and there exists a constant $\gamma > 0$ such that

$$\|\Gamma\| \leqslant \gamma, \quad \max\{\|X\Gamma Y\|_*, \|(\Gamma X)Y\|_*\} \leqslant \gamma\|X\|_*\|Y\|_*;$$

5) $J((\Gamma X)JY) = 0$ for all $X, Y \in \mathcal{M}$;

6) for all $X \in \mathcal{M}$ and $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A)$ such that $\|X(A - \lambda_\varepsilon I)^{-1}\|_\infty < \varepsilon$.

Let is fox an admissible triple $(\mathcal{M}, J, \Gamma)$ for an unperturbed operator A.

Theorem 11 ([21], [22])**.** *Let $(\mathcal{M}, J, \Gamma)$ be an admissible triple for the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and B be some operator in $\mathcal{M}$. If*

$$4\|J\|\|B\|_*\gamma < 1, \tag{17}$$

the operator $A - B$ is similar to the operator $A - JX_*$, where $X_* \in \mathcal{M}$ solves the nonlinear equation

$$X = B\Gamma X - (\Gamma X)(JB) - (\Gamma X)J(B\Gamma X) + B = \Phi(X). \tag{18}$$

A solution X_* can be found by the method of simple iterations letting $X_0 = 0$, $X_1 = B$, $\dots$. The similarity transformation of the operator $A - B$ into the operator $A - JX_*$ is made by the invertible operator $I + \Gamma X_* \in \mathrm{End}\,\mathcal{H}$. The mapping $\Phi : \mathcal{M} \to \mathcal{M}$ is contracting in the ball $\{X \in \mathcal{M} : \|X - B\|_* \leqslant 3\|B\|_*\}$.

Lemma 1 and Theorem 11 imply the following theorem; in its formulation we employ the notations of Theorem 11.

Theorem 12 ([22])**.** *Let $(\mathcal{M}, J, \Gamma)$ be an admissible triple for the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, the operator $B \in \mathcal{M}$ satisfies condition (17) and the operator $A - JX_*$ is a generator of a strongly continuous group of operators $\widetilde{T} : \mathbb{R} \to \mathrm{End}\,\mathcal{H}$. Then the operator $A - B$ is a generator of a group of operators $T : \mathbb{R} \to \mathrm{End}\,\mathcal{H}$, defined by identities:*

$$T(t) = (I + \Gamma X_*)\widetilde{T}(t)(I + \Gamma X_*)^{-1}, \quad t \geqslant 0,$$

where X_* is a solution to equation (18).

Quite often it is difficult to construct the space of admissible perturbations containing a considered perturbation. In this case one can make a preliminary similarity transformation of the considered operator, whose perturbation belongs to the space of admissible perturbations $\mathcal{M}$ in some admissible triple $(\mathcal{M}, J, \Gamma)$. Such transformation is possible under the assumptions of the next proposition.

Assumption 1 ([22]). *Given an operator $C \in \mathfrak{L}_A(\mathcal{H})$, there exist operators ΓC, $JC \in \mathfrak{L}_A(\mathcal{H})$ satisfying the conditions:*
1) $\Gamma C \in \operatorname{End} \mathcal{H}$ and $\|\Gamma C\| < 1$;
2) $(\Gamma C) D(A) \subset D(A)$;
3) $C\Gamma C$, $(\Gamma C) J C \in \mathcal{M}$;
4) $A(\Gamma C) x - (\Gamma C) A x = C x - (JC) x$, $x \in D(A)$;
5) for each $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A)$ such that $\|C(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.

Theorem 13 ([22]). *Under the assumptons of Proposition 1, the operator $A - C$ is similar to the operator $A - JC - C_0$, where $C_0 = (I + \Gamma C)^{-1}(C\Gamma C - (\Gamma C) JC)$, and the identity holds:*

$$(A - C)(I + \Gamma C) = (I + \Gamma C)(A - JC - C_0).$$

Let an unperturbed operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a skew-adjoint operator and its spectrum $\sigma(A)$ admits the representation of form

$$\sigma(A) = \bigcup_{k \in \mathbb{Z}} \Delta_k,$$

where Δ_k, $k \in \mathbb{Z}$, are compact mutually disjoint sets. Let P_k, $k \in \mathbb{Z}$, be the Riesz projector constructed by the spectral set Δ_k, $k \in \mathbb{Z}$. Then the system of projectors $\{P_k, k \in \mathbb{Z}\}$ and system of subspaces $\mathcal{H}_n = \operatorname{Im} P_n$, $n \in \mathbb{Z}$, satisfy the properties provided after Lemma 1.

Let us fix an admissible triple $(\mathcal{M}, J, \Gamma)$ for the operator A, in which the transformer $J : \mathcal{M} \to \mathcal{M}$ is well-defined by means of the formula

$$JX = \sum_{k \in \mathbb{Z}} P_k X P_k, \quad X \in \mathcal{M}.$$

Thus, each of the operators JX, $X \in \mathcal{M}$, is an orthogonal direct sum of the operators $X_k = P_k X | \mathcal{H}_k$, $X_k \in \operatorname{End} \mathcal{H}_k$, $k \in \mathbb{Z}$. Moreover, and this is very important, the operator $A - JX$ possesses the same property and

$$A - JX = \bigoplus_{k \in \mathbb{Z}} (A_k - X_k), \tag{19}$$

where $A_k = A | \mathcal{H}_k$ is the restriction of the operator A on $\mathcal{H}_k$.

Theorem 14. *Under the assumptions of Theorem 12, the operator $A - B$, where $B \in \mathcal{M}$, is similar to the operator $A - JX_*$, $X_* \in \mathcal{M}$, being an orthogonal direct sum of the operators (19), where $X = X_*$, with respect to the orthogonal expansion of the space $\mathcal{H}$ of form (9), where $\mathcal{H}_i = \operatorname{Im} P_i$, $i \in \mathbb{Z}$. The operator $A - B$ is an quasi-orthogonal direct sum of bounded operators with respect to the quasi-orthogonal expansion of the space H of form (9), where U is an operator of transformation of the operator $A - B$ into the operator $A - JX_*$.*

We denote the operator $A - JX_*$ by A_0. For operators A_0 of form (19) the following theorem holds.

Lemma 2. *An operator A_0 of form (19) is a generator of some strongly continuous group $T_0 : \mathbb{R} \to \operatorname{End} \mathcal{H}$, if and only if the condition*

$$\sup_{|t| \leqslant b} \sup_{n \in \mathbb{Z}} \|e^{A_{0,n} t}\|_{\operatorname{End} \mathcal{H}_n} = C(b) < \infty, \tag{20}$$

holds, where $b \geqslant 1$. If condition (20) holds, the operator $T_0(t)$, $t \in \mathbb{R}$, are represented as an orthogonal direct sum

$$T_0(t) = \bigoplus_{n \in \mathbb{Z}} e^{A_{0,n}t}, \quad t \in \mathbb{R},$$

with respect to the expansion of the space $\mathcal{H}$ of form (9).

Proof. If condition (20) holds, the formula

$$T_0(t)x = \sum_{n \in \mathbb{Z}} e^{A_{0,n}t} P_n x, \quad t \in \mathbb{R},$$

defines a bounded operator that is due to the Parseval identity and the estimate

$$\|T_0(t)x\|^2 = \sum_{n \in \mathbb{Z}} \|e^{A_{0,n}t} P_n x\|^2 \leqslant C^2(b) \sum_{n \in \mathbb{Z}} \|P_n x\|^2 = C^2(b)\|x\|^2, \quad x \in \mathcal{H}, \quad |t| \leqslant b.$$

It is straightforward to check that the operators $T_0(t) \in \operatorname{End}\mathcal{H}$, $t \in \mathbb{R}$, form a group of operators. Since this group is strongly continuous on a dense subset of vectors represented as $x = \sum_{|n| \leqslant m} P_n x$, $m \in \mathbb{Z}_+$, it is strongly continuous on the entire space $\mathcal{H}$.

The inverse statement is obvious. The proof is complete. $\qquad\qquad\square$

Thus, the construction of the group of operators for the initial operator is reduced to constructing a group of operators for the operator being an orthogonal direct sum of operators with respect to expansion (19) by using Lemma 2.

3. First transformation of similarity

In what follows as the space of admissible perturbations $\mathcal{M}$ in the method of similar operators, the space $\mathfrak{S}_2(L_2)$ will serve, while as a system of projectos $\mathcal{P} = (\mathcal{P}_n)$, $n \in \mathbb{Z}$, we choose a system of spectral projectors of an unperturbed operator L_0 (see formula (13)).

Hereinafter the symbol $\mathcal{H}$ stands for a Hilbert space L_2.

In what follows we employ the matrices of the operators of two forms: operator and scalar ones. To each operator $X \in \operatorname{End}\mathcal{H}$ we associate an operator matrix $X \sim (X_{ij})$, where $X_{ij} = \mathcal{P}_i X \mathcal{P}_j \in \operatorname{End}\mathcal{H}$, $i, j \in \mathbb{Z}$, and the projectors $\mathcal{P}_i$, $i \in \mathbb{Z}$, are defined by formula (13).

The scalar matrix $X \sim (x_{ij})$ consists of the entries x_{ij}, $i, j \in \mathbb{Z}$, defined by the formula $x_{ij} = (X e_j, e_i)$, and as a basis e_i, $i \in \mathbb{Z}$, we choose normalized eigenvectors of the unperturbed operator L_0.

In what follows we employ the following properties of the ideal $\mathfrak{S}_2(\mathcal{H})$ (see [1]):

1) The operator $X \in \operatorname{End}\mathcal{H}$ is a Hilbert-Schmidt operator if and only if its matrix $x_{kn} = (X g_n, g_k)$, $n, k \in \mathbb{Z}$, is a Hilbert-Schmidt matrix for some orthonormalized basis $\{g_k, k \in \mathbb{Z}\}$ and $\|X\|_2^2 = \sum_{k,n \in \mathbb{Z}} |x_{kn}|^2$.

2) The product XY of the operators $X, Y \in \mathfrak{S}_2(\mathcal{H})$ is a nuclear operator and $\|XY\|_1 \leqslant \|X\|_2 \|Y\|_2$.

3) Let $\{Q_n, n \geqslant 0\}$ be the system of orthoprojectors in $\operatorname{End}\mathcal{H}$ forming the partition of the unity. Then $\|X\|_2^2 = \sum_{n,m=0}^{\infty} \|Q_n X Q_m\|_2^2$.

4) Let the operator $X : D(X) \subset \mathcal{H} \to \mathcal{H}$ belongs the space $\mathfrak{L}_A(\mathcal{H})$ and hence, it has a domain $D(X)$ dense in $\mathcal{H}$. If the quantity $\sum_{n,k \in \mathbb{Z}} |(X e_n, e_k)|^2$ is finite for some orthonormalized basis $\{e_k, k \in \mathbb{Z}\}$ with the property $e_k \in D(X)$, $k \in \mathbb{Z}$, then the operator X has the unique extension on $\mathcal{H}$. This extension is a Hilbert-Schmidt operator and it is denoted by the same symbol X.

The notion of an operator matrix is naturally extended to the operators relatively bounded with respect to the operator L_0, not necessarily bounded.

To each operator $X : D(A) \subset L_2 \to L_2$, $X \in \mathfrak{L}_{L_0}(\mathcal{H})$, we associate an operator matrix (X_{ij}), $i, j \in \mathbb{Z}$, formed by bounded operator $X_{ij} = \mathcal{P}_i X \mathcal{P}_j \in \operatorname{End} \mathcal{H}$, $i, j \in \mathbb{Z}$.

We also note that in the considered case ($\dim \operatorname{Im} P_j = 1$ for all $j \in \mathbb{Z}$):

$$X_{ij}x = (P_i X P_j)x = (XP_j x, e_i)e_i = (Xe_i, e_j)(x, e_j)e_i = (x_{ij})\widehat{x}(j)e_i, \quad i, j \in \mathbb{Z}.$$

The operator V defined by formula (12) is relatively bounded with respect the operator L_0 and it has the matrix (V_{kn}) with respect the partition of the unity $(\mathcal{P}_k, k \in \mathbb{Z})$, whose elements are defined by the formulae

$$(V_{ln}x)(s) = (Ve_l, e_n)\widehat{x}(n)e^{i\frac{2\pi l}{\omega}s} = \frac{1}{\omega}\int_0^\omega v(\tau)e^{-i\frac{2\pi l}{\omega}(\omega-\tau)}\,d\tau \cdot \widehat{x}(n)e^{i\frac{2\pi l}{\omega}s} = \widehat{v}(l+n)\widehat{x}(n)e^{i\frac{2\pi l}{\omega}s}.$$

Thus, the operator matrix of the operator V is of the form

$$V \sim \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \widehat{v}(-2) & \widehat{v}(-1) & \widehat{v}(0) & \widehat{v}(1) & \cdots \\
\cdots & \widehat{v}(-1) & \widehat{v}(0) & \widehat{v}(1) & \widehat{v}(2) & \cdots \\
\cdots & \widehat{v}(0) & \widehat{v}(1) & \widehat{v}(2) & \widehat{v}(3) & \cdots \\
\cdots & \widehat{v}(1) & \widehat{v}(2) & \widehat{v}(3) & \widehat{v}(4) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

In what follows by $\mathcal{P}_{(m)}$ we denote the projector $\mathcal{P}_{(m)} = \sum\limits_{|i|\leqslant m} \mathcal{P}_i$. We introduce operators $J_m V$, $\Gamma_m V$, $m \geqslant 0$, by the formulae

$$JV = J_0 V = \sum_{i \in \mathbb{Z}} \mathcal{P}_i V \mathcal{P}_i,$$

$$J_m V = \mathcal{P}_{(m)} V \mathcal{P}_{(m)} + \sum_{|i|>m} \mathcal{P}_i V \mathcal{P}_i, \quad m \geqslant 0,$$

$$\Gamma V = \Gamma_0 V = \frac{\omega}{2\pi i} \sum_{n \in \mathbb{Z}\setminus\{0\}} \widetilde{V}_n, \qquad \widetilde{V}_n = \sum_{k-j=n} \frac{\mathcal{P}_k V \mathcal{P}_j}{k-j},$$

$$\Gamma_m V = \Gamma V - \mathcal{P}_{(m)}(\Gamma V)\mathcal{P}_{(m)}.$$

The matrices of the operators JV and ΓV with respect the partition of the unity $(\mathcal{P}_k, k \in \mathbb{Z})$ are of the form

$$JV \sim \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \widehat{v}(-2) & 0 & 0 & \cdots \\
\cdots & 0 & \widehat{v}(0) & 0 & \cdots \\
\cdots & 0 & 0 & \widehat{v}(2) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

$$\Gamma V \sim \frac{\omega}{2\pi i} \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & -\widehat{v}(-1) & -\frac{1}{2}\widehat{v}(0) & -\frac{1}{3}\widehat{v}(1) & \cdots \\
\cdots & \widehat{v}(-1) & 0 & -\widehat{v}(1) & -\frac{1}{2}\widehat{v}(2) & \cdots \\
\cdots & \frac{1}{2}\widehat{v}(0) & \widehat{v}(1) & 0 & -\widehat{v}(3) & \cdots \\
\cdots & \frac{1}{3}\widehat{v}(1) & \frac{1}{2}\widehat{v}(2) & \widehat{v}(3) & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.$$

The definition of the operators $J_m V$ and $\Gamma_m V$, $m \geqslant 0$, imply immediately that they belong to $\mathfrak{S}_2(\mathcal{H})$. They are well-defined, that is, the series defining these operators converge in $\mathfrak{S}_2(\mathcal{H})$.

For the preliminary similarity transformation we shall also the operators $Z_m = V\Gamma_m V$, $m \geqslant 0$. The entries of the operator matrix of the operator $Z = V\Gamma V$ are defined by the formula

$$Z_{ij} = \frac{\omega}{2\pi i} \sum_{k \neq j} \frac{\widehat{v}(i+k)\widehat{v}(k+j)}{j-k}, \quad i,j \in \mathbb{Z}.$$

Lemma 3. *The operators $J_m V$, $\Gamma_m V$ and $V\Gamma_m V$, $m \geqslant 0$, are the operators in $\mathfrak{S}_2(\mathcal{H})$.*

Proof. It remains to show that $Z \in \mathfrak{S}_2(\mathcal{H})$. This is implied by an integral representation of the operator $Z = V\Gamma V$ of form

$$(Zy)(s) = \frac{1}{2\omega} \int_0^{2\omega} f\left(\frac{s-\tau}{2}\right)\widehat{v}\left(\frac{2\omega - s - \tau}{2}\right) v(s) y(\tau)\, d\tau \tag{21}$$

given in [17], [21]. Here the function f is defined as $f(s) = i\left(s - \frac{\omega}{2}\right)$. □

Let $L_\infty([0,\omega], \mathbb{C})$ be a Banach space of (classes of) functions with values in $\mathbb{C}$ and essentially bounded on $[0,\omega]$; the norm is $\|x\|_\infty = \operatorname{vrai} \sup\limits_{s \in [0,\omega]} \|x(s)\|$.

Lemma 4. *The operators $\Gamma_m V$, $m \in \mathbb{Z}_+$, possess the properties:*
1) $(\Gamma_m V)D(L_0) \subset D(L_0)$;
2) $L_0(\Gamma_m V)x - (\Gamma_m V)L_0 x = (V - J_m V)x$, $x \in D(L_0)$;
3) For each $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(L_0)$ such that $\|V(L_0 - \lambda_\varepsilon I)^{-1}\|_2 < \varepsilon$.

Proof. Let $\lambda_\varepsilon \in \rho(L_0)$. We consider a sequence o projectors $Q_{(n)} = \sum\limits_{|j| \leqslant n} P_j$, $n \geqslant 0$. For each vector $y \in \mathcal{H}$ the identities hold:

$$Q_{(n)} L_0 (\Gamma_m V)(L_0 - \lambda_\varepsilon I)^{-1} y = Q_{(n)}(\Gamma_m V)L_0(L_0 - \lambda_\varepsilon I)^{-1} y +$$
$$+ Q_{(n)}(V - J_m V)(L_0 - \lambda_\varepsilon I)^{-1} y = Q_{(n)}\mathcal{L}y,$$

where the operator $\mathcal{L}$ can be represented as

$$\mathcal{L}y = (\Gamma_m V)L_0(L_0 - \lambda_\varepsilon I)^{-1} y + (V - J_m V)(L_0 - \lambda_\varepsilon I)^{-1} y.$$

The mentioned identities can be checked on the basis vectors. Since the operator $\mathcal{L}$ is an operator in $\operatorname{End}\mathcal{H}$, the sequence of the operators in the right hand side converges to the operator $\mathcal{L}$ in the norm of the space $\operatorname{End}\mathcal{H}$. The closedness of the operator L_0 implies that $\Gamma_m Vx \in D(L_0)$ as $x \in D(L_0)$ and the identity

$$(\Gamma_m V)L_0(L_0 - \lambda_\varepsilon I)^{-1} = L_0(\Gamma_m V)(L_0 - \lambda_\varepsilon I)^{-1} + (V - J_m V)(L_0 - \lambda_\varepsilon I)^{-1}$$

holds. Thus, properties 1) and 2) hold true.

It remains to check property 3). Let $Y = V(L_0 - \lambda_\varepsilon I)^{-1}$, $\lambda_\varepsilon \in \rho(A)$. In this representation of the operator Y we consider it as a produce of two operators $(L_0 - \lambda_\varepsilon I)^{-1} \in \operatorname{End}\mathcal{H}$, $V_\infty : L_\infty([0,\omega], \mathbb{C}) \to L_2$, $(V_\infty x)(s) = (Vx)(s)$, $s \in [0,\omega]$, $x \in L_\infty([0,\omega], \mathbb{C})$. It is clear that $\|V\|_\infty = \|v\|_{L_2}$. It remains to prove that $\|(L_0 - \lambda_\varepsilon I)^{-1}\| < \varepsilon/\|v\|_{L_2}$ for a number λ_ε of form $\lambda_\varepsilon = k$ for sufficiently large $k \in \mathbb{N}$ and a given $\varepsilon > 0$. For each function $x \in \mathcal{H}$ the representation

$$(L_0 - kI)^{-1} x = \sum_{l \in \mathbb{Z}} \frac{\widehat{x}(l) e^{i\frac{2\pi l}{\omega}s}}{\frac{i2\pi l}{\omega} - k}$$

holds. This is why

$$\|(L_0 - kI)^{-1} x\|_{L_2} \leqslant \left(\sum_{l \in \mathbb{Z}} \frac{1}{k^2 + \left(\frac{2\pi l}{\omega}\right)^2}\right)^{\frac{1}{2}} \|x\|_2 \leqslant \frac{\varepsilon \|x\|_2}{\|V\|_{L_2}}$$

for sufficiently large k.

Therefore, we have proved the property $\|(L_0 - kI)^{-1}x\|_2 \leqslant \varepsilon/\|v\|_{L_2}$. Thus,

$$\|V(L_0 - kI)^{-1}\| \leqslant \|v\|_{L_2}\|(L_0 - kI)^{-1}\| < \varepsilon.$$

The proof is complete. $\qquad\square$

In the proof the next theorem, being a main result of this section, we shall make use of the following lemma.

Lemma 5. *The identity holds:* $\lim\limits_{m\to\infty} \|\Gamma_m V\|_2 = \lim\limits_{m\to\infty} \|\Gamma V - P_{(m)}(\Gamma V)P_{(m)}\| = 0.$

Proof. Since the operator ΓV is the Hilbert-Schmidt operator, the statement of the lemma follows immediately the definition of the orthoprojectors $P_{(m)}$, $m \geqslant 0$. $\qquad\square$

The obtained statements of the sequence of the operators $J_m V$, $\Gamma_m V$, $m \geqslant 0$, and Theorem 13 implies the following theorem.

Theorem 15. *There exists a number $m \in \mathbb{Z}_+$ such that $\|\Gamma_m V\|_2 < 1$, that is, the operator $I + \Gamma_m V$ is invertible and the operator $L_0 - V$ is similar to the operator $L_0 - \widetilde{V}$, where*

$$\widetilde{V} = J_m V + (I + \Gamma_m V)^{-1}(V\Gamma_m V - (\Gamma_m V)J_m V) \in \mathfrak{S}_2(\mathcal{H}, \mathcal{P}), \qquad (22)$$

and the identity holds:

$$(L_0 - V)(I + \Gamma_m V) = (I + \Gamma_m V)(L_0 - \widetilde{V}).$$

The operator $\widetilde{V}$ can be represented as

$$\widetilde{V} = J_m V + V\Gamma_m V - (\Gamma_m V)J_m V + C = JV + V\Gamma V - (\Gamma V)JV + C_1, \qquad (23)$$

where C, $C_1 \in \mathfrak{S}_1(\mathcal{H})$.

Proof. It remains to prove formula (23). The representation

$$(I + \Gamma_m V)^{-1}(V\Gamma_m V - (\Gamma_m V)J_m V) = (I + \Gamma_m V)^{-1}(\Gamma_m V)(V\Gamma_m V$$
$$- (\Gamma_m V)J_m V) + V\Gamma_m V - (\Gamma_m V)J_m V$$

and the fact that a product of two operators in $\mathfrak{S}_2(\mathcal{H})$ is an operator in $\mathfrak{S}_1(\mathcal{H})$ yield the first part of identity (23). Since the operators $V\Gamma_m V - V\Gamma V$, $J_m V - JV$, $\Gamma_m V - \Gamma V$ are of finite rank, they belong to $\mathfrak{S}_1(\mathcal{H})$ and the second part of formula (23) holds. The proof is complete. $\qquad\square$

4. Construction of transformers J_k and Γ_k, $k \geqslant 0$, in $\mathfrak{S}_2(\mathcal{H})$

It follows from Theorem 15 that the studied differential operator with involution is similar to the operator $L_0 - \widetilde{V}$, where the operator $\widetilde{V}$ defined by formula (22) belongs to $\mathfrak{S}_2(\mathcal{H})$. Below, see Section 5, we shall apply the method of similar operator to the operator $L_0 - \widetilde{V}$, cf. Theorem 3. At that we shall employ essentially the sequences of transformers

$$J_k, \Gamma_k : \mathfrak{S}_2(\mathcal{H}) \to \mathfrak{S}_2(\mathcal{H}), \quad k \geqslant 0.$$

For each operator $X \in \mathfrak{S}_2(\mathcal{H})$ we define these transformers by the following formulae:

$$JX = JX_0 = \sum_{n \in \mathbb{Z}} \mathcal{P}_n X \mathcal{P}_n,$$

$$J_k X = P_{(k)} X P_{(k)} + \sum_{|j| > k} \mathcal{P}_j X \mathcal{P}_j, \quad k \geqslant 1, \tag{24}$$

$$\Gamma X = \Gamma_0 X = \frac{\omega}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \widetilde{X}_n, \quad \text{where} \quad \widetilde{X}_n = \frac{1}{n} \sum_{p - j = n} X_{pj}, \tag{25}$$

$$\Gamma_k X = \Gamma X - P_{(k)}(\Gamma X) P_{(k)}, \quad k \geqslant 1.$$

Here $X_{pj} = \mathcal{P}_p X \mathcal{P}_j$, $p, j \in \mathbb{Z}$, are matrix entries of the operator X.

The transformers J_k, $\Gamma_k \in \mathrm{End}\,(\mathfrak{S}_2(\mathcal{H}))$, $k \geqslant 0$, are well-defined and all written series converge in $\mathfrak{S}_2(\mathcal{H})$.

Definition 15. *The operator $X_k = \sum\limits_{\substack{i,j \in \mathbb{Z} \\ i-j=k}} X_{ij}$, $k \in \mathbb{Z}$ is called a kth diagonal of the operator matrix of the operator $X \in \mathfrak{S}_2(\mathcal{H})$.*

Remark 2. *An nth diagonal X_n of an operator $X \in \mathfrak{S}_2(\mathcal{H})$ satisfy the formulae*

$$\|X_n\|_2^2 = \sum_{\substack{i,j \in \mathbb{Z} \\ i-j=n}} \|X_{ij}\|^2,$$

at that,

$$\|X\|_2^2 = \sum_{n \in \mathbb{Z}} \|X_n\|_2^2.$$

In papers [17], [21] there was employed another approach used for determining the transformers J and Γ acting $\mathrm{End}\,\mathcal{H}$; that approach was based on the representation theory. To describe this approach, we mention that the differentiation operator $L_0 = \frac{d}{ds}$ is a generator of an ω-periodic group $S : \mathbb{R} \to \mathrm{End}\, L_{2,\omega}$, $(S(t)x)(s) = x(s+t)$, $t, s \in \mathbb{R}$, $x \in L_{2,\omega}$. This group can be represented as

$$S(t)x = \sum_{n \in \mathbb{Z}} e^{i \frac{2\pi n}{\omega} t} P_n x.$$

To each operator $X \in \mathrm{End}\,\mathcal{H}$ we associate an ω-periodic strongly continuous operator-valued function

$$t \mapsto S(t) X S(-t) : \mathbb{R} \to \mathrm{End}\,\mathcal{H}.$$

At that, there arises a isometric representation of period ω:

$$\widetilde{S} : \mathbb{R} \to \mathrm{End}\,(\mathrm{End}\,\mathcal{H}), \quad \widetilde{S}(t) = S(t) X S(-t), \quad t \in \mathbb{R}.$$

Given a function $\widetilde{S} : \mathbb{R} \to \mathrm{End}\,(\mathrm{End}\,\mathcal{H})$, we consider its Fourier series

$$\widetilde{S}(t)x \sim \sum_{n \in \mathbb{Z}} X_n x e^{i \frac{2\pi n}{\omega} t}, \quad x \in \mathcal{H}, \quad t \in \mathbb{R},$$

where the operators X_n read as

$$X_n x = \frac{1}{\omega} \int_0^\omega S(t) X S(-t) e^{-i \frac{2\pi n}{\omega} t}\, dt, \quad x \in \mathcal{H}, \quad n \in \mathbb{Z}.$$

The series $\sum\limits_{n \in \mathbb{Z}} X_n$ is called the Fourier series of the operator $X \in \mathrm{End}\,\mathcal{H}$ with respect to the group of the operators $\widetilde{S}$ and the operators X_n, $n \in \mathbb{Z}$ are called the Fourier coefficients of this

operator (see [43], [44]). It is important to note that X_n is the nth diagonal of the operator $X \in \mathfrak{S}_2(\mathcal{H})$ in the sense of definition 15.

Remark 3. *The group of isometries $\widetilde{S}$ can be considered with respect to the subspace $\mathfrak{S}_2(\mathcal{H})$ invariant for this group. In this case the Fourier coefficients X_n, $n \in \mathbb{Z}$, are also operators belonging to $\mathfrak{S}_2(\mathcal{H})$.*

We note that according works [17] and [21], the transformers J and Γ can be defined by the formulae

$$(JX)x = X_0 x = \frac{1}{\omega} \int_0^\omega S(\tau) X S(-\tau) x \, d\tau, \quad x \in \mathcal{H},$$

$$(\Gamma X)x = \frac{1}{\omega} \int_0^\omega f(\tau) S(\tau) X S(-\tau) x \, d\tau, \quad x \in \mathcal{H}, \tag{26}$$

where $f : \mathbb{R} \to \mathbb{C}$ is an ω-periodic function $f(s) = i\left(s - \frac{\omega}{2}\right)$, $s \in [0, \omega]$, with the Fourier series of form $f(s) \sim \sum_{n \neq 0} \frac{\omega}{2\pi n} e^{i \frac{2\pi n}{\omega} s}$.

It is important to observe that the definition of the transformers J and Γ by integral representation coincide with the definition on matrix entries by (24) and (25).

The definition of the transformer $\Gamma : \mathcal{M} \to \mathcal{M}$ by (26) implies the following lemma.

Lemma 6. *The operator ΓV belongs to $\mathfrak{S}_2(\mathcal{H})$ and is an integral operator of form*

$$(\Gamma V)x(s) = \frac{1}{2\omega} \int_0^{2\omega} f\left(\frac{\omega - s - \tau}{2}\right) v\left(\frac{\omega + s - \tau}{2}\right) x(\tau) \, d\tau, \quad x \in L_2.$$

At that, $(\Gamma V)x \in L_\infty([0, \omega], H)$, $x \in L_2$.

5. SECOND SIMILARITY TRANSFORMATION

In order to prove Theorem 3, we shall need another space of admissible perturbations $\mathcal{M} \subset \mathfrak{S}_2(\mathcal{H})$. To describe it, for each non-zero operator $X \in \mathfrak{S}_2(\mathcal{H})$ we introduce a two-sided sequence of real numbers of form

$$\alpha_n(X) = \|X\|_2^{-\frac{1}{2}} \max\left\{ \left(\sum_{\substack{|k| \geqslant n \\ k \in \mathbb{Z}}} \|\mathcal{P}_k X\|_2^2 \right)^{\frac{1}{4}}, \left(\sum_{\substack{|k| \geqslant n \\ k \in \mathbb{Z}}} \|X \mathcal{P}_k\|_2^2 \right)^{\frac{1}{4}} \right\}, \ n \in \mathbb{Z}. \tag{27}$$

The sequence $(\alpha_n(X))$, $n \in \mathbb{Z}$, possesses the following properties:
1) $\alpha_n(X) = \alpha_{-n}(X)$, $n \in \mathbb{Z}$;
2) $\lim\limits_{|n| \to \infty} \alpha_n(X) = 0$, $n \in \mathbb{Z}$;
3) $\alpha_n(X) \leqslant 1$ for each $n \in \mathbb{Z}$;
4) $\alpha_n(X) \geqslant \alpha_{n+1}(X)$, $n \geqslant 0$;
5) $\alpha_n(X) \neq 0$ for each $n \in \mathbb{Z}$ if $P_{(m)} X P_{(m)} \neq X$ for each $m \in \mathbb{Z}_+$;
6) the quantity

$$\sum_{n \in \mathbb{Z}} \frac{\|X \mathcal{P}_n\|_2^2 + \|\mathcal{P}_n X\|_2^2}{(\alpha_n(X))^2}$$

is finite.

As a perturbation of the operator L_0, the operator $\widetilde{V}$ in $\mathfrak{S}_2(\mathcal{H})$ defined by formula (22) in Theorem 15 will serve. Without loss of generality we can assume that $P_{(n)} \widetilde{V} P_{(n)} \neq 0$ for all $n \in \mathbb{Z}_+$. Otherwise the operator $L_0 - \widetilde{V}$ is an orthogonal sum of operators of finite rank $(L_0 - \widetilde{V})|_{\mathcal{H}_{(n)}}$ for some $n \geqslant 0$ and of the operator $L_0|_{\mathcal{H}^{(n)}}$, where $\mathcal{H}^{(n)} = I - \mathcal{H}_{(n)}$.

Thus, in what follows we consider the operator $L_0 - \widetilde{V}$.

For each operator $X \in \mathfrak{S}_2(\mathcal{H})$ we consider a self-adjoint compact operator F_X defined by the formula

$$F_X = \sum_{n \in \mathbb{Z}} \alpha_n(X) \mathcal{P}_n.$$

We observe that $F_X \in \operatorname{End} \mathcal{H}$ is a function of a self-adjoint operator L_0 of form $F_X = f_X(L_0)$, where $f_X : \sigma(L_0) \to \mathbb{R}_+$, $f_X(\lambda_n) = \alpha_n(X)$, $n \in \mathbb{Z}$, and $\|F_X\|_\infty = \max\limits_{n \in \mathbb{Z}} |\alpha_n(X)| = 1$.

To simplify the writing, we redenote the operator $F_{\widetilde{V}}$ by F.

We introduce the set of the operators $\mathcal{M}$ in $\mathfrak{S}_2(\mathcal{H})$ admitting the representation of form

$$X = X_l F, \quad X = F X_r,$$

where $X_l, X_r \in \mathfrak{S}_2(\mathcal{H})$. We let $\|X\|_{\mathcal{M}} = \max\{\|X_l\|_2, \|X_r\|_2\}$. It is obvious that $\|X\|_2 \leqslant \|X\|_{\mathcal{M}}$, $X \in \mathcal{M}$.

The property $\operatorname{Ker} F = 0$ (cf. Property 5 of the sequence $\alpha_n(X)$) allows us to conclude that the introduced set of the operators $\mathcal{M}$ is a Banach space.

It is obvious that for each operator X in $\mathfrak{S}_2(\mathcal{H})$ the representation holds:

$$X = \Big(\sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n(X)} X \mathcal{P}_n \Big) F_X = F_X \Big(\sum_{n \in \mathbb{Z}} \frac{1}{\alpha_n(X)} \mathcal{P}_n X \Big).$$

This is why the perturbation $\widetilde{V}$ is in the space $\mathfrak{S}_2(\mathcal{H})$.

Since $\mathcal{M} \subset \mathfrak{S}_2(\mathcal{H})$, the transformers J_k and Γ_k, $k \geqslant 0$, are defined also for the operators in $\mathcal{M}$. Moreover, the subspace $\mathcal{M}$ is invariant with respect to J_k and Γ_k, $k \geqslant 0$, and

$$J_k(X_l F) = (J_k X_l) F, \quad J_k(F X_r) = F(J_k X_r),$$
$$\Gamma_k(X_l F) = (\Gamma_k X_l) F, \quad \Gamma_k(F X_r) = F(\Gamma_k X_r),$$

where $X_r, X_l \in \mathfrak{S}_2(\mathcal{H})$.

To estimate the norms $\|\Gamma_k(XF)\|_2$ and $\|\Gamma_k(FX)\|_2$, $X \in \mathfrak{S}_2(\mathcal{H})$, we consider two sequences (α'_n), $n \in \mathbb{N}$, and $(\widetilde{\alpha}_n)$, $n \in \mathbb{N}$, defined by the formulae

$$\alpha'_n = \max_{\substack{|i| \geqslant n \\ |j| < n}} \frac{|\alpha_i - \alpha_j|}{|i - j|}, \quad \widetilde{\alpha}_n = \frac{\omega}{2\pi}(2\alpha_n + \alpha'_n), \quad n \in \mathbb{N}.$$

The sequences (α'_n) and $(\widetilde{\alpha}_n)$ belong to the space $c_0(N)$ of converging to zero sequences.

The next lemma is an analogue of Lemma 3 in [21].

Lemma 7. *For all $k \in \mathbb{Z}_+$ and each operator $X \in \mathfrak{S}_2(\mathcal{H})$ the estimates hold:*

$$\max\{\|\Gamma_k(XF)\|_2, \|\Gamma_k(FX)\|_2\} \leqslant \widetilde{\alpha}_{k+1}\|X\|_2.$$

Proof. Let $\mathcal{P}^{(k)} = I - \mathcal{P}_{(k)}$, $k \in \mathbb{Z}_+$, be the additional to $\mathcal{P}_{(k)}$ projector. Then

$$\|F\mathcal{P}^{(k)}\|_\infty = \|\sum_{n \in \mathbb{Z}} \alpha_n \mathcal{P}_n \mathcal{P}^{(k)}\|_\infty \leqslant \alpha_{k+1}.$$

The definition of the transformer Γ_k implies the estimate

$$\|\Gamma_k(XF)\|_2 = \|\Gamma_k(XF\mathcal{P}^{(k)}) + \Gamma_k(\mathcal{P}^{(k)}XF\mathcal{P}_{(k)})\|_2 \leqslant \frac{\omega}{2\pi}\alpha_{k+1}\|X\|_2 + \|\Gamma_k(\mathcal{P}^{(k)}XF\mathcal{P}_{(k)})\|_2.$$

We represent the operator $\Gamma_k(\mathcal{P}^{(k)}XF\mathcal{P}_{(k)})$ as

$$\Gamma_k(\mathcal{P}^{(k)}XF\mathcal{P}_{(k)}) = \Gamma_k(\mathcal{P}^{(k)}FX\mathcal{P}_{(k)}) + \Gamma_k(\mathcal{P}^{(k)}(XF - FX)\mathcal{P}_{(k)}).$$

At that, the latter operator has an operator matrix $(\mathcal{P}_i Z_{(k)} \mathcal{P}_j)$, $i, j \in \mathbb{Z}$, $k \geqslant 0$, where

$$Z_{(k)} = \Gamma_k(\mathcal{P}^{(k)}(XF - FX)\mathcal{P}_{(k)}),$$

formed by the entries of form

$$\mathcal{P}_i Z_{(k)} \mathcal{P}_j = \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \mathcal{P}_i X \mathcal{P}_j = \frac{\omega}{2\pi} \frac{\alpha_i - \alpha_j}{i - j} \mathcal{P}_i X \mathcal{P}_j,$$

where $|i| \geqslant k+1$, $|j| \leqslant k$, and $\mathcal{P}_i Z_{(k)} \mathcal{P}_j = 0$ otherwise. Thus,

$$\|\Gamma_k(\mathcal{P}^{(k)} X F \mathcal{P}_{(k)})\|_2 \leqslant \frac{\omega}{2\pi} \alpha_{k+1} \|X\|_2 + \frac{\omega}{2\pi} \max_{\substack{|i| \geqslant k+1 \\ |j| \leqslant k}} \frac{|\alpha_i - \alpha_j|}{|i - j|} \|X\|_2$$

$$= \frac{\omega}{2\pi} (\alpha_{k+1} + \alpha'_{k+1}) \|X\|_2 = \widetilde{\alpha}_{k+1} \|X\|_2.$$

In the same way the norm $\Gamma_k(FX)$, $X \in \mathfrak{S}_2(\mathcal{H})$, is estimated. The proof is complete. $\quad\square$

Lemma 8. *A triple $(\mathcal{M}, J_k, \Gamma_k)$ is admissible for the unperturbed operator L_0 for each $k \geqslant 0$ and the constant $\gamma = \gamma_k$ in Definition 14 obeys the estimate*

$$\gamma_k \leqslant \widetilde{\alpha}_{k+1}, \quad k \in \mathbb{Z}.$$

Proof. We have established above that the introduced space of admissible perturbations $\mathcal{M}$ is a Banach space. It follows from the embedding $\mathcal{M} \subset \mathfrak{S}_2(\mathcal{H})$ that $\mathcal{M}$ is embedded into $\mathfrak{L}_{L_0}(\mathcal{H})$ since each bounded operator is relatively bounded with respect to the unperturbed operator L_0. This is why Property 1) in Definition 14 holds true.

Properties 2) and 5) follow the construction of transformers J_k, Γ_k, $k \geqslant 0$, and Lemma 5.

Properties 3) and 6) are checked in the same way as similar properties were established in Lemma 4.

We proceed to proof of property 4). Let $X = X_l F \in \mathcal{M}$, $Y = Y_l F \in \mathcal{M}$, where X_l, $Y_l \in \mathfrak{S}_2(\mathcal{H})$. Then $X\Gamma_k Y = Z_l F$, where $Z_l = X_l \Gamma_k(FY_l)$. It follows from Lemma 5 that

$$\|Z_l\|_2 \leqslant \widetilde{\alpha}_{k+1} \|X_l\|_2 \|Y_l\|_2 \leqslant \widetilde{\alpha}_{k+1} \|X\|_\mathcal{M} \|Y\|_\mathcal{M}.$$

Let $X = FX_r$, $Y = FY_r$, X_r, $Y_r \in \mathfrak{S}_2(\mathcal{H})$. Then $X\Gamma_k Y = FZ_r$, where $Z_r = X_r \Gamma_k(FY_r)$, and again it follows from Lemma 5 that

$$\|Z_r\|_2 \leqslant \widetilde{\alpha}_{k+1} \|X_r\|_2 \|Y_r\|_2 \leqslant \widetilde{\alpha}_{k+1} \|X\|_\mathcal{M} \|Y\|_\mathcal{M}.$$

A similar estimate holds for the norm of the operator $(\Gamma_k X)Y$. The proof is complete. $\quad\square$

Theorem 16. *Let an integer $k \geqslant m$ be such that the condition*

$$4\widetilde{\alpha}_{k+1} \|\widetilde{V}\|_\mathcal{M} < 1$$

is satisfied. Then the operator $L_0 - \widetilde{V}$ is similar to the operator $L_0 - J_k X_* = L_0 - V_0$, where the operator $X_* \in \mathcal{M}$ is a solution to nonlinear operator equation (18) in Theorem 11 with transformers $J = J_k$ and $\Gamma = \Gamma_k$ defined by formulae (24), (25) and $B = \widetilde{V}$. Moreover, the operator V_0 is an orthogonal direct sum

$$V_0 = V_{0(k)} \oplus \bigoplus_{|i| > k} V_{0i}$$

with respect to the representation of the space L_2 of the form

$$\mathcal{H} = \mathcal{H}_{(k)} \oplus \bigoplus_{|i| > k} \mathcal{H}_i,$$

where $\mathcal{H}_{(k)} = \operatorname{Im} P_{(k)}$, $\mathcal{H}_i = \operatorname{Im} P_i$, $|i| > k$, and the projectors $P_{(k)}$, P_i, $|i| > k$, are the spectral projectors of the unperturbed operator L_0 defined by formulae (13). The operator of transformation the operator $L_0 - \widetilde{V}$, $\widetilde{V} \in \mathfrak{S}_2(\mathcal{H})$ into $L_0 - V_0$ is the operator $I + \Gamma_k X$, where $X \in \mathcal{M} \subset \mathfrak{S}_2(\mathcal{H})$ and $\Gamma_k X \in \mathfrak{S}_2(\mathcal{H})$.

Theorem 16 follows Theorems 11, 14, Lemmata 7, 8 and the property $\lim\limits_{k\to\infty} \widetilde{\alpha}_k = 0$, which ensures Condition (17) of Theorem 11.

6.   Proof of Theorems 3, 4, 5

Theorems 15 and 16 imply Theorem 3 given in Section 1. We stress the operator U in Theorem 3 is of the form

$$U = U_{mk} = (I + \Gamma_m V)(I + \Gamma_k X_*) = I + W_{mk}, \tag{28}$$

where the operator $W_{mk} = \Gamma_m V + \Gamma_k X_* + (\Gamma_m V)(\Gamma_k X_*)$ belongs to $\mathfrak{S}_2(\mathcal{H})$.

We proceed to estimating the eigenvalues and the eigenprojectors. The next theorem follows Theorem 16 and Lemma 1.

Theorem 17. *The spectrum of the operator L coincides with the spectrum of the operator*

$$L_0 - V_0 = L_0 - P_{(k)} X_* P_{(k)} - \bigoplus_{|j|>k} P_j X_* P_j.$$

Moreover, the identity holds:

$$\sigma(L) = \sigma(L_{(k)}) \cup \Big(\bigcup_{|j|>k} \sigma(L_j) \Big) = \sigma(L_{(k)}) \cup \Big\{ i\frac{2\pi j}{\omega} + x_{*jj}, |j| > k \Big\},$$

where $L_{(k)}$ is the restriction of the operator $L_0 - P_{(k)} X_* P_{(k)}$ on $\mathcal{H}_{(k)} = \mathrm{Im}\, P_{(k)}$ and L_j is the restriction of the operator $L_0 - P_j X_* P_j$ and $\mathrm{Im}\, P_j$, $|j| > k$, while x_{*jj}, $|j| > k$, are the diagonal entries of the matrix of the operator X_*.

Theorem 17 implies immediately Theorem 4. In Theorem 4, it is taken into consideration that for each bounded operator, its spectral radius does not exceeds the norm and the sequence (x_{*jj}), $|j| > k$, is square summable since $X_* \in \mathfrak{S}_2(\mathcal{H})$. All statements on the eigenvectors follow Lemma 1 and the representation of the invertible operator of transformation (28) as $U = I + \Gamma_{km} X_*$, where $\Gamma_{km} X_* \in \mathfrak{S}_2(\mathcal{H})$.

Proof of Theorem 5. The arguing is made under the assumptions of Theorem 3. We fix numbers k, $m \in \mathbb{Z}_+$ such that the assumptions of Theorems 15, 16 hold. Let $\mathcal{P}_n = P(\{\lambda_n\}, L_0)$, $\lambda_n = \{\frac{i2\pi n}{\omega}\}$, $n \in \mathbb{Z}$, $\mathcal{P}_{(m)} = \sum\limits_{|n|\leqslant m} \mathcal{P}_n$, $\widetilde{\mathcal{P}}_{(m)} = U_{km} \mathcal{P}_{(m)} U_{km}^{-1}$, $\widetilde{\mathcal{P}}_n = U_{km} \mathcal{P}_n U_{km}^{-1}$. According Lemma 1, the projectors $\widetilde{\mathcal{P}}_{(m)}$ and $\widetilde{\mathcal{P}}_n$ are the spectral projectors of the operator L. By the symbols $\mathcal{P}(\Omega)$ and $\widetilde{\mathcal{P}}(\Omega)$ we respectively denote the following projectors:

$$\mathcal{P}(\Omega) = \sum_{n\in\Omega} \mathcal{P}_n, \quad \widetilde{\mathcal{P}}(\Omega) = \sum_{n\in\Omega} \widetilde{\mathcal{P}}_n = \sum_{n\in\Omega} U_{km} \mathcal{P}_n U_{km}^{-1}, \quad \Omega = \mathbb{Z} \setminus \{-m, \dots, m\}.$$

We note that $\mathcal{P}(\Omega)$ is a spectral projector constructed by the spectral set $\{\lambda_n, n \in \Omega\}$ of the operator L_0.

For each $X \in \mathfrak{S}_2(\mathcal{H})$ we define a quantity

$$\alpha(\Omega, X) = \max_{n\in\Omega} \alpha_n(X), \quad \Omega \subset \mathbb{Z},$$

where the sequence α_n, $n \in \mathbb{Z}$, is defined by formula (27).

Let $X \in \mathcal{M}$, that is, $X = X_l F_X = F_X X_r$, where X_l, $X_r \in \mathfrak{S}_2(\mathcal{H})$. We estimate the norm

$$\|\mathcal{P}(\Omega) X\|_2 = \|\mathcal{P}(\Omega) F_X X_r\|_2 = \Big\| \Big(\sum_{n\in\mathbb{Z}} \alpha_n(X) \mathcal{P}_n \Big) X_r \Big\|_2 \leqslant \alpha(\Omega, X) \|X\|_2.$$

In the same way we obtain the estimate for the operator $X\mathcal{P}(\Omega)$. Therefore,

$$\max\{\|\mathcal{P}(\Omega) X\|_2, \|X\mathcal{P}(\Omega)\|_2\} \leqslant \|X\|_{\mathcal{M}} \alpha(\Omega, X).$$

We proceed to estimating the quantities $\|\mathcal{P}(\Omega)\Gamma_m X\|_2$, $\|\Gamma_m X \mathcal{P}(\Omega)\|_2$:

$$\|\mathcal{P}(\Omega)\Gamma_m X\|_2^2 = \sum_{\substack{i \in \Omega, j \in \mathbb{Z} \\ i \neq j}} \frac{\|\mathcal{P}_i \mathcal{P}(\Omega) X \mathcal{P}_j\|_2^2}{|\lambda_i - \lambda_j|^2} \leqslant \left(\frac{\omega}{2\pi}\right)^2 \sum_{i \in \Omega} \|\mathcal{P}_i X\|_2^2 \leqslant \left(\frac{\omega}{2\pi}\right)^2 \alpha^4(\Omega, X)\|X\|_2^2.$$

Thus,

$$\|\mathcal{P}(\Omega)\Gamma_m X\|_2 \leqslant \frac{\omega}{2\pi}\alpha^2(\Omega, X)\|X\|_2.$$

In the same way and by the same quantity we estimate $\|\Gamma_m X \mathcal{P}(\Omega)\|_2$. We note that the definition of the sequence α and the constructing of the space of admissible perturbations $\mathcal{M}$ imply the identities $\|F\mathcal{P}(\Omega)\| = \|\mathcal{P}(\Omega)F\| = \alpha(\Omega, B)$. This is why $\alpha(\Omega, X) \leqslant \|X\|_{\mathcal{M}}\alpha(\Omega, B)$.

Also the definition of the sequence α_n, $n \in \mathbb{Z}$, imply the following properties:

1) If $X = \sum\limits_{l \geqslant 1} X_l$, where $X_l \in \mathfrak{S}_2(\mathcal{H})$, when this series converges absolutely and

$$\|X\|_2 \alpha_n(X) \leqslant \sum_{l \geqslant 1} \|X_l\|_2 \alpha_n(X_l).$$

2) If $X = X_1 \cdot \cdots \cdot X_l$, $X_j \in \mathfrak{S}_2(\mathcal{H})$, $1 \leqslant j \leqslant l$, then

$$\|X\|_2 \alpha_n(X) \leqslant (\alpha_n(X_1) + \cdots + \alpha_n(X_l)) \bigcap_{j=1}^{l} \|X_j\|_2.$$

We proceed to estimating the norm of the difference $\widetilde{\mathcal{P}}(\Omega) - \mathcal{P}(\Omega)$, where $\Omega \subset \mathbb{Z} \setminus \{-m, \ldots, m\}$. We recall that the operator U in Theorem 3 making the similarity transformation can be represented as $U_{mk} = I + W_{mk}$, where $W_{mk} = \Gamma_m V + \Gamma_k X_* + (\Gamma_m V)(\Gamma_k X_*)$. Lemma 1 implies the identity

$$\widetilde{\mathcal{P}}(\Omega) - \mathcal{P}(\Omega) = (W_{mk}\mathcal{P}(\Omega) - \mathcal{P}(\Omega)W_{mk})(I + W_{mk})^{-1}.$$

Then

$$\begin{aligned}
\|\mathcal{P}(\Omega)W_{mk}\|_2 &\leqslant \|\mathcal{P}(\Omega)(\Gamma_m V)\|_2 + \|\mathcal{P}(\Omega)(\Gamma_k X_*)\|_2 + \|\mathcal{P}(\Omega)(\Gamma_m V)(\Gamma_k X_*)\|_2 \\
&\leqslant \|\mathcal{P}(\Omega)(\Gamma V)\|_2 + \|\mathcal{P}(\Omega)(\Gamma X_*)\|_2 + \|\mathcal{P}(\Omega)(\Gamma V)(\Gamma X_*)\|_2 \\
&\leqslant C_1(\alpha(\Omega, \Gamma V) + \alpha^2(\Omega, X_*)) \leqslant C_2(\alpha(\Omega, \Gamma V) + \alpha^2(\Omega, V)).
\end{aligned}$$

We note that the constants C_1 and C_2 are independent of Ω. By a similar quantity we can estimate the norm $\|W_{mk}\mathcal{P}(\Omega)\|_2$.

The operator $(I + W_{mk})^{-1}$ can be represented as $(I + W_{mk})^{-1} = I + \sum\limits_{j=1}^{\infty} (-1)^j W_{mk}^j$, this is why

$$\|(I + W_{mk})^{-1} - I\|_2 \leqslant \frac{\|W_{mk}\|_2}{1 - \|W_{mk}\|_2}.$$

As a result we obtain

$$\|\widetilde{\mathcal{P}}(\Omega) - \mathcal{P}(\Omega)\|_2 \leqslant C_3(\alpha(\Omega, \Gamma V) + \alpha^2(\Omega, V)), \tag{29}$$

where the constant C_3 is independent of Ω.

We observe that there two partitions of the unity in the space $\mathcal{H}$

$$I = \sum_{|i| > k} \mathcal{P}_i + \mathcal{P}_{(k)}, \qquad I = \sum_{|i| > k} \widetilde{\mathcal{P}}_i + \widetilde{\mathcal{P}}_{(k)}.$$

The statement of Theorem 5 follows immediately estimate (29) if as Ω we consider the set $\Omega = \{n \in \mathbb{Z}, |n| > N\}$, where N is a sufficiently large natural number, and Property 2) of the sequence $\{\alpha_n\}$. $\qquad\square$

Corollary 1. *The operator L is a spectral operator of scalar type (see [45]).*

7. Proof of Theorems 1, 6, 8

We employ Theorem 3 and its more detailed version, Theorem 16, in order to construct a group of the operators $T : \mathbb{R} \to \operatorname{End} L_2$, whose generator is the studied operator with the involution $L : W_2^1 \subset L_2 \to L_2$. It follows from Theorem 16, Theorem 3, 12 and Lemma 2 that the group of the operators $T(t)$, $t \in \mathbb{R}$, is an $U = U_{mk}$-orthogonal direct sum

$$T(t) = U \Big(\bigoplus_{j=-\infty}^{-k-1} e^{(\frac{i2\pi j}{\omega} + x_{*jj})t} \oplus e^{X_{*(m)}t} \bigoplus_{j=k+1}^{\infty} e^{(\frac{i2\pi j}{\omega} + x_{*jj})t} \Big) U^{-1}, \quad t \in \mathbb{R}, \tag{30}$$

with respect to the U-orthogonal expansion

$$L_2 = \mathcal{H} = \bigoplus_{j=-\infty}^{-k-1} U\mathcal{H}_j \oplus U\mathcal{H}_{(k)} \oplus \Big(\bigoplus_{j=k+1}^{\infty} U\mathcal{H}_j \Big).$$

We note that the numbers m, $k \in \mathbb{Z}_+$, are defined in Theorems 15 and 16, respectively. At that, the operator U_{mk} can be represented as (28), that is, $U_{mk} = I + W_{mk}$, where $W_{mk} \in \mathfrak{S}_2(\mathcal{H})$.

Thus, we have constructed a group $T : \mathbb{R} \to \operatorname{End} L_2$, whose generator is the operator $L = L_0 - V$. The statement of Theorem 1 on representation of classical and mild solutions to problem (1) follows the general theory of semi-groups of the operators (see [29], [30], [45]). The proved representation (30) implies the statement of Theorem 6, where $b_j = x_{jj}$, $|j| \geqslant m + 1$, $B_{(k)} = X_*$. We note that Theorems 3, 16 imply the property $\sum\limits_{|j| \geqslant m+1} |b_j|^2 < \infty$.

In order to prove Theorem 8, we employ representation (30) of the group T and the Parseval identity. The made estimates are in fact given in the formulation of Theorem 8.

In conclusion we mention that the results of the present work were partially announced in [46] and [47]. The study in [48] follows the lines of the present work.

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