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# UNIFORM CONVERGENCE OF LAGRANGE-STRUM-LIOUVILLE PROCESSES ON ONE FUNCTIONAL CLASS

**A.Yu. TRYNIN**

**Abstract.** We establish the uniform convergence inside an arbitrary interval $(a, b) \subset [0, \pi]$ for the values of the Lagrange-Sturm-Liouville operators for functions in a class defined by one-side moduli of continuity and oscillations. Outside this interval, the sequence of values of the Lagrange-Sturm-Liouville operators may diverge. The conditions describing this functional class contain a restriction only on the rate and magnitude of the increasing (or decreasing) of the continuous function. Each element of the proposed class can decrease (or, respectively, increase) arbitrarily fast. Popular sets of functions satisfying the Dini-Lipschitz condition or the Krylov criterion are proper subsets of this class, even if, under their conditions, the classical modulus of continuity and the variation are replaced by the one-sided ones. We obtain sharp upper bounds for functions and Lebesgue constants of the Lagrange-Sturm-Liouville processes. We establish sufficient conditions of the uniform convergence of the Lagrange-Sturm-Liouville processes in terms of the maximal absolute value of the sum and the maximal sum of the absolute values of the weighted first order differences. We prove the equiboundedness of the sequence of fundamental functions of Lagrange-Sturm-Liouville processes. Three new operators are proposed, which are modifications of the Lagrange-Sturm-Liouville operator and they allow one to approximate uniformly an arbitrary continuous function vanishing at the ends on the segment $[0, \pi]$. All the results of the work remain valid if the one-sided moduli of continuity and oscillations are replaced by the classical ones.

**Keywords:** sinc approximation, interpolation functions, uniform approximation.

**Mathematics Subject Classification:** Primary 41A05, 41A58; Secondary 94A12

## 1. Introduction

In [1], G.I. Natanson obtained the Dini-Lipschitz condition of the uniform convergence inside the interval $(0, \pi)$, that is, uniformly on each compact set in $(0, \pi)$, for the Lagrange-Sturm-Liouville processes of form

$$L_n^{SL}(f, x) = \sum_{k=1}^{n} f(x_{k,n}) \frac{U_n(x)}{U_n'(x_{k,n})(x - x_{k,n})} = \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}^{SL}(x), \tag{1}$$

where $U_n$ is the $n$th eigenfunction of the regular Sturm-Liouville problem

$$\begin{cases} U'' + [\lambda - q]U = 0, \\ U'(0) - hU(0) = 0, \\ U'(\pi) + HU(\pi) = 0 \end{cases} \tag{2}$$

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with a continuous potential $q$ of a bounded variation on $[0,\pi]$ and boundary conditions implying that the leading term in the asymptotics for $U_n$ is the cosine, that is, $h \neq \pm\infty$, $H \neq \pm\infty$. Here $0 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < \pi$ are the zeroes of the function $U_n$. To the study of the approximative properties of Lagrange-Strum-Liouville operators (1), works [2]–[4] were also devoted. In work [2], there was established the existence of a continuous on $[0,\pi]$ function, whose interpolation Lagrange-Sturm-Liouville process (1) diverges unboundedly almost everywhere in $[0,\pi]$. Studies made in [3], [5], [6] showed that under arbitrary small variation of the parameters in Sturm-Liouville problem (2) (potential $q$ or constants $h$, $H$), the approximative properties of processes (1) can change substantially.

The properties of interpolation operators for functions of Lagrange type (1) are closely related with the behavior of sinc-approximations

$$L_n(f,x) = \sum_{k=0}^{n} \frac{\sin(nx-k\pi)}{nx-k\pi} f\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n} \frac{(-1)^k \sin nx}{nx-k\pi} f\left(\frac{k\pi}{n}\right) \tag{3}$$

used in the the Whittaker-Kotelnikov-Shannon sampling theory, see [7]–[10]. The most complete survey of the results obtained in studying of the properties of sinc-approximations (3) of analytic on the real axis functions decaying exponentially at infinity as well as many important applications of sinc-approximations can be found, for instance, in [9] and [11].

Sinc-approximations are widely used in various numerical methods in mathematical physics and in approximating functions of both one and several variables [12]–[14], in the theory of quadrature formulae [9] and in wavelet theory [7], [8], [10].

Before appearing of works [15]–[21], the approximation of such operators on a segment or on a bounded interval was made only for some classes of analytic functions [9], [22] via reducing to the case of the axis by means of a conformal mapping. In [21], there was found the upper estimate for the best approximation of continuous functions by linear combinations of sincs.

The results of the studies in [23] showed that while approximating non-smooth continuous functions by the values of operators (3), a “resonance” can arise, which leads to an unbounded growth of the approximation error on the entire interval $(0,\pi)$. In [24]–[27] there were proposed various modifications of sinc-approximations (3) allowing to approximate continuous functions on the segment $[0,\pi]$. The study of the completeness of the system of syncs (3) in [26] in the spaces $C[0,\pi]$ and $C_0[0,\pi] = \{f : f \in C[0,\pi], f(0) = f(\pi) = 0\}$ allows one to conclude that it is vainly to construct an operator as linear combinations of sincs approximating uniformly an arbitrary continuous function on a segment.

The study of Lagrange-Sturm-Liouville operators (1) is also closely related with the approximative properties of the interpolation operators constructed by the solutions of the Cauchy problem for second order differential equations [28]. The operators proposed in [28] are generalizations of Lagrange-Sturm-Liouville operators (1) and of classical sinc-approximation (3). In [29], a series of applications of the results in work [28] was given to studying approximative properties of classical algebraic interpolation Lagrange polynomials with a matrix of interpolation nodes, each row of which is formed by the zeroes of Jacobi polynomials $P_n^{\alpha_n,\beta_n}$ with the parameters depending on $n$.

In monograph [4], detailed proofs were given and the misprints made in some formulae in earlier publications were corrected.

Following the lines of [30]—[37], in the present work we obtain sufficient conditions for the uniform convergence in the interval $(0,\pi)$ of interpolation processes (1) constructed by solutions to problem (2) in terms of one-sided moduli of continuity and oscillation.

## 2. Main results

Throughout the work, the potential $q$ in Sturm-Liouville problem (2) is assumed to be a continuous function with a bounded variation on $[0,\pi]$. We also suppose that the eigenfunction is normalized by the condition $U_n(0) = 1$. We consider Robin conditions (2) excluding Dirichlet type conditions are excluded, that is, $h \neq \pm\infty$, $H \neq \pm\infty$. For each $0 \leqslant a < b \leqslant \pi$, $0 < \varepsilon < (b-a)/2$, we define the indices $p_1$, $p_2$, $m_1$ and $m_2$ by means of relations

$$\begin{aligned} &x_{p_1,n} \leqslant a + \varepsilon < x_{p_1+1,n}, && x_{p_2,n} \leqslant b - \varepsilon < x_{p_2+1,n}, \\ &x_{k_1-1,n} < a \leqslant x_{k_1,n}, && x_{k_2+1,n} \leqslant b < x_{k_2+2,n}, \\ &m_1 = \left[\frac{k_1}{2}\right] + 1, && m_2 = \left[\frac{k_2}{2}\right] \end{aligned} \tag{4}$$

after adding the points $x_{0,n} = 0$ and $x_{n+1,n} = \pi$ to the set of zeroes $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ of the $n$th eigenfunction $U_n$. Here $[z]$ denotes the integer part of a number $z$. If else is not said, the prime at the sum denotes the absence of the term with the zero denominator.

We denote by $\mathbf{\Omega}$ the set of all real non-decreasing concave $[0, b-a]$ functions $\omega$ vanishing at zero. Let $C(\omega^l, [a,b])$ be $C(\omega^r, [a,b])$ the set of the elements in the space $C[a,b]$ such that for arbitrary $x$ and $x+h$ ($a \leqslant x < x+h \leqslant b$) the inequalities hold:

$$f(x+h) - f(x) \geqslant -K_f\omega(h) \quad \text{or} \quad f(x+h) - f(x) \leqslant K_f\omega(h), \tag{5}$$

respectively, where $\omega \in \mathbf{\Omega}$. Here the choice of a positive constant $K_f$ depends only on the function $f$. In this case the function $\omega(h)$ is called respectively the left or right modulus of continuity.

The classical modulus of continuity of a function $f \in C[a,b]$ is denoted as usually by

$$\omega(f,\delta) = \sup_{|h|<\delta; x, x+h \in [a,b]} |f(x+h) - f(x)|.$$

In the case $a = 0$, $b = \pi$, the modulus of continuity of $f \in C[0,\pi]$ is denoted by

$$\omega_1(f,\delta) = \sup_{|h|<\delta; x, x+h \in [0,\pi]} |f(x+h) - f(x)|.$$

The modulus of variation of a function $f$ on a segment $[a,b]$ is the function of a natural variable

$$v(n,f) = \sup_{T_n} \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|,$$

where $T_n = \{a \leqslant t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \leqslant b\}$, $n \in \mathbb{N}$. We take a non-negative non-decreasing concave function of a natural argument $v(n)$. If the modulus of variation of the function $f$ on an interval $[a,b]$ is such that $v(n,f) = O(v(n))$ as $n \to \infty$, we say that $f$ belongs to the class $V(v)$. Here the choice of the constant in $O$-symbol depends only on the function $f$.

Similar to the positive (negative) variation of the function, we call a positive (negative) modulus of variation of a function $f$ on a segment $[a,b]$ respectively the functions of a natural argument:

$$v^+(n,f) = \sup_{T_n} \sum_{k=0}^{n-1} \big(f(t_{k+1}) - f(t_k)\big)_+ \quad \text{and} \quad v^-(n,f) = \inf_{T_n} \sum_{k=0}^{n-1} \big(f(t_{k+1}) - f(t_k)\big)_-,$$

where

$$z_+ = \frac{z+|z|}{2}, \quad z_- = \frac{z-|z|}{2}, \quad T_n = \{a \leqslant t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \leqslant b\}, \quad n \in \mathbb{N}.$$

We say that $f$ belongs to a class $V^+(v)$ or $V^-(v)$ if there exists a function $M_f$ such that for each natural $n$ the inequality holds:

$$v^+(n,f) \leqslant M_f v(n) \quad \text{or} \quad v^-(n,f) \geqslant -M_f v(n),$$

respectively.

**Theorem 1.** *Let $0 \leqslant a < b \leqslant \pi$, $0 < \varepsilon < (b-a)/2$. If a non-decreasing convex function of a natural argument $v(n)$ and a function $\omega \in \mathbf{\Omega}$ are such that*

$$\lim_{n\to\infty} \min_{1\leqslant m\leqslant k_2-k_1-1} \left\{ \omega\left(\frac{\pi}{n}\right) \sum_{k=1}^{m} \frac{1}{k} + \sum_{k=m+1}^{k_2-k_1-1} \frac{v(k)}{k^2} \right\} = 0, \tag{6}$$

*where $k_1$ and $k_2+1$ are the indices of the smallest and greatest zero of the eigenfunction $U_n$ in the segment $[a,b]$, then for each function $f \in C(\omega^l[a,b]) \cap V^-(v)$ ($f \in C(\omega^r[a,b]) \cap V^+(v)$) the relation holds:*

$$\lim_{n\to\infty} \|f - L_n^{SL}(f,\cdot)\|_{C[a+\varepsilon,b-\varepsilon]} = 0, \tag{7}$$

*where the Lagrange-Sturm-Liouville operator $L_n^{SL}(f,\cdot)$ was defined in* (1).

**Remark 1.** *At that, the relation*

$$\lim_{n\to\infty} |f(x) - L_n^{SL}(f,x)| = 0$$

*can fail on the set $[0,\pi]\setminus[a,b]$, see, for instance,* [2], [3] *and* [4].

In the next theorem we obtain an order sharp upper estimate for the growth rate of the sequence of norms of the Lagrange-Sturm-Liouville operators and functionals (1) acting from $C[0,\pi]$ into $C[0,\pi]$ and from $C[0,\pi]$ into $\mathbb{R}$, respectively. Such sequences are called sequences of Lebesgue constants and Lebesgue functions. The approximative properties of Lagrange-Sturm-Liouville operators (1) in the sense of uniform and pointwise convergence depend essentially on their behavior.

**Theorem 2.** *Let $U_n$ be the eigenfunction associated with the eigenvalue $\lambda_n$ of regular Sturm-Liouville problem* (2). *Then there exist constants $C_1$, $C_2$ and $C_3$ depending only on the parameters of the Sturm-Liouville problem such that for all $x \in [0,\pi]$ and all $n = 2,3,4,\dots$ Lebesgue constants and Lebesgue constants of interpolation processes* (1) *satisfy the inequalities*

$$L_n^{SL}(x) = \sum_{k=1}^{n} |l_{k,n}^{SL}(x)| \leqslant C_1 |U_n(x)| \ln n + C_3, \tag{8}$$

$$L_n^{SL} = \max_{x\in[0,\pi]} L_n^{SL}(x) \leqslant C_2 \ln n. \tag{9}$$

**Remark 2.** *The order sharpness of estimates* (8) *and* (9) *follows Theorem 2 and the results of works* [2, Lm. 2] *or* [4].

The proof of these statements are given in Section 4.

## 3. Auxiliary statements

Before proving the theorems, we prove a series of auxiliary statements.

**Lemma 1.** *Let $U_n$ be the eigenfunction associated with the eigenvalue $\lambda_n$ of regular Sturm-Liouville problem* (2). *By $0 < x_{1,n} < x_{2,n} < \dots < x_{n,n} < \pi$ we denote the zeroes of the function*

$U_n$. *Then the following asymptotic formulae hold:*

$$U_n(x) = \cos nx + \frac{\beta(x)}{n}\sin nx + O(n^{-2}), \tag{10}$$

$$U_n'(x) = -n\sin nx + \beta(x)\cos nx + O(n^{-1}), \tag{11}$$

$$U_n''(x) = -n^2\cos nx - n\beta(x)\sin nx + O(1), \tag{12}$$

$$U_n'(x_{k,n}) = (-1)^k n + O(n^{-1}), \tag{13}$$

$$x_{k,n} = \frac{2k-1}{2n}\pi + n^{-2}\beta\Big(\frac{2k-1}{2n}\pi\Big) + O(n^{-3}), \tag{14}$$

$$\sqrt{\lambda_n} = n + O(n^{-1}), \tag{15}$$

*where*

$$\beta(x) = -cx + h + \frac{1}{2}\int_0^x q(\tau)\,d\tau, \qquad c = \frac{1}{\pi}\Big(h + H + \frac{1}{2}\int_0^\pi q(\tau)\,d\tau\Big),$$

*and the estimate for the error term in formulae* (10)-(14) *is uniform either in* $x \in [0,\pi]$ *or* $1 \leqslant k \leqslant n$.

*Proof.* For the proof of (10), (11) and (15) see, for instance, [39]. Let us make sure that (14) is true. Let $x_{k,n}$ be the $k$th zero of the eigenfunction $U_n$. Asymptotic formula (10) implies the relation

$$|\cos nx_{k,n} + \frac{\beta(x_{k,n})}{n}\sin nx_{k,n}| = O(n^{-2}).$$

Letting

$$\cos\alpha_{k,n} := \frac{n}{\sqrt{n^2 + \beta^2(x_{k,n})}},$$

we obtain the asymptotic formula

$$\Big|\sin\Big(\frac{\pi}{2} + nx_{k,n} - \alpha_{k,n}\Big)\Big| = O(n^{-2}).$$

Therefore, the relation

$$\Big|\frac{\pi}{2} + nx_{k,n} - \alpha_{k,n} - \pi k\Big| = O(n^{-2})$$

holds. But the function $\beta$ is at least continuously differentiable and this is why the asymptotic formula

$$x_{k,n} = \frac{2k-1}{2n}\pi + n^{-2}\beta\left(\frac{2k-1}{2n}\pi\right) + O(n^{-3})$$

holds. Formula (12) follows (10) and (2), and (13) is implied by (11) and (14). □

**Remark 3.** *By asymptotic formula* (10) *we see that the chosen normalization of the eigenfunctions* $U_n$ *ensures their equiboundedness.*

We denote

$$\mathbb{M} = \sup\big\{|U_n(x)|, x \in [0,\pi], n \in \mathbb{N}\big\} < \infty. \tag{16}$$

Let $\rho_\lambda = o\big(\frac{\sqrt{\lambda}}{\ln\lambda}\big)$ as $\lambda \to +\infty$. We suppose that that $h(\lambda) \in \mathbb{R}$ for arbitrary non-negative $\lambda$. We denote by $q_\lambda$ an arbitrary function in the ball $V_{\rho_\lambda}[0,\pi]$ of the radius $\rho_\lambda$ in the space of functions of bounded variation vanishing at zero, that is,

$$V_0^\pi[q_\lambda] \leqslant \rho_\lambda, \quad \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln\lambda}\right), \quad \text{as} \quad \lambda \to \infty, \quad q_\lambda(0) = 0. \tag{17}$$

For an arbitrary potential $q_\lambda \in V_{\rho_\lambda}[0,\pi]$, as $\lambda \to +\infty$, the zeroes of the solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 1, \qquad y'(0,\lambda) = h(\lambda), \end{cases} \tag{18}$$

or, under the additional condition $h(\lambda) \neq 0$,

$$V_0^{\pi}[q_\lambda] \leqslant \rho_\lambda, \quad \rho_\lambda = o\left(\frac{\sqrt{\lambda}}{\ln \lambda}\right), \quad \text{as} \quad \lambda \to \infty, \quad q_\lambda(0) = 0, \quad h(\lambda) \neq 0, \tag{19}$$

the zeroes of the solution to the Cauchy problem

$$\begin{cases} y'' + \big(\lambda - q_\lambda(x)\big)y = 0, \\ y(0,\lambda) = 0, \qquad y'(0,\lambda) = h(\lambda), \end{cases} \tag{20}$$

located in the segment $[0,\pi]$ are indexed as follows:

$$0 \leqslant x_{0,\lambda} < x_{1,\lambda} < \ldots < x_{n(\lambda),\lambda} \leqslant \pi \quad (x_{-1,\lambda} < 0, x_{n(\lambda)+1,\lambda} > \pi). \tag{21}$$

Here $x_{-1,\lambda} < 0,$ and $x_{n(\lambda)+1,\lambda} > \pi$ stand for the zeroes of some continuation of the solution to the Cauchy problem (18) or (20) provided the variation of the potential $q_\lambda$ is bounded outside $[0,\pi]$. In [28], [4], there was described the set of continuous on the segment $[0,\pi]$ functions $f$, which can be approximated by the values of the following operator uniformly inside the interval $(0,\pi)$. We consider the operator constructed by the solutions to the Cauchy problem (20) or (21) and mapping each function with finitely many values on the sets $\{x_{k,\lambda}\}_{k=0,n=1}^{n,\infty}$ into a continuous function by the rule

$$S_\lambda(f,x) = \sum_{k=0}^{n} \frac{y(x,\lambda)}{y'(x_{k,\lambda},\lambda)(x - x_{k,\lambda})} f(x_{k,\lambda}) = \sum_{k=0}^{n} s_{k,\lambda}(x) f(x_{k,\lambda}). \tag{22}$$

It is obvious that the value of operator (22) interpolate the function $f$ at the nodes $\{x_{k,\lambda}\}_{k=0}^{n}$.

We denote

$$C_0[0,\pi] = \{f : f \in C[0,\pi],\ f(0) = f(\pi) = 0\}.$$

While approximating the functions $f \in C[0,\pi] \setminus C_0[0,\pi]$ by means of operators (1), in the vicinity of the ends of the segments $[0,\pi]$, the Gibbs phenomenon arises (see, for instance, [25, Thm. 2], [4]). This problem is solved by the generalization of operator (22) proposed in [28, Form. (1.9)], [4] of form

$$T_\lambda(f,x) = \sum_{k=0}^{n} \frac{y(x,\lambda)}{y'(x_{k,\lambda})(x - x_{k,\lambda})} \left\{ f(x_{k,\lambda}) - \frac{f(\pi) - f(0)}{\pi} x_{k,\lambda} - f(0) \right\} + \frac{f(\pi) - f(0)}{\pi} x + f(0), \tag{23}$$

where $y(x,\lambda)$ is the solution to the Cauchy problem (18) or (20) and $x_{k,\lambda}$ are the zeroes of this solution.

**Proposition 1** ([28, Prop. 9], [4]). *Let $y(x,\lambda)$ be the solution to Cauchy problem* (18) *or* (20) *and assume that in the case of Cauchy problem* (18) *conditions* (17) *are satisfied, while in the case of Cauchy problem* (20), *conditions* (19) *are satisfied. If $f \in C_0[0,\pi]$, then the relation*

$$\lim_{\lambda\to\infty} \left( f(x) - S_\lambda(f,x) - \frac{1}{2} \sum_{k=0}^{n-1} \big( f(x_{k+1,\lambda}) - f(x_{k,\lambda}) \big) s_{k,\lambda}(x) \right) = 0 \tag{24}$$

*holds uniformly on* $[0,\pi]$.

**Remark 4.** *It follows from Proposition 1 that the values of the operators*

$$A_\lambda(f,x)=\frac{1}{2}\sum_{k=0}^{n-1}\big(f(x_{k+1,\lambda})+f(x_{k,\lambda})\big)s_{k,\lambda}(x),$$
$$B_\lambda(f,x)=\frac{1}{2}\sum_{k=1}^{n}\big(f(x_{k-1,\lambda})+f(x_{k,\lambda})\big)s_{k,\lambda}(x),$$
$$C_\lambda(f,x)=\frac{1}{4}\sum_{k=1}^{n-1}\big(f(x_{k-1,\lambda})+2f(x_{k,\lambda})+f(x_{k+1,\lambda})\big)s_{k,\lambda}(x)$$

*proposed in* [28], [4] *approximate an arbitrary element in the space* $C_0[0,\pi]$ *uniformly on the entire segment* $[0,\pi]$.

**Lemma 2.** *Let* $U_n$ *be the eigenfunction associated with the eigenvalue* $\lambda_n$ *of regular Sturm-Liouville problem* (2)*. Then there exists a constant* $C_4$ *depending only on* $q$, $h$, $H$ *such that for all* $x\in[0,\pi]$ *and all* $n=1,2,3,\dots$ *the inequality*

$$|l^{SL}_{k,n}(x)|=\left|\frac{U_n(x)}{U'_n(x_{k,n})(x-x_{k,n})}\right|\leqslant C_4 \tag{25}$$

*holds.*

*Proof.* If for some $1\leqslant k\leqslant n$ and $n\in\mathbb{N}$ we have $x=x_{k,n}$, then $|l^{SL}_{k,n}(x)|=1$. Consider the case $x\neq x_{k,n}$. Assume first that $0<|x-x_{k,n}|\leqslant n^{-1}$, $x\in[0,\pi]$, then by Taylor formula with the error term in the Lagrange form and (12) and (13) we get the inequality

$$|l^{SL}_{k,n}(x)|\leqslant\left|\frac{U'_n(x_{k,n})(x-x_{k,n})+U''_n(\xi_{k,n})(x-x_{k,n})^2/2}{U'_n(x_{k,n})(x-x_{k,n})}\right|=1+\frac{O(n^2)}{n+O(n^{-1})}\frac{1}{n}\leqslant C_{4,1}$$

for some constant $C_{4,1}$ depending only on the parameters $q$, $h$ and $H$ of the Sturm-Liouville problem. It remains to consider the case $|x-x_{k,n}|>n^{-1}$, $x\in[0,\pi]$. By asymptotic formulae (10) and (13) there exists a constant $C_{4,2}$ satisfying the inequality

$$|l^{SL}_{k,n}(x)|\leqslant n\left|\frac{U_n(x)}{U'_n(x_{k,n})}\right|\leqslant\left|\frac{\cos nx+\frac{\beta(x)}{n}\sin nx+O(n^{-2})}{n+O(n^{-1})}\right|n\leqslant C_{4,2}.$$

Letting $C_4=\max(C_{4,1},C_{4,2})$, we complete the proof. □

In what follows we shall need the proof of Theorem 2

*Proof of Theorem 2.* We choose arbitrary $x\in[0,\pi]$. We denote by $k_0$ the index of the node closest to $x$. If there are two such nodes, we choose any of them, say, the left one. We represent the Lebesgue function of Lagrange-Sturm-Liouville interpolation process (1) as three terms

$$L^{SL}_n(x)=\sum_{k=1}^{k_0-3}|l^{SL}_{k,n}(x)|+\sum_{k=k_0-2}^{k_0+2}|l^{SL}_{k,n}(x)|+\sum_{k=k_0+3}^{n}|l^{SL}_{k,n}(x)|.$$

As $k_0=1,2,3$, there is no first term in this representation. As $k_0=n-2,n-1,n$, the third sum is absent. At most five terms in the second sum are estimated by means of Lemma 2. Then, employing (11), (13) Lagrange formula of finite increments, we estimate the Lebesgue function as follows:

$$L_n^{SL}(x) \leqslant \frac{|U_n(x)|}{n}\left(\sum_{k=1}^{k_0-3}\frac{1}{|x-x_{k,n}|}+\sum_{k=k_0+3}^{n}\frac{1}{|x-x_{k,n}|}\right)+5C_4$$
$$+\sum_{k=1}^{k_0-3}\left|\frac{|U_n(x)|}{|U_n'(x_{k,n})||x-x_{k,n}|}-\frac{|U_n(x)|}{n|x-x_{k,n}|}\right|+\sum_{k=k_0+3}^{n}\left|\frac{|U_n(x)|}{|U_n'(x_{k,n})||x-x_{k,n}|}-\frac{|U_n(x)|}{n|x-x_{k,n}|}\right|$$
$$\leqslant\frac{|U_n(x)|}{n}\left(\sum_{k=1}^{k_0-3}\frac{1}{|x-x_{k,n}|}+\sum_{k=k_0+3}^{n}\frac{1}{|x-x_{k,n}|}\right)+5C_4+\sum_{k=1}^{n}\left|\frac{U_n(x)}{x-x_{k,n}}\right|\left|\frac{n-(n+O(n^{-1}))}{n(n+O(n^{-1}))}\right|$$
$$=\frac{|U_n(x)|}{n}\left(\sum_{k=1}^{k_0-3}\frac{1}{|x-x_{k,n}|}+\sum_{k=k_0+3}^{n}\frac{1}{|x-x_{k,n}|}\right)+5C_4+O(n^{-1}).$$

By asymptotic formula (14) for the zeroes of the eigenfunction $U_n$ we find an index $n_0$ depending only on the parameters of the Sturm-Liouville problem such that starting from this index, the inequality holds

$$\min_{1\leqslant k\leqslant n-1}|x_{k+1,n}-x_{k,n}|\geqslant\frac{\pi}{2n}. \tag{26}$$

Therefore, the relation

$$|x-x_{k_0\pm2,n}|\geqslant\min_{1\leqslant k\leqslant n-1}|x_{k+1,n}-x_{k,n}|\geqslant\frac{\pi}{2n} \tag{27}$$

is true. By (26) and (27), the Lebesgue function of Lagrange-Sturm-Liouville interpolation process (1) can be estimated as

$$L_n^{SL}(x)\leqslant\frac{|U_n(x)|}{n}\left(\sum_{k=1}^{k_0-3}\frac{1}{x_{k+1,n}-x_{k,n}}\int_{x_{k,n}}^{x_{k+1,n}}\frac{dt}{x-t}+\sum_{k=k_0+3}^{n}\frac{1}{x_{k,n}-x_{k-1,n}}\int_{x_{k-1,n}}^{x_{k,n}}\frac{dt}{t-x}\right)+C_{3,0}$$
$$\leqslant\frac{2|U_n(x)|}{\pi}\left(2\ln n-2\ln\frac{\pi}{2}+\ln|x(x-\pi)|\right)+C_{3,0} \tag{28}$$

uniformly on the entire segment $[0,\pi]$. The identity $\max\Big(\ln|x(\pi-x)|, x\in(0,\pi)\Big)=2\ln\frac{\pi}{2}$ and asymptotic formula (10) imply (8) and (9) in the case $n\geqslant n_0$. To make estimates (8) and (9) true for all $n=2,3,4,\dots$, we let, for instance,

$$C_3=\max(C_{3,0},L_2^{SL},L_3^{SL},L_4^{SL},\dots,L_{n_0-1}^{SL},),\quad C_2=C_1\mathbb{M}+C_3/\ln 2,$$

where the constant $\mathbb{M}$ is determined in relation (16). □

For each $0\leqslant a<b\leqslant\pi$, $0<\varepsilon<(b-a)/2$ we denote

$$Q_n(f,[a,b],\varepsilon):=\max_{p_1\leqslant p\leqslant p_2}\left|\sum_{m=m_1}^{m_2}{}'\frac{f(x_{2m+1,n})-f(x_{2m,n})}{p-2m}\right|. \tag{29}$$

**Proposition 2.** *For a function $f\in C[0,\pi]$, relation*

$$\lim_{\lambda\to\infty}Q_n(f,[a,b],\varepsilon)=0 \tag{30}$$

*implies* (7).

*Proof.* We introduce the notation

$$\psi_{k,n}=f(x_{k+1,n})-f(x_{k,n}),\qquad 1\leqslant k\leqslant n-1,\quad n\in\mathbb{N}. \tag{31}$$

Taking into consideration that $f \in C[0,\pi]$ and (14), we confirm that there exists a constant $C_5$ such that the estimate

$$|\psi_{k,n}| \leqslant C_5\omega_1\Big(f, \frac{\pi}{n}\Big) \quad \text{for all} \quad 1 \leqslant k \leqslant n-1 \quad n \in \mathbb{N}. \tag{32}$$

holds.

We note that (31), (11) and (13) imply the uniform on the entire segment $[0,\pi]$ estimate

$$\begin{aligned} &\left|\sum_{k=k_1}^{k_2}\big(f(x_{k+1,n}) - f(x_{k,n})\big)l_{k,n}^{SL}(x) - \sum_{k=k_1}^{k_2}\psi_{k,n}\frac{(-1)^k U_n(x)}{n(x-x_{k,n})}\right| \\ &\leqslant \sum_{k=k_1}^{k_2}|\psi_{k,n}|\left|\frac{U_n(x)}{(x-x_{k,n})}\right|\left|\frac{(-1)^k n - U_n'(x_{k,n})}{nU_n'(x_{k,n})}\right| = \omega\Big(f,\frac{\pi}{n}\Big)O(n^{-1}). \end{aligned} \tag{33}$$

Arguing as in the proof of Theorem 2, multiplying $|U_n(x)|$ by $C_5\omega_1\Big(f, \frac{\pi}{n}\Big)$ in (28) and removing the terms with the indices $k_1 \leqslant k \leqslant k_2$ from the sum in (28), by (31) and (32) we see that there exists a constant $C_6$ and an index $n_0 \in \mathbb{N}$ independent of the function $f \in C[0,\pi]$, $0 \leqslant a < b \leqslant \pi$ and $0 < \varepsilon < (b-a)/2$ such that for arbitrary $x \in [a+\varepsilon, b-\varepsilon]$ and $n > n_0$ the inequality

$$\left|\frac{1}{2}\sum_{k\in[1,n-1]\setminus[k_1,k_2]}\psi_{k,n}l_{k,n}^{SL}(x)\right| \leqslant C_6\omega_1\left(f,\frac{\pi}{n}\right)\sum_{k\in[1,n-1]\setminus[k_1,k_2]}\left|l_{k,n}^{SL}(x)\right| \leqslant C_6\omega_1\left(f,\frac{\pi}{n}\right)\ln\frac{2\pi}{\varepsilon} \tag{34}$$

holds.

In the case of Cauchy problem (18) we let $\lambda = \lambda_n$, where $\lambda_n$ is an eigenvalue of Sturm-Liouville problem (2), we obtain the identity $U_n(x) \equiv y(x,\lambda_n)$. Therefore, the values of operators (1) and (22) coincide identically as $\lambda = \lambda_n$. By (34), Proposition 1 in the case of Cauchy problem (18), $\lambda = \lambda_n$ and (33) we get the relation

$$\begin{aligned} &\lim_{n\to\infty}\Big(f(x) - L_n^{SL}(f,x) - \frac{1}{2}\sum_{k=0}^{n-1}\big(f(x_{k+1,n}) - f(x_{k,n})\big)l_{k,n}^{SL}(x)\Big) \\ &= \lim_{n\to\infty}\left(f(x) - L_n^{SL}(f,x) - \frac{1}{2}\sum_{k=k_1}^{k_2}\psi_{k,n}\frac{(-1)^k U_n(x)}{n(x-x_{k,n})}\right) = 0. \end{aligned} \tag{35}$$

We fix arbitrary $x \in [a+\varepsilon, b-\varepsilon]$. We choose an index $p = p(x,\lambda)$ such that $x \in [x_{p,n}, x_{p+1,n})$. Then $x = x_{p,n} + \alpha(x_{p+1,n} - x_{p,n})$, where $\alpha = \alpha(x,\lambda) \in [0,1)$,

$$x - x_{k,n} = \frac{p-k+\alpha+\beta_{k,n}}{n}\pi.$$

By (14), the estimate $\beta_{k,n} = \beta_{k,n}(x) = O(n^{-1})$ holds uniformly in all $1 \leqslant k \leqslant n$ and $x \in [0,\pi]$.

By (32) and (14), for all $x \in [a+\varepsilon, b-\varepsilon]$ and sufficiently large $n$ such that for all $1 \leqslant k \leqslant n-1$ the inequality $|\beta_{k,n}| < 1$ holds, the estimate

$$\begin{aligned} &\Big|\sum_{\substack{k:k_1\leqslant k\leqslant k_2\\|p-k|\geqslant 3}}\frac{(-1)^k\psi_{k,n}}{p-k+\alpha+\beta_{k,n}} - \sum_{\substack{k:k_1\leqslant k\leqslant k_2\\|p-k|\geqslant 3}}\frac{(-1)^k\psi_{k,n}}{p-k}\Big| \\ &\leqslant C_5\omega\left(f,\frac{\pi}{n}\right)\sum_{\substack{k:k_1\leqslant k\leqslant k_2\\|p-k|\geqslant 3}}\frac{\alpha}{|p-k|(|p-k|-2)} \leqslant 3C_5\omega\Big(f,\frac{\pi}{n}\Big) \end{aligned} \tag{36}$$

holds true.

Taking into consideration notation (31), we rewrite the sum in (35) as follows:

$$\frac{1}{2}\sum_{k=k_1}^{k_2}\big(f(x_{k+1,n})-f(x_{k,n})\big)l^{SL}_{k,n}(x)=\frac{1}{2}\sum_{\substack{k:k_1\leqslant k\leqslant k_2\\ |p-k|\geqslant 3}}\psi_{k,n}l^{SL}_{k,n}(x)+\frac{1}{2}\sum_{\substack{k:k_1\leqslant k\leqslant k_2\\ |p-k|<3}}\psi_{k,n}l^{SL}_{k,n}(x). \qquad (37)$$

Now by the triangle inequality, (31), (36) and (35), we obtain the estimate

$$\begin{aligned}&\Big|\frac{1}{2}\sum_{k=k_1}^{k_2}\big(f(x_{k+1,n})-f(x_{k,n})\big)l^{SL}_{k,n}(x)-\frac{U_n(x)}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{(-1)^k\psi_{k,n}}{p-k}\Big|\\ &\leqslant\frac{1}{2\pi}\Big|\sum_{k:|p-k|\geqslant 3}\frac{(-1)^k\psi_{k,n}}{p-k+\alpha}-\sum_{k:|p-k|\geqslant 3}\frac{(-1)^k\psi_{k,n}}{p-k}\Big| \qquad (38)\\ &+\frac{1}{2\pi}\sum_{k:|p-k|<3}\Big|\psi_{k,n}l^{SL}_{k,n}(x)\Big|+\frac{1}{2\pi}{\sum_{k:|p-k|<3}}'\frac{|\psi_{k,n}|}{|p-k|}=o(1).\end{aligned}$$

uniform in $x\in[a+\varepsilon,b-\varepsilon]$.

By (38) and (35) we obtain the relation

$$\lim_{n\to\infty}\left(f(x)-L^{SL}_n(f,x)-\frac{U_n(x)}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{(-1)^k\psi_{k,n}}{p-k}\right)=0 \qquad (39)$$

uniform in $x\in[a+\varepsilon,b-\varepsilon]$. We estimate the last term in (39) by means of (16), (10), (32) and the triangle inequality

$$\Big|\frac{U_n(x)}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{(-1)^k\psi_{k,n}}{p-k}\Big|\leqslant 2\Big|\frac{\mathbb{M}}{2\pi}{\sum_{m=m_1}^{m_2}}'\frac{\psi_{2m,n}}{p-2m}\Big|+\Big|\frac{\mathbb{M}}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{\psi_{k,n}}{p-k}\Big|+O\Big(\omega_1\Big(f,\frac{\pi}{n}\Big)\Big). \qquad (40)$$

Since the function $f$ is continuous on the segment $[0,\pi]$, we can find a sequence of natural numbers $\{l_n\}_{n=1}^{\infty}$ such that

$$l_n=o(n),\quad \lim_{n\to\infty}l_n=\infty,\quad \lim_{\lambda\to\infty}\omega_1\Big(f,\frac{\pi}{n}\Big)\sum_{k=1}^{l_n}\frac{1}{k}=0. \qquad (41)$$

We estimate the second sum in (40):

$$\Big|\frac{1}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{\psi_{k,n}}{p-k}\Big|\leqslant\Big|\frac{1}{2\pi}{\sum_{k:|p-k|\leqslant l_n}}'\frac{\psi_{k,n}}{p-k}\Big|+\Big|\frac{1}{2\pi}{\sum_{k:|p-k|>l_n}}'\frac{\psi_{k,n}}{p-k}\Big|. \qquad (42)$$

It follows from inequality (32) that

$$\Big|\frac{1}{2\pi}{\sum_{k:|p-k|\leqslant l_n}}'\frac{\psi_{k,n}}{p-k}\Big|\leqslant\frac{1}{2\pi}{\sum_{k:|p-k|\leqslant l_n}}'\Big|\frac{\psi_{k,n}}{p-k}\Big|\leqslant\frac{C_5}{\pi}\omega_1\Big(f,\frac{\pi}{n}\Big)\sum_{k=1}^{l_n}\frac{1}{k}. \qquad (43)$$

After the Abel transform, in the case $k\in[k_1,k_2]$, $|p-k|>l_n$, the second sum in (42) can be estimated as

$$\Big|\frac{1}{2\pi}{\sum_{k:|p-k|>l_n}}'\frac{\psi_{k,n}}{p-k}\Big|\leqslant\frac{4\|f\|_{C[a,b]}}{l_n+1}+4\|f\|_{C[a,b]}\sum_{k=l_n}^{\infty}\frac{1}{k(k+1)}.$$

Therefore, by (41), (42) and (43) we obtain the relation

$$\Big|\frac{\mathbb{M}}{2\pi}{\sum_{k=k_1}^{k_2}}'\frac{\psi_{k,n}}{p-k}\Big|=o(1) \qquad (44)$$

uniformly in $x\in[a+\varepsilon,b-\varepsilon]$.

By (39), (40), (44) and the triangle inequality we get the estimate

$$\Big|f(x) - L_n^{SL}(f,x)\Big| \leqslant \Big|f(x) - L_n^{SL}(f,x) - \frac{U_n(x)}{2\pi}\sum_{k=k_1}^{k_2}{}' \frac{(-1)^k\psi_{k,n}}{p-k}\Big| + \Big|\frac{\mathbb{M}}{\pi}\sum_{m=m_1}^{m_2}{}' \frac{\psi_{2m,n}}{p-2m}\Big|$$
$$+\Big|\frac{\mathbb{M}}{2\pi}\sum_{k=k_1}^{k_2}{}' \frac{\psi_{k,n}}{p-k}\Big| + o(1) \leqslant \frac{\mathbb{M}}{\pi}Q_n(f,[a,b],\varepsilon) + o(1).$$

Therefore, condition (30) yields uniform convergence (7). The proof is complete. □

For arbitrary $0 \leqslant a < b \leqslant \pi$, $0 < \varepsilon < (b-a)/2$ we denote

$$Q_n^*(f,[a,b],\varepsilon) := \max_{p_1 \leqslant p \leqslant p_2} \sum_{m=m_1}^{m_2}{}' \Big|\frac{f(x_{2m+1,n}) - f(x_{2m,n})}{p-2m}\Big|. \tag{45}$$

**Corollary 1.** *If $f \in C[0,\pi]$, then the relation*

$$\lim_{n\to\infty} Q_n^*(f,[a,b],\varepsilon) = 0 \tag{46}$$

*implies* (7).

*Proof.* Condition (46) ensures condition (30), and in its turn, by Proposition 2, this implies (7). □

**Remark 5.** *Proposition 2 and Corollary 1 are analogues of known Privalov test of uniform convergence of trigonometric interpolation polynomials and classical Lagrange interpolation polynomials over the matrix of Chebyshev interpolation nodes* [30].

## 4. Sufficient condition of uniform convergence of Lagrange-Sturm-Liouville processes inside $(0,\pi)$

Now we can proceed to proving the above formulated Theorem 1.

*Proof of Theorem 1.* Assume that the functions $v$ and $\omega$ satisfy condition (6) and $f \in C(\omega^l[a,b]) \cap V^-(v)$. Let us show that relation (46) holds. By the uniform continuity of the function $f$ on the segment $[0,\pi]$, for each positive $\tilde{\epsilon}$ there exist natural numbers $\nu$ and $n_1$ such that for all $n \geqslant n_1$ ($n \in \mathbb{N}$) two inequalities hold:

$$\omega\Big(\frac{\pi}{n}\Big)\sum_{k=1}^{\nu}\frac{1}{k} < \frac{\tilde{\epsilon}}{6} \tag{47}$$

and

$$24\|f\|_{C[a,b]} < \tilde{\epsilon}\nu. \tag{48}$$

Let $n \geqslant n_1$. We find an index $p_0$ depending on $n$, $a$, $b$, $\varepsilon$ and $f$, at which the maximum is attained in relation (45):

$$Q_n^*(f,[a,b],\varepsilon) = \sum_{m=m_1}^{m_2}{}' \Big|\frac{f(x_{2m+1,n}) - f(x_{2m,n})}{p_0-2m}\Big|.$$

We denote

$$Q_n^{**}(f,[a,b],\varepsilon) := \sum_{k=k_1}^{k_2}{}' \Big|\frac{f(x_{k+1,n}) - f(x_{k,n})}{p_0-k}\Big|.$$

Since $Q_n^{**}(f,[a,b],\varepsilon)$ is obtained from $Q_n^*(f,[a,b],\varepsilon)$ by additing non-negative terms, the inequality

$$Q_n^*(f,[a,b],\varepsilon) \leqslant Q_n^{**}(f,[a,b],\varepsilon) \tag{49}$$

is true. We partition $Q_n^{**}(f,[a,b],\varepsilon)$ into two terms

$$Q_n^{**}(f,[a,b],\varepsilon)=\sum_{k=k_1}^{k_2}{}'\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}-2\sum_{k=k_1}^{k_2}{}''\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}=S_1(p_0)+S_2(p_0),\quad (50)$$

where two primes denotes the absence of non-negative terms in the usm and the term with the index $k=p_0$.

We begin with estimating the first sum. In order to do this, we represent it as

$$\begin{aligned}S_1(p_0)=&\sum_{\substack{k:k\in[k_1,k_2],\\ 0<|p_0-k|<\nu}}\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}+\sum_{\substack{k:k\in[k_1,k_2],\\ |p_0-k|\geqslant\nu\\ |p_0-k|>0}}\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}\\ =&S_{1,1}(p_0)+S_{1,2}(p_0).\end{aligned}\quad (51)$$

In the case $\{k:k\in[k_1,k_2],|p_0-k|\geqslant\nu,|p_0-k|>0\}=\varnothing$ we second term is supposed to be zero.

Inequality (47) implies the relation

$$|S_{1,1}(p_0)|\leqslant 2\omega\left(\frac{\pi}{n}\right)\sum_{k=1}^{\nu}\frac{1}{k}<\frac{\tilde{\epsilon}}{3}\quad (52)$$

for all $n\geqslant n_1$. Let us estimate $S_{1,2}(p_0)$. If $p_0$ satisfies relation $k_1\leqslant p_0-\nu<p_0<p_0+\nu\leqslant k_2$, the inequalities $p_0-k_1\geqslant\nu$ and $k_2-p_0\geqslant\nu$ hold. We employ (48) and the Abel transform to obtain the estimate

$$\begin{aligned}|S_{1,2}(p_0)|\leqslant&\left|\sum_{k=k_1}^{p_0-\nu}\frac{f(x_{k+1,n})-f(x_{k,n})}{p_0-k}\right|+\left|\sum_{k=p_0+\nu}^{k_2}\frac{f(x_{k+1,n})-f(x_{k,n})}{k-p_0}\right|\\ \leqslant&\left|\sum_{k=k_1}^{p_0-\nu-1}\frac{f(x_{k+1,n})-f(x_{k_1,n})}{(p_0-k)(p_0-k-1)}\right|+\left|\frac{f(x_{p_0-\nu+1,n})-f(x_{k_1,n})}{p_0-k_1}\right|\\ &+\left|\sum_{k=p_0+\nu}^{k_2-1}\frac{f(x_{k+1,n})-f(x_{p_0+\nu,n})}{(k-p_0)(k+1-p_0)}\right|+\left|\frac{f(x_{k_2,n})-f(x_{p_0+\nu,n})}{k_2-p_0}\right|\\ \leqslant&4\|f\|_{C[a,b]}\sum_{i=\nu}^{\infty}\frac{1}{i(i+1)}+\frac{4\|f\|_{C[a,b]}}{\nu}\leqslant\frac{8\|f\|_{C[a,b]}}{\nu}<\frac{\tilde{\epsilon}}{3}.\end{aligned}\quad (53)$$

In the same way we prove (53) in the situation when the index $p_0$ satisfies one of the relations $p_0-\nu<k_1\leqslant p_0<p_0+\nu\leqslant k_2$ or $k_1\leqslant p_0-\nu<p_1\leqslant k_2<p_0+\nu$. There remains the only possible case $p_0-\nu<k_1\leqslant p_1\leqslant k_2<p_0+\nu$. Here $|S_{1,2}(p_0)|=0$.

By (51), (52) and (53), for all $n\geqslant n_1$ we have the estimate

$$|S_1(p_0)|\leqslant\frac{2\tilde{\epsilon}}{3}.\quad (54)$$

We proceed to studying the properties of the sum $S_2(p_0)$. We take arbitrary integer $m$ such $1\leqslant m\leqslant k_2-k_1-2$ and represent $S_2(p_0)$ as

$$\begin{aligned}0\leqslant S_2(p_0)=&-2\sum_{\substack{k:k\in[k_1,k_2]\\ |p_0-k|\leqslant m}}{}''\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}-2\sum_{\substack{k:k\in[k_1,k_2]\\ |p_0-k|>m}}{}''\frac{f(x_{k+1,n})-f(x_{k,n})}{|p_0-k|}\\ =&S_{2,1}(p_0)+S_{2,2}(p_0).\end{aligned}\quad (55)$$

We choose sufficiently large number $n_2 \geqslant n_1$ depending only on the parameters of the Sturm-Liouville problem such that starting from this index, by (14), the inequalities hold:

$$\max_{1 \leqslant k \leqslant n} |x_{k+1,n} - x_{k,n}| \leqslant \frac{3\pi}{2n}.$$

Since $f \in C(\omega^l[a,b])$, according definition (5), starting from $n_2$ we have the relation

$$f(x_{k+1,n}) - f(x_{k,n}) \geqslant -10K_f\omega\Big(\frac{\pi}{n}\Big). \tag{56}$$

This is why,

$$0 \leqslant S_{2,1}(p_0) = -2 \sum_{\substack{k: k \in [k_1, k_2], \\ |p_0 - k| \leqslant m}}'' \frac{f(x_{k+1,n}) - f(x_{k,n})}{|p_0 - k|} \leqslant 10K_f\omega\Big(\frac{\pi}{n}\Big) \sum_{k=1}^{m} \frac{1}{k}. \tag{57}$$

Let us estimate the sum $S_{2,2}(p_0)$.

$$\begin{aligned} 0 \leqslant &S_{2,2}(p_0) = -2 \sum_{\substack{k: k \in [k_1, k_2] \\ |p_0 - k| > m}}'' \frac{f(x_{k+1,n}) - f(x_{k,n})}{|p_0 - k|} \\ \leqslant &2 \sum_{k=k_1}^{p_0 - m - 1} \frac{-(f(x_{k+1,n}) - f(x_{k,n}))_-}{p_0 - k} + 2 \sum_{k=p_0+m+1}^{k_2} \frac{-(f(x_{k+1,n}) - f(x_{k,n}))_-}{k - p_0}. \end{aligned} \tag{58}$$

If $p_0 - m \leqslant k_1$ or $p_0 + m \geqslant k_2$, then in (58) the first or second term disappears, respectively. In the case $p_0 - m < k_1 < k_2 < p_0 + m$, there is no sum $S_{2,2}(p_0)$ in (55). Taking into account that $f \in V(v)$, by means of the Abel transform (56) we estimate (58):

$$\begin{aligned} 0 \leqslant &S_{2,2}(p_0) \leqslant 2\Bigg( \frac{\sum\limits_{k=k_1}^{p_0-m-1} -(f(x_{k+1,n}) - f(x_{k,n}))_-}{p_0 - k_1} + \sum_{k=k_1+1}^{p_0-m-1} \frac{\sum\limits_{j=k}^{p_0-m-1} -(f(x_{j+1,n}) - f(x_{j,n}))_-}{(p_0-k)(p_0-k+1)} \\ &+ \frac{\sum\limits_{k=p_0+m+1}^{k_2} -(f(x_{k+1,n}) - f(x_{k,n}))_-}{k_2 - p_0} + \sum_{k=p_0+m+1}^{k_2-1} \frac{\sum\limits_{j=p_0+m+1}^{k} -(f(x_{j+1,n}) - f(x_{j,n}))_-}{(p_0-k)(p_0-k-1)} \Bigg) \\ \leqslant &2\Bigg( \frac{\big((p_0 - k_1) - m - 1\big)2,5K_f\omega\big(\frac{\pi}{n}\big)}{p_0 - k_1} + M_f \sum_{k=k_1+1}^{p_0-m-1} \frac{v(p_0 - m - k)}{(p_0-k)(p_0-k+1)} \\ &+ \frac{\big((k_2 - p_0) - m - 1\big)2,5K_f\omega\big(\frac{\pi}{n}\big)}{k_2 - p_0} + M_f \sum_{k=p_0+m+1}^{k_2-1} \frac{v(k - p_0 - m)}{(p_0-k)(p_0-k-1)} \Bigg) \\ \leqslant &2M_f \Bigg( \sum_{k=m+1}^{p_0-k_1-1} \frac{v(k-m)}{k(k+1)} + \sum_{k=m+1}^{k_2-p_0-1} \frac{v(k-m)}{k(k+1)} \Bigg) + 10K_f\omega\Big(\frac{\pi}{n}\Big) \\ \leqslant &4M_f \sum_{k=m+1}^{k_2-k_1-1} \frac{v(k)}{k^2} + 10K_f\omega\Big(\frac{\pi}{n}\Big). \end{aligned}$$

Hence, by (55), (57) and (58) we have

$$0 \leqslant S_2(p_0) \leqslant 10K_f\omega\Big(\frac{\pi}{n}\Big) \sum_{k=1}^{m} \frac{1}{k} + 4M_f \sum_{k=m+1}^{k_2-k_1-1} \frac{v(k)}{k^2} + 10K_f\omega\Big(\frac{\pi}{n}\Big).$$

Thanks to the non-negativity of both terms, condition (6) is equivalent to

$$\lim_{n\to\infty}\min_{1\leqslant m\leqslant k_2-k_1-1}\max\Big\{\omega\Big(\frac{\pi}{n}\Big)\sum_{k=1}^{m}\frac{1}{k},\sum_{k=m+1}^{k_2-k_1-1}\frac{v(k)}{k^2}\Big\}=0.$$

This is why there exists a number $n_3\in\mathbb{N}$, $n_3\geqslant n_2$, such that for an arbitrary $n\geqslant n_3$ there exists $m$ obeying $1\leqslant m\leqslant k_2-k_1-1$, for which the inequality holds

$$0\leqslant S_2(p_0)\leqslant\frac{\tilde\epsilon}{3}. \tag{59}$$

By (49), (50), (51), (54) and (59) we obtain that for an arbitrary $\tilde\epsilon>0$ there exists an index $n_3\in\mathbb{N}$ such that for each $n>n_3$ there exists $m$ obeying $1\leqslant m\leqslant k_2-k_1-2$, for which the inequalities

$$Q_n^*(f,[a,b],\varepsilon)\leqslant Q_n^{**}(f,[a,b],\varepsilon)<\tilde\epsilon$$

are true. Now, in the case$f\in C(\omega^l[a,b]\cap V^-(v)$, Theorem 1 follows Proposition 1.

To prove Theorem 1 in the case $f\in C(\omega^r[a,b])\cap V^+(v)$, it is sufficient to observe that if $f\in C(\omega^r[a,b])\cap V^+(v)$, then $-f\in C(\omega^l[a,b])\cap V^-(v)$ and the operator $L_n^{SL}(f,\cdot)$ is linear. The proof is complete. □

**Remark 6.** *In the case $f\in C(\omega^l[a,b])\cap V(v)$ or $f\in C(\omega^r[a,b])\cap V(v)$ ($v$ is the majorant of the classical variation modulus $v(n,f)$) there was established in* [30] *that condition* (6) *for the uniform convergence of trigonometric interpolation polynomials and algebraic Lagrange interpolation in the case of matrix of Chebyshev interpolation nodes.*

*In paper* [31]*, the uniform convergence was established for the classical trigonometric Fourier series of $2\pi$-periodic functions $f\in C(\omega[a,b])\cap V(v)$, where the functions $\omega$ and $v$ are the classic continuity modulus and and the variation modulus of the function $f$.*

**Remark 7.** *It follows from Theorem 1 that if $f_1\in C(\omega_1^r[a,b])\cap V^+(v_1)$ and $f_2\in C(\omega_2^l[a,b])\cap V^-(v_2)$ and the pairs of functions $(v_i,\,\omega_i)$, $i=1,2$, satisfy relation* (6)*, then, despite the linear combination $f=\alpha f_1+\beta f_2$ can be out of each of these classes, the Lagrange-Sturm-Liouville interpolation process approximates the function $f$, see* (7)*.*

**Remark 8.** *Each of the Dini-Lipschitz class of functions ( $\lim\limits_{n\to\infty}\omega(f,1/n)\ln n=0$, see* [1]*)) and the Krylov class (continuous functions of bounded variation) are proper subset of the functional class defined by relation* (6)*.*

**Remark 9.** *If $f\in C[0,\pi]$, the two-sided estimates hold:*

$$v^+(n,f)\leqslant v(n,f)\leqslant 2\left(v^+(n,f)+\|f\|_{C[0,\pi]}\right),$$
$$-v^-(n,f)\leqslant v(n,f)\leqslant 2\left(-v^-(n,f)+\|f\|_{C[0,\pi]}\right).$$

**Corollary 2.** *It follows from Theorem 1 that each of the conditions $\lim\limits_{n\to\infty}\omega^l(f,1/n)\ln n=0$ or $\lim\limits_{n\to\infty}\omega^r(f,1/n)\ln n=0$ implies condition* (7)*.*

**Corollary 3.** *If a non-decreasing concave function of a natural variable $v$ is such that*

$$\sum_{k=1}^{\infty}\frac{v(k)}{k^2}<\infty, \tag{60}$$

*then for each function $f\in C[0,\pi]\cap V^{\pm}(v)$ relation* (7) *holds.*

*Proof.* Indeed, the continuity of the function $f$ implies the existence of a sequence of natural numbers $\{m_n\}_{n=1}^{\infty}$ satisfying simultaneously two conditions $\lim\limits_{n\to\infty}m_n=\infty$ and $\lim\limits_{n\to\infty}\omega(f,\pi/n)\ln m_n=0$. Therefore, the convergence of series (60) ensures condition (6) for each function $f$ belonging to one of the classes $C[0,\pi] \cap V^+(v)$ or $C[0,\pi] \cap V^-(v)$. The proof is complete. □

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Aleksandr Yurievich Trynin,
Saratov State University named after N.G. Chernyshevskii,
Astrakhanskaya str. 83
410012, Saratov, Russia
E-mail: `atrynin@gmail.com`