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NEVANLINNA'S FIVE-VALUE THEOREM FOR ALGEBROID FUNCTIONS

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Abstract. By using the second main theorem of the algebroid function, we study the following problem. Let $W_1(z)$ and $W_2(z)$ be two ν-valued non-constant algebroid functions, $a_j\,(j=1,2,\ldots,q)$ be $q \geqslant 4\nu+1$ distinct complex numbers or ∞. Suppose that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q, m$ are positive integers or ∞, $1 \leqslant m \leqslant q$ and $\delta_j \geqslant 0$, $j=1,2,\ldots,q$, are such that

$$\left(1+\frac{1}{k_m}\right)\sum_{j=m}^{q}\frac{1}{1+k_j}+3\nu+\sum_{j=1}^{q}\delta_j<(q-m-1)\left(1+\frac{1}{k_m}\right)+m.$$

Let $B_j=\overline{E}_{k_j}(a_j,f)\backslash\overline{E}_{k_j}(a_j,g)$ for $j=1,2,\ldots,q$. If

$$\overline{N}_{B_j}(r,\frac{1}{W_1-a_j})\leqslant\delta_j T(r,W_1)$$

and

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_1-a_j})}{\sum\limits_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_2-a_j})}>\frac{\nu k_m}{(1+k_m)\sum\limits_{j=1}^{q}\frac{k_j}{k_j+1}-2\nu(1+k_m)+(m-2\nu-\sum\limits_{j=1}^{q}\delta_j)k_m},$$

then $W_1(z)\equiv W_2(z)$. This result improves and generalizes the previous results given by Xuan and Gao.

Keywords: value distribution theory, Nevanlinna theory, algebroid functions, uniqueness.

Subject Classification: 30D35

1. Introduction

The value distribution theory of meromorphic functions was extended to the corresponding theory of algebroid functions by Ullarich [1] and Valiron [2] around 1930, and important results on uniqueness for algebroid functions were obtained. It is well known that Valiron obtained a famous $(4\nu+1)$-valued theorem. The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. Many researchers like Valiron [2], Baganas [3], He [4] and others ([6],[7],[9-27]) made lot of work in this area. In this article, we extend a result by Indrajit Lahiri and Rupa Pal [5] in the Nevanlinna's value distribution theory of meromorphic functions on Nevanlinna's five values theorem to algebroid functions

Let $A_\nu(z), A_{\nu-1}(z), \ldots, A_0(z)$ be analytic functions with no common zeros in the complex plane and consider the equation

$$A_\nu(z)W^\nu+A_{\nu-1}(z)W^{\nu-1}+\ldots+A_1(z)W+A_0(z)=0. \tag{1}$$

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This equation defines a ν-valued algebroid function $W(z)$ [8].

It is well known [8] that on the complex plane with the projection of the critical points of the function W cut out, the Nevanlinna characteristic $T(r, W)$ is defined as

$$T(r, W) = m(r, W) + N(r, W),$$

where

$$m(r, W) = \frac{1}{2\pi\nu}\sum_{j=1}^{\nu}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N(r, W) = \frac{1}{\nu}\int_0^r \frac{n(t, W) - n(0, W)}{t}dt + \frac{n(0, W)}{\nu}\log r.$$

Let $w_i(z)$ and $m_j(z)$ be one-valued branches of two (μ-valued and ν-valued) algebroid functions. Following Prokopovich [15], we consider their quotient in the domain of the complex plane with the projection of the critical points of both functions cut out. The one-valued branches of th function W/M $(W \cdot M)$ are defined as w_i/m_j $(w_i \cdot m_j)$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. The Nevanlinna's characteristic $T(r, W/M)$ is defined by $T(r, W) + T(r, M)$.

Lemma 1 (8)**.** *Let $W(z)$ be a ν-valued algebroid function and $\{a_j\}_{j=1}^{q} \subset \overline{\mathbb{C}}$ be q distinct complex numbers and let $\{k_j\}_{j=1}^{q} \subset \mathbb{N}$ be q positive integers. Then*

$$(q - 2\nu)T(r, W) \leqslant \sum_{k=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r, W = a_j) + \sum_{k=1}^{q}\frac{1}{k_j+1}N(r, W = a_j) + S(r, W),$$

$$\left(q - 2\nu - \sum_{k=1}^{q}\frac{1}{k_j+1}\right)T(r, W) \leqslant \sum_{k=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r, W = a_j) + N(r, W = a_j) + S(r, W).$$

In 2006 Zu-Xing Xuan and Zong-Sheng Gao [18] improved this statement as follows.

Theorem 1. *Let $W(z)$ and $M(z)$ be two ν-valued non-constant algebroid functions, let $a_j\,(j = 1, 2, \ldots, 4\nu + 1)$ be $4\nu + 1$ distinct complex numbers in $\overline{\mathbb{C}}$. If*

$$\overline{E}_{2\nu+1)}(a_j, W) = \overline{E}_{2\nu+1)}(a_j, M), \qquad j = 1, 2, \ldots, 2\nu + 1$$

and

$$\overline{E}_{2\nu)}(a_j, W) = \overline{E}_{2\nu)}(a_j, M), \qquad j = 1, 2, \ldots, 4\nu + 1,$$

then $W(z) = M(z)$

2. Main Results

Let $W(z)$ be a ν-valued algebroid function and $a \in \overline{\mathbb{C}}$ be a complex number. The symbol $\overline{E}_k(W = a)$ denotes the set of zeros of $W(z) - a$, whose multiplicities are not greater than k. The symbol $\overline{n}_{k)}(W = a)$ stands for the number of distinct zeros of $W(z) - a$ in $|z| \leqslant r$, whose multiplicities do not exceed k and are counted only once. Similarly, we define the functions $\overline{n}_{(k+1}(r, W = a)$, $\overline{N}_{k)}(r, W = a)$ and $\overline{N}_{(k+1}(r, W = a)$.

In this paper, we study the problem on the Nevanlinna's five value theorem for algebroid functions. To state our main theorem, we first introduce the following definition.

Definition 1. *For $B \subset \mathbb{A}$ and $a \in \overline{\mathbb{C}}$, we denote by $\overline{N}_B(r, \frac{1}{f-a})$ the reduced counting function of the zeros of $f - a$ on $\mathbb{A}$ belonging to the set B.*

Theorem 2. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued non-constant algebroid functions, let $a_j\,(j = 1, 2, \ldots, q)$ be $q \geqslant 4\nu + 1$ distinct complex numbers or ∞. Suppose that*

$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q, m$ are positive integers or ∞; $1 \leqslant m \leqslant q$ and $\delta_j \geqslant 0$, $j = 1, 2, \ldots, q$, are such that

$$\left(1+\frac{1}{k_m}\right)\sum_{j=m}^{q}\frac{1}{1+k_j}+3\nu+\sum_{j=1}^{q}\delta_j<(q-m-1)\left(1+\frac{1}{k_m}\right)+m. \tag{2}$$

Let $B_j = \overline{E}_{k_j}(a_j, f)\backslash\overline{E}_{k_j}(a_j, g)$ for $j = 1, 2, \ldots, q$. If

$$\overline{N}_{B_j}(r, \frac{1}{W_1-a_j}) \leqslant \delta_j T(r, W_1) \tag{3}$$

and

$$\liminf_{r\to\infty}\frac{\sum\limits_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_1-a_j})}{\sum\limits_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_2-a_j})}>\frac{\nu k_m}{(1+k_m)\sum\limits_{j=1}^{q}\frac{k_j}{k_j+1}-2\nu(1+k_m)+(m-2\nu-\sum\limits_{j=1}^{q}\delta_j)k_m} \tag{4}$$

then $W_1(z) \equiv W_2(z)$.

Proof. Suppose that $W_1(z) \neq W_2(z)$. The by Lemma 1 for each integer m, $1 \leqslant m \leqslant q$, we have

$$\begin{aligned}
(q-2\nu)\,T(r,W_1) &\leqslant \sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W_1-a_j}\right)+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+\overline{N}_{(k_{j+1}}\left(r,\frac{1}{W_1-a_j}\right)\right\}+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+\frac{1}{1+k_j}N_{(k_{j+1}}\left(r,\frac{1}{W_1-a_j}\right)\right\}+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\left\{\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+\frac{1}{1+k_j}N\left(r,\frac{1}{W_1-a_j}\right)\right\}+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+\sum_{j=1}^{q}\frac{1}{1+k_j}T(r,W_1)+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\left(\frac{k_j}{1+k_j}-\frac{k_m}{1+k_m}\right)\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+\sum_{j=1}^{q}\frac{1}{1+k_j}T(r,W_1)\\
&\quad+\sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+S(r,W_1)\\
&\leqslant \sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)\\
&\quad+\left(m-1-\frac{(m-1)k_m}{k_m+1}+\sum_{j=1}^{q}\frac{k_j}{1+k_j}\right)T(r,W_1)+S(r,W_1)
\end{aligned}$$

Therefore

$$\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu+\frac{(m-1)k_m}{k_m+1}\right)T(r,W_1)\leqslant\sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)+S(r,W_1). \tag{5}$$

Similarly,

$$\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu+\frac{(m-1)k_m}{k_m+1}\right)T(r,W_2)\leqslant\sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}\left(r,\frac{1}{W_2-a_j}\right)+S(r,W_2). \quad (6)$$

Since $B_j=\overline{E}_{k_j)}(a_j,W_1)\ \overline{E}_{k_j)}(a_j,W_2)$, let $D_j=\overline{E}_{k_j)}(a_j,W_1)\ B_j$ for $j=1,2,\ldots,q$. Thus, by (5) and (6), for a sequence of values of r tending to ∞ we get:

$$\begin{aligned}\sum_{j=m}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)&=\sum_{j=m}^{q}\overline{N}_{B_j)}\left(r,\frac{1}{W_1-a_j}\right)+\sum_{j=m}^{q}\overline{N}_{D_j)}\left(r,\frac{1}{W_1-a_j}\right)\\&\leqslant\sum_{j=m}^{q}\delta_jT(r,W_1)+\nu N\left(r,\frac{1}{W_1-W_2}\right)\\&\leqslant\left(\nu+\sum_{j=m}^{q}\delta_j\right)T(r,W_1)+\nu T(r,W_2)+O(1)\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu+\frac{(m-1)k_m}{k_m+1}+O(1)\right)\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)\\&\leqslant\left(\nu+\sum_{j=m}^{q}\delta_j\right)\sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)\\&\quad+(\nu+O(1))\sum_{j=1}^{q}\frac{k_m}{k_m+1}\overline{N}_{k_j)}\left(r,\frac{1}{W_2-a_j}\right).\end{aligned} \quad (7)$$

Since

$$1\geqslant\frac{k_1}{k_1+1}\geqslant\ldots\geqslant\frac{k_q}{k_q+1}\geqslant\frac{1}{2},$$

by (7), for a sequence of values of r tending to $+\infty$, we get

$$\begin{aligned}&\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu+\frac{(m-1)k_m}{k_m+1}-\frac{k_m}{k_m+1}\left(\nu+\sum_{j=m}^{q}\delta_j\right)+O(1)\right)\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)\\&\leqslant(\nu+O(1))\frac{k_m}{k_m+1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_2-a_j}\right).\end{aligned}$$

This implies that

$$\begin{aligned}\liminf_{r\to\infty}\frac{\overline{N}_{k_j)}\left(r,\frac{1}{W_1-a_j}\right)}{\overline{N}_{k_j)}\left(r,\frac{1}{W_2-a_j}\right)}&\leqslant\frac{\nu\frac{k_m}{k_m+1}}{\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu+\frac{k_m}{k_m+1}\left(m-2\nu-\sum_{j=m}^{q}\delta_j\right)\right)},\\&\leqslant\frac{\nu k_m}{(1+k_m)\left(\sum_{j=m}^{q}\frac{k_j}{k_j+1}-2\nu(1+k_m)+k_m\left(m-2\nu-\sum_{j=m}^{q}\delta_j\right)\right)}.\end{aligned}$$

This i contradicts equation (2). Thus, we have $f(z)\not\equiv g(z)$. The proof is complete. □

Theorem 2 yield the following corollaries.

Corollary 1. *Let $m = 1$, $k_j = \infty$ for $j = 1, 2, 3, \ldots, q$ and*

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_{k_j)}\left(r, \frac{1}{W_1-a_j}\right)}{\overline{N}_{k_j)}\left(r, \frac{1}{W_2-a_j}\right)} > \frac{1}{q-2\nu+1}$$

If $\overline{N}_{B_j}(r, \frac{1}{W_1-a_j}) \leqslant \delta_j T(r, W_1)$, where $\delta \geqslant 0$ satisfies

$$0 \leqslant \sum_{j=1}^{q} \delta_j < k - (2\nu+1) - \frac{1}{\gamma},$$

then $f(z) \equiv g(z)$

If we take $q = 4\nu+1$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, f)$, then $B_j = \emptyset$ for $j = 1, 2, \ldots, 4\nu+1$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, \ldots, 4\nu+1$ and take any constant γ obeying $0 \leqslant 2\nu - \frac{1}{\gamma}$ in Corollary 1, we can get that $f \equiv g$. Moreover, if $q = 4\nu+1$ and $\overline{E}(a_j, f) = \overline{E}(a_j, g)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, \ldots, 4\nu+1$; this implies $f \equiv g$. Then Corollary 1 is an improvement of Theorem 1.

Corollary 2. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued non-constant algebroid functions, let a_j, $j = 1, 2, \ldots, q$, be $q \geqslant 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, \ldots, k_q$ are positive integers or ∞ with $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q$ if $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ and*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{\gamma(k_1+1)} - 2\nu > 0,$$

where γ is as stated in Corollary 1. Then $f(z) \equiv g(z)$.

Corollary 3. *Under the assumptions of Corollary 2, we have $\overline{E}_{k_j)}(a_j, W_1) = \overline{E}_{k_j)}(a_j, W_2)$ and*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{(k_1+1)} - 2\nu > 0,$$

Corollary 4. *Let $W_1(z)$ and $W_2(z)$ be two ν-valued non-constant algebroid functions, let a_j, $j = 1, 2, \ldots, q$, be $q \geqslant 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, \ldots, k_q$ are positive integers or ∞ with $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_q$ if $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ and*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - 2\nu + \frac{(m-2\nu-\frac{1}{\gamma})k_m}{\gamma(k_m+1)} - 2\nu > 0,$$

where γ is as stated a in Corollary 1. Then $f(z) \equiv g(z)$.

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