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SYSTEMS OF CONVOLUTION EQUATIONS IN COMPLEX DOMAINS

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Abstract. In this paper we study the systems of convolution equations in spaces of vector-valued functions of one variable. We define an analogue of the Leontiev interpolating function for such systems, and we provide a series of the properties of this function. In order to study these systems, we introduce a geometric difference of sets and provide its properties.

We prove a theorem on the representation of arbitrary vector-valued functions as a series over elementary solutions to the homogeneous system of convolution equations. These results generalize some well-known results by A.F. Leontiev on methods of summing a series of elementary solutions to an arbitrary solution and strengthen the results by I.F. Krasichkov-Ternovskii on summability of a square system of convolution equations.

We describe explicitly domains in which a series of elementary solutions converges for arbitrary vector-valued functions. These domains depend on the domains of the vector-valued functions, on the growth of the Laplace transform of the elements in this system, and on the lower bound of its determinant. We adduce examples showing the sharpness of this result.

Similar results are obtained for solutions to a homogeneous system of convolution equations, and examples are given in which the series converges in the entire domain of a vector-valued function.

Keywords: Systems of convolution equations, vector-valued functions, Leontiev interpolating function, series of elementary solutions.

Mathematics Subject Classification: 30B50

1. Introduction

Let $q, r \in \mathbb{N}$, $U_1, U_2, \ldots, U_q$ be domains in the complex plane, $H(U_j)$ be the space of the functions holomorphic in the domain U_j with the topology of uniform convergence on compact sets, S_j^p be linear continuous functionals on the space $H(U_j), j = 1, \ldots, q$, $p = 1, \ldots, r$.

We consider the system of convolution equations

$$\sum_{j=1}^{q} \left\langle S_j^p, f_j(z+h) \right\rangle = 0, \qquad p = 1, \ldots, r, \qquad f = (f_1, \ldots, f_q) \in \prod_{j=1}^{q} H(U_j).$$

In works by I.F. Krasichkov-Ternovskii [1]–[7], the approximation of an arbitrary solution to this system by linear combinations of elementary solutions was studied, while works [8]–[10] were devoted to the summability of a series of elementary solutions.

In the present paper we extend the known results by A.F. Leontiev on methods of the summation of series of elementary solutions to an arbitrary solution in the case $q = 1$ and one

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convolution equation [11] to the solutions of the above system and we strengthen some results by I.F. Krasichkov-Ternovskii on summability.

2. Notations, preliminaries and results

Given a set M in the complex plane, by $\mathrm{conv}M$, $\mathrm{int}\, M$, $\overline{M}$, M_w, where $w \in \mathbb{C}$, we denote respectively its convex hull, interior, closure and a connected component containing the point w.

The sum and geometric difference of sets $M_1, M_2 \subset \mathbb{C}$ are defined respectively as the sets

$$M_1 + M_2 = \{z_1 + z_2 : z_1 \in M_1, z_2 \in M_2\}, \qquad M_1 \stackrel{*}{-} M_2 = \{z \in \mathbb{C} : z + M_2 \subset M_1\}.$$

For the operations with empty sets we obviously have

$$M + \varnothing = \varnothing, \quad M \stackrel{*}{-} \varnothing = \mathbb{C}, \quad \varnothing \stackrel{*}{-} M_1 = \varnothing, \ M_1 \neq \varnothing.$$

The support function of a set $M \subset \mathbb{C}$ is defined by the formula

$$h(\theta, M) = \sup_{a \in M} \mathrm{Re}(ae^{-i\theta}), \quad \theta \in [0, 2\pi].$$

This function possesses the following properties (see [12], [13]):

Lemma 1. *The following properties are true:*

1) $h(\theta, M) = h(\theta, \mathrm{conv}\overline{M})$.
2) $(h(\theta, M_1) \leqslant h(\theta, M_2), \ \theta \in [0, 2\pi]) \iff \big(M_1 \subset \overline{\mathrm{conv}M_2}\big)$.
3) $h(\theta, M_1 + M_2) = h(\theta, M_1) + h(\theta, M_2)$.

Lemma 2. *The difference of sets possesses the following relations:*

1) $(M + K \subset U) \iff (M \subset U \stackrel{*}{-} K)$.
2) $(U_1 \subset U_2, \ K_1 \supset K_2) \implies (U_1 \stackrel{*}{-} K_1 \subset U_2 \stackrel{*}{-} K_2)$.
3) $[(U_1 + U_2) \stackrel{*}{-} K] \supset [(U_1 \stackrel{*}{-} K) + U_2]$.
4) $U \stackrel{*}{-} (K_1 + K_2) = (U \stackrel{*}{-} K_1) \stackrel{*}{-} K_2$.
5) For arbitrary sets of indices A and B the identity holds:

$$\bigcap_{\alpha \in A, \beta \in B} (U_\alpha \stackrel{*}{-} K_\beta) = \left(\bigcap_{\alpha \in A} U_\alpha\right) \stackrel{*}{-} \left(\bigcup_{\beta \in B} K_\beta\right).$$

6) If sets U_n are open and the sets K_m are compact and

$$U_n \subset U_{n+1}, \qquad K_m \supset K_{m+1}, \qquad n, m = 1, 2, \ldots,$$

then

$$\bigcup_{n,m} (U_n \stackrel{*}{-} K_m) = \left(\bigcup_n U_n\right) \stackrel{*}{-} \left(\bigcap_m K_m\right).$$

7) If a set U is open and a set K is compact, then the set $U \stackrel{*}{-} K$ is open.
8) If a set U is convex, then the set $U \stackrel{*}{-} K$ is convex and

$$U \stackrel{*}{-} K = U \stackrel{*}{-} \mathrm{conv}K.$$

9) For sets $U, K \subset \mathbb{C}$, $K \neq \varnothing$, the inequality holds:

$$h(\theta, U \stackrel{*}{-} K) \leqslant h(\theta, U) - h(\theta, K), \quad \theta \in [0, 2\pi].$$

10) If a convex set U is either closed or open and a set K is compact and non-empty, then

$$(U + K) \stackrel{*}{-} K = U.$$

11) Let an open set U be simply-connected, then all connected components of the set $U \stackrel{*}{-} K$ are also simply-connected.

Proof. Properties 1)–7) are easily implied by the definition of the difference of sets and the properties of compact sets.

8) The convexity of the set $U \overset{*}{-} K$ follows property 5):

$$U \overset{*}{-} K = \bigcap_{z\in K} (U - z).$$

Since the set U is convex, then

$$(z + K \subset U) \Longleftrightarrow (z + \operatorname{conv} K) \subset U.$$

Property 9) is implied by

$$(U \overset{*}{-} K) + K \subset U$$

and property 3) of support functions.

10) By the previous property,

$$\begin{aligned} h(\theta, (U + K) \overset{*}{-} K) \leqslant & h(\theta, U + K) - h(\theta, K) \\ = & h(\theta, U) + h(\theta, K) - h(\theta, K) = h(\theta, U), \qquad \theta \in [0, 2\pi]\,, \end{aligned}$$

and this is why by property 2) of support functions in the case of a closed set U we have the relation

$$(U + K) \overset{*}{-} K \subset U.$$

The opposite inclusion is also obvious.

If the set U is open, it can be exhausted by an increasing sequence of convex compact sets and the desired identity can be easily obtained by property 6).

We observe that if the set U is non-convex, the latter property is generally speaking not true. For instance, let

$$U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}\,, \ \ K = \{z \in \mathbb{C} : -1 \leqslant \operatorname{Re} z \leqslant 1, \ \operatorname{Im} z = 0\}\,.$$

As one can easily confirm,

$$U + K = \mathbb{C},$$

and

$$\mathbb{C} = (U + K) \overset{*}{-} K \neq U.$$

11) Let $C \subset U \overset{*}{-} K$ be a closed contour enveloping the point z. In this case for each point $w \in K$ we have $w + C \subset U$ and the closed contour $w + C$ envelops the point $w + z$.

Since the domain U is simply-connected, we have

$$w + z \in U,$$

and this implies easily the desired statement. □

Let U and K be respectively an open and a compact subsets of the complex plane. By $H(U)$ and $H(K)$ we denoted respectively the space of holomorphic functions in the domain U and the space of holomorphic germs on the compact set K with natural topologies. By $H^*(U)$ and $H^*(K)$ we denote the space of linear continuous functional respectively on the spaces $H(U)$ and $H(K)$ with strong topologies.

As it is known [14], given an arbitrary functional $S \in H(\mathbb{C})$, there exists a compact set $K \subset \mathbb{C}$ and a function γ holomorphic outside K and obeying $\gamma(\infty) = 0$ such that

$$\langle S, f\rangle = \frac{1}{2\pi i}\int_C \gamma(t) f(t) dt, \quad f \in H(\mathbb{C}),$$

where C is a contour enveloping the compact set K.

The Laplace transform $\widehat{S}(\lambda)$ of a functional S is defined by the formula

$$\widehat{S}(\lambda) = \left\langle S_z, e^{\lambda z}\right\rangle$$

and is an entire functional of exponential type. The smallest convex compact sets containing all singularities of the function γ is called the conjugate diagram of the function $\widehat{S}(\lambda)$.

Vice versa, for each entire function of exponential type there exists a functional in the space $H^*(\mathbb{C})$, whose Laplace transform coincides with this function.

Let a compact set $K \subset \mathbb{C}$ be a conjugate diagram of the function $\widehat{S}(\lambda)$. If a domain $U \subset \mathbb{C}$ is such that $K \subset U$, in the space $H(U)$ we can define the convolution operator $S*$ with the characteristic function $\widehat{S}(\lambda)$ by the rule:

$$(S * f)(z) = \frac{1}{2\pi i} \int_C \gamma(t) f(t+z)dt, \qquad f \in H(U).$$

As one can show easily, this operator maps the space $H(U)$ into the space $H\left([U \stackrel{*}{-} K]_0\right)$ linearly and continuously.

If a functional F belongs to the space $H^*(\mathbb{C})$, a compact set R is a conjugate diagram of the function $\widehat{F}$ and $K + R \subset U$, then we can defined the convolution of the functionals F and S as a functional $F * S$ on the space $H(U)$ acting by the formula

$$\langle F * S, f \rangle = \langle F, S * f \rangle, \qquad f \in H(U).$$

It was shown in monograph [15] that this is a linear and continuous functional on the space $H(U)$. It is easy to see that it satisfies the relations:

$$F * S = S * F, \qquad \widehat{F * S} = \widehat{F}\widehat{S}.$$

Let U_j be domains in the complex plane, $S_j^p \in H^*(U_j)$, $\varphi_j^p(\mu) = \widehat{S_j^p}(\mu)$, compact sets $K_j^p \subset U_j$ are conjugate diagrams of the functions $\varphi_j^p(\mu)$, $j = 1, \ldots, q$, $p = 1, \ldots, r$, and

$$\varphi(\mu) = \begin{pmatrix} \varphi_1^1(\mu) & \varphi_1^2(\mu) & \ldots & \varphi_1^q(\mu) \\ \varphi_2^1(\mu) & \varphi_2^2(\mu) & \ldots & \varphi_2^q(\mu) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_r^1(\mu) & \varphi_r^2(\mu) & \ldots & \varphi_r^q(\mu) \end{pmatrix}.$$

On the space $\prod_{j=1}^q H(U_j)$ we define a linear continuous convolution operator $S*$ with values in the space

$$\prod_{p=1}^r H\left(\left[\bigcap_{j=1}^q \left(U_j \stackrel{*}{-} K_j^p\right)\right]_0\right)$$

by the formula:

$$(S * f)_p = \sum_{j=1}^q S_j^p * f_p,$$

where $(S * f)_p$ is the pth component of the vector function $S * f$, $p = 1, \ldots, r$.

We consider the homogeneous system of convolution equations

$$S * f = 0. \tag{1}$$

A solution to this system is elementary if it is represented as

$$\sum_{m=1}^s e^{\lambda z} z^m c_m, \qquad s \in \mathbb{N},$$

where $c_m \in \mathbb{C}^r$, $m = 1, \ldots, s$. The number λ is called the exponent of this solution.

The rank of system (1) is the number

$$\operatorname{rg} S = \max_{\lambda \in \mathbb{C}} \operatorname{rg} \varphi(\lambda).$$

Let F_p^m be linear continuous functionals on the space of entire functions, compact sets R_p^m are conjugate diagrams of the functions $\widehat{F}_p^m$, $p = 1, \dots, r$, $m = 1, \dots, l$, $l \in \mathbb{N}$, and

$$\operatorname{conv} \bigcup_{p=1}^{r} \left(R_p^m + K_j^p\right) \subset U_j, \qquad m = 1, \dots, l, \qquad j = 1, \dots, q.$$

In this case it is easy to show that the matrix of the functionals $F * S$ with entry $\sum_{p=1}^r F_p^m * S_j^p$ at the jth column and mth row generates the convolution operator

$$(F * S)* : \prod_{j=1}^{q} H(U_j) \to \prod_{m=1}^{l} H\left(\left\{\bigcap_{j=1}^{q}\left[U_j \overset{*}{-} \operatorname{conv}\bigcup_{p=1}^{r}\left(R_p^m + K_j^p\right)\right]\right\}_0\right),$$

and

$$(F * S) * f = F * (S * f)\,, f \in \prod_{j=1}^{q} H(U_j), \qquad \widehat{F * S} = \widehat{F}\widehat{S}.$$

Assume that system (1) satisfy the identities

$$q = p = \operatorname{rg} S = n. \tag{2}$$

We introduce the following notations: $L(\lambda) = \det \varphi(\lambda)$, $\varphi^*(\lambda)$ is the adjunct of $\varphi(\lambda)$, B_m^j are conjugate diagrams of the entries of the matrix $\varphi^*(\lambda)$,

$$K_m = \operatorname{conv}\bigcup_{j=1}^{n} K_m^j, \;\; B^j = \operatorname{conv}\bigcup_{m=1}^{n} B_m^j, \qquad m, j = 1, \dots, n,$$

K is the conjugate diagram of the function $L(\lambda)$.

The properties of the adjunct imply easily the following relations:

$$B^j \subset \sum_{p \neq j} K_p, \quad \operatorname{conv}\bigcup_{j=1}^{n}\left(K_m^j + B_j^m\right) \supset K, \quad j, m = 1, \dots, n. \tag{3}$$

Hereafter we assume that $K_p \subset U_p$, $p = 1, \dots, n$.

For a convex compact set $B \subset K$ we define sets $(U, \varphi, B)_p$ as the unions of the sets

$$\bigcap_{j=1}^{n}\left[(B + R_j) \overset{*}{-} B_j^p\right] \tag{4}$$

over all systems of convex compact sets $(R_1, \dots, R_n)$ such that for some simply-connected domains $G_p \subset U_p$

$$\begin{gathered} \operatorname{conv}\bigcup_{j=1}^{n}\left(R_j + K_m^j\right) \subset G_m, \qquad m = 1, \dots, n, \\ R_j \subset \left[\bigcap_{m=1}^{n}\left(G_m \overset{*}{-} K_m^j\right)\right]_0, \qquad j = 1, \dots, n, \qquad p = 1, \dots, n. \end{gathered} \tag{5}$$

It is clear that the sets $(R_1, \dots, R_n)$ with conditions (5) can be varied a little and this is why the sets $(U, \varphi, B)_p$, $p = 1, \dots, n$, are obviously open.

Lemma 3. *The following relations hold:*

1) $(U, \varphi, B)_p \subset U_p$, $p = 1, \dots, n$.

2) *For simply-connected domains $G_p \subset U_p$ and each convex subdomain D of the domain*

$$\left[\bigcap_{m=1}^{n}\left(G_m \overset{*}{-} K_m\right)\right]_0$$

the inclusion holds:

$$(B+D) \stackrel{*}{-} B^p \subset (U,\varphi,B)_p\,, \quad p=1,\dots,n.$$

3) *For convex domains U_p the identities hold:*

$$(U,\varphi,B)_p = \bigcap_{j=1}^{n} \left\{ \left[B + \bigcap_{m=1}^{n} \left(U_m \stackrel{*}{-} K_m^j\right) \right] \stackrel{*}{-} B_j^p \right\}, \qquad p=1,\dots,n.$$

Proof. It is clear that it is sufficient to restrict ourselves by the case $p=1$.

1) Assume that a point z belongs to the set $(U,\varphi,B)_1$. In this case there exists a system of convex compact sets $(R_1,\dots,R_n)$ with property (5) so that the point z belongs to set (4). We have

$$z + B_j^1 \subset B + R_j \subset K + R_j, \quad j=1,\dots,n. \tag{6}$$

We denote by M_1 the left hand side in the first relation in (5). It is clear that this set is a convex compact set and

$$R_j + K_1^j \subset M_1,$$

or

$$R_j \subset M_1 \stackrel{*}{-} K_1^j,$$

and this is why by relations (6) we obtain

$$z + B_j^1 \subset K + (M_1 \stackrel{*}{-} K_1^j), \quad j=1,\dots,n.$$

By property 3) of the difference of sets

$$K + (M_1 \stackrel{*}{-} K_1^j) \subset (K+M_1) \stackrel{*}{-} K_1^j, \quad j=1,\dots,n,$$

therefore,

$$z \in \bigcap_{j=1}^{n} \left\{ \left[(K+M_1) \stackrel{*}{-} K_1^j\right] \stackrel{*}{-} B_j^1 \right\} = \bigcap_{j=1}^{n} \left[(K+M_1) \stackrel{*}{-} \left(K_1^j + B_j^1\right)\right]$$

$$\subset (K+M_1) \stackrel{*}{-} \bigcup_{j=1}^{n} \left(K_1^j + B_j^1\right) \subset (K+M_1) - K = M_1.$$

Here we have applied properties 2), 4), 5), 10) of the difference of sets and relations (3).

Since the set M_1 lies in the domain G_1, this proves the desired statement.

2) If R is a convex compact set in the domain D, by property 2) of the difference of sets we have

$$R \subset \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m) \right]_0 \subset \left[\bigcap_{m=1}^{n} \left(G_m \stackrel{*}{-} K_m^j\right) \right]_0, \quad j=1,\dots,n,$$

and by property 1) thanks to the convexity of the compact set R we get

$$G_m \supset R + K_m = \operatorname{conv} \bigcup_{j=1}^{n} \left(R + K_m^j\right), \ m=1,\dots,n.$$

In this case, if we let $R_p = R$, $p=1,\dots,n$, the definition of the set $(U,\varphi,B)_1$ yields

$$\bigcap_{j=1}^{n} \left[(B+R) \stackrel{*}{-} B_j^1\right] \subset (U,\varphi,B)_1\,,$$

and the desired statement follows properties 5), 6) and 8) of the difference of sets.

3) The inclusion "$\subset$" is implied immediately by the definition and property 2) of the difference of sets.

In the case of convex domains U_p, they serve as the domains G_p, $p=1,\dots,n$, and as one can show easily, conditions (5) on the sets R_j are equivalent to the relation

$$R_j \subset \bigcap_{m=1}^{n} \left(U_m \stackrel{*}{-} K_m^j\right), \qquad j=1,\dots,n.$$

Let R_j^l, $l \in \mathbb{N}$, be a sequence of convex compact sets such that

$$R_j^l \subset \operatorname{int} R_j^{l+1},\ l \in \mathbb{N}, \quad \bigcup_{l=1}^{\infty} R_j^l = \bigcap_{m=1}^{n} \left(U_m \stackrel{*}{-} K_m^j\right), \qquad j=1,\dots,n.$$

In this case

$$(U,\varphi,B)_1 \supset \bigcup_{l=1}^{\infty}\bigcap_{j=1}^{n} \left[\left(B+R_j^l\right) \stackrel{*}{-} B_j^1\right] \supset \bigcup_{l=1}^{\infty}\bigcap_{j=1}^{n} \left[\left(B+\operatorname{int} R_j^l\right) \stackrel{*}{-} B_j^1\right].$$

The sequence in the square brackets in the latter relation is increasing w.r.t. the variable l. Hence, one can show easily that

$$\bigcup_{l=1}^{\infty}\bigcap_{j=1}^{n} \left[\left(B+\operatorname{int} R_j^l\right) \stackrel{*}{-} B_j^1\right] \supset \bigcap_{j=1}^{n}\bigcup_{l=1}^{\infty} \left[\left(B+\operatorname{int} R_j^l\right) \stackrel{*}{-} B_j^1\right]$$

and the desired statement follows property 6) of the difference of sets. □

Corollary 1. *Simply-connected domains $G_p \subset U_p$ satisfy the inclusions:*

$$B + \left\{ \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m)\right]_0 \stackrel{*}{-} B^p \right\} \subset (U,\varphi,B)_p,$$

$$(B \stackrel{*}{-} B^p) + \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m)\right]_0 \subset (U,\varphi,B)_p, \qquad p=1,\dots,n.$$

Proof. Here we also assume that $p=1$. If a point z belongs to the left hand side of the first relation, then $z \in B + z_0$ for some point

$$z_0 \in \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m)\right]_0 \stackrel{*}{-} B^1,$$

or

$$z_0 + B^1 \subset \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m)\right]_0.$$

Letting $R = z_0 + B^1$, by the proved above we have

$$\left(R + z_0 + B^1\right) \stackrel{*}{-} B^1 \subset (U,\varphi,B)_1.$$

The non-degeneracy of the matrix φ implies that the set B^1 is non-empty and this is why property 10) of the difference of sets we have

$$\left(R + z_0 + B^1\right) \stackrel{*}{-} B^1 = R + z_0$$

that proves the first relation.

As it has been shown above, for each convex domain

$$D \subset \left[\bigcap_{m=1}^{n} (G_m \stackrel{*}{-} K_m)\right]_0,$$

the inclusion

$$(B + D) \stackrel{*}{-} B^1 \subset (U,\varphi,B)_1$$

holds and the second relation follows property 3) of the difference of sets and the arbitrariness of the convex domain D. □

3. Properties of functions ω and P

Assume that simply-connected domains G_p contain compact sets K_p, $p = 1, \ldots, n$, and system (1) obeys identities (2), so the function $L(\lambda)$ is not identically zero.

On the space

$$\prod_{p=1}^{n} H(G_p) \tag{7}$$

we introduce two vector functions

$$\omega(\mu, f, \varphi, G, a) = \varphi^*(\mu) \left(\sum_{p=1}^{n} \left\langle S_p^j, \int_{a_p}^{z} e^{\mu(z-t)} f_p(t) dt \right\rangle \right)_{j=1}^{n},$$

$$P(z, f, \varphi, C) = \frac{1}{2\pi p} \int_C \frac{e^{\mu z} \omega(\mu, f, \varphi, G, a)}{L(\mu)} d\mu, \tag{8}$$

where $a \in \prod_{p=1}^{n} G_p$, C is a closed contour not passing through the zeroes of the function $L(\lambda)$. In the first case the integration is made along curves in the domain G_p, $p = 1, \ldots, n$.

We note that the function ω is a generalization of the known Leontiev interpolation function for the vector case.

Lemma 4. *The function $\omega(\mu, f, \varphi, G, a)$ possesses the following properties:*

1) *W.r.t. the variable μ, the function $\omega(\mu, f, \varphi, G, a)$ is an entire function of exponential type for each component; w.r.t. the variable f it is a linear continous functional on space* (7).

2) *The convolution operator $S*$ satisfies the following identity*

$$S * e^{\mu z} \omega(\mu, f, \varphi, G, a) = L(\mu) \left(\sum_{p=1}^{n} \left\langle S_p^j, \int_{a_p}^{z} e^{\mu(z-t)} f_p(t)\, dt \right\rangle \right)_{j=1}^{n}.$$

3) *The vector function*

$$f(z) = (\exp \lambda z)\, b, \quad b \in \mathbb{C}^n, \tag{9}$$

satisfies the relation

$$\omega(\mu, f, \varphi, G, a) = \frac{\varphi^*(\mu)\varphi(\lambda) b - L(\mu) E(\lambda - \mu, a, b)}{\lambda - \mu}.$$

Here

$$E(\lambda - \mu, a, b) = \left(e^{(\lambda-\mu)a_1} b_1, \ldots, e^{(\lambda-\mu)a_n} b_n \right).$$

The function $g_p^j(\mu, \lambda)$ at the intersection of the pth and the jth row of the matrix $\varphi^*(\mu)\varphi(\lambda)$ obeys the estimate

$$|g_p^j(\mu, \lambda)| \leqslant C(\varepsilon) C_1(\varepsilon_1) e^{\left[h\left(-\arg\mu, B^j\right)+\varepsilon\right]|\mu| + [h(-\arg\lambda, K_p)+\varepsilon_1]|\lambda|} \tag{10}$$

and satisfies the relation

$$g_p^p(\mu, \mu) = L(\mu), \quad g_p^j(\mu, \mu) = 0, \qquad p \neq j, p, j = 1, \ldots, n. \tag{11}$$

4) *Assume that* $\psi(\mu)$ *is* $n\times n$ *square matrix, whose entries are entire functions of exponential type, compact sets* R_j^m *are conjugate diagrams of the functions* ψ_j^m *and inclusions hold:*

$$\operatorname{conv}\bigcup_{j=1}^{n}\left(R_j^m+K_p^j\right)\subset G_p, \tag{12}$$

$$R_j^m\subset\left[\bigcap_{p=1}^{n}\left(G_p \stackrel{*}{-} K_p^j\right)\right]_0 \stackrel{\text{def}}{=} O_j, \qquad p,j,m=1,\ldots,n. \tag{13}$$

Then for each vector function f *in space* (7) *the identity holds:*

$$\omega(\mu,f,\psi\varphi,G,a)=\det\psi(\mu)\omega(\mu,f,\varphi,G,a)+\varphi^*(\mu)\omega(\mu,S*f,\psi,O,0).$$

5) *Let* M_p *be a compact convex polygon* $K_p\subset\operatorname{int}M_p$, $M_p\subset G_p$, $\arg z=-\alpha_{m,p}$, $m=1,\ldots,p_p$, *be perpendiculars to the sides of the polygon, and* $l_{m,p}$ *be the rays* $\arg z=\alpha_{m,p}$*; here we assume that the passage from the ray* $l_{m,p}$ *to the ray* $l_{m+1,p}$ *goes via the shortest way counterclockwise,* $p=1,\ldots,n$.

For an arbitrary vector function f *in space* (7) *the representation holds:*

$$\omega(\mu,f,\varphi,G,a)=L(\mu)A(\mu)-\varphi^*(\mu)D(\mu),$$

where $A(\mu)$ *and* $D(\mu)$ *are meromorphic vector functions, whose poles are located at the rays* $l_{m,p}$, $m=1,\ldots,p_p$, $p=1,\ldots,n$.

For each number $\varepsilon>0$ *there exists a number* $c(\varepsilon)>0$ *such that outside the angles*

$$P_{m,p}:|\arg(z-\alpha_{m,p})|<\varepsilon,$$

the inequality holds:

$$|D_j(\mu)|<\frac{c(\varepsilon)}{|\mu|}\sum_{p=1}^{n}\max_{t\in M_p}|f_p(t)|, \qquad m=1,\ldots,p_p, \quad p,j=1,\ldots,n.$$

Proof. Properties 1)–3) are proved by simple calculations.

4) Let us show that all three operators in this relations are continuous w.r.t. the variable f in space (7).

Indeed, the second operator is continuous by property 1), the first is also continuous by this property thanks to inclusions (12).

The third operator is a superposition of the other two, and the internal operator, the convolution $S*$, maps space (7) continuously into the space $\prod_{p=1}^n H(O_p)$, while the external operator is the product of the matrix φ^* by the vector function ω defined on this space.

As one can see easily, the intersection of finitely many open sets with simply-connected components is the same set and this is why, according property 11) of the difference of sets, the domains O_p, $p=1,\ldots,n$, are also simply-connected and the continuity of the third operator is implied by property 1).

As one can easily get by property 2), the needed identity is true for the vector functions $f(z)=(\exp\lambda z)b$, $b\in\mathbb{C}^n$ and the linear combinations of these functions are dense in space (7) since the domains G_p, $p=1,\ldots,n$, are simply-connected.

5) This statement is implied by Theorem 4.6.10 in monograph [16] and remarks on this theorem. □

2. Let us provide properties of the vector functions $P(z,f,\varphi,C)$.

Lemma 5. 1) *In representation* (8)*, the vector function* P *is independent of the parameter* a*, is a linear continuous functional on the space*

$$\prod_{p=1}^{n}H(K_p)$$

w.r.t. the variable f, while w.r.t. the variable z it is a linear combination of elementary solutions to system (1) *with exponents inside the contour C.*

2) *Assume that the matrix $\psi(\mu)$ satisfies Statement 4) of the previous lemma.*

If $\det\psi(\mu) \not\equiv 0$ *and the vector function f satisfies system* (1)*, then this vector function satisfies the system with the characteristic matrix $\psi(\mu)\varphi(\mu)$ and*

$$P(z, f, \psi\varphi, C) = P(z, f, \varphi, C)$$

for each contour C not passing through the zeroes of the function $\det\psi(\mu)\varphi(\mu)$.

3) *Let $G_p \subset U_p$ be convex domains $G_p \supset K_p$ and for some number $m, 1 \leqslant m \leqslant n$, the inclusions hold:*

$$B_j^m + K_p^j \subset G_p, \quad p, j = 1, \ldots, n.$$

If a vector function f from space (7) *satisfies system* (1)*, then mth component of this function satisfies the convolution equation with the characteristic function $L(\mu)$ and*

$$P_m(z, f, \varphi, C) = P(z, f_m, L, C)$$

for each contour C not passing the zeroes of the function $L(\mu)$, where P_m is the mth component of the vector function P.

4) *If a vector function f is a linear combination of elementary solutions to system* (1) *with exponents inside the contour C, then*

$$P(z, f, \varphi, C) = f(z).$$

Proof. 1) The independence of the parameter a for a vector function f of form (9) follows easily property 3) of vector function ω, while for other functions this is implied by the completeness of the above vector functions.

We choose points a_p in the compact sets K_p, $p = 1, \ldots, n$. In view of the representation for vector function ω it is clear that it is sufficient to integrate in infinitesimal neighbourhoods of the mentioned compact sets that implies the continuity of the vector function P in the needed topology.

Employing representation (8) and property 2) of the vector function ω, we obtain easily that the vector function P is a linear combination of elementary solutions to system (1) with exponents inside the contour C.

2) If the vector function f in space (7) solves system (1), as one can confirm easily, the last term in the identity in Statement 3) of the previous lemma is identically zero.

1) For a vector function f of form (9) the independence of the parameter a follows easily property 3) of the vector function ω.

Property 3) for the function $(\exp\lambda z)b$, $b \in \mathbb{C}^n$ is implied by property 2) of the function ω and the desired statement follows the completeness of linear combinations of exponential vector functions in the space $\prod_{p=1}^n H(K_p)$.

4) Let

$$a_p \in \operatorname{conv}\bigcup_{j=1}^{n}\left(R_j + K_p^j\right), \quad a_p' \in K_p, \quad a_p'' \in R_p.$$

This follows the previous property and property 12) of the interpolating function in monograph [16]. □

4. Summation of series of elementary solutions

1. Here we make use of the following result.

Lemma 6. *Let $g(\mu,\lambda), L(\mu)$ be entire functions obeying the estimates*

$$\begin{aligned}|g(\mu,\lambda)| \leqslant & c_1(\varepsilon_1)c_2(\varepsilon_2)\exp\left[H(\arg\lambda)+\varepsilon\right]|\lambda|+\left[h(-\arg\mu,K)+\varepsilon_1\right]|\mu|,\\ |L(\mu)| \geqslant & c_2(\varepsilon_2)\exp\left[h(-\arg\mu,B)-\varepsilon_2\right]|\mu|,\ |\mu|=r_p,\end{aligned} \tag{14}$$

where $r_p \nearrow \infty$, $H(\theta)$ is the support function of some compact set, B, K are convex compact sets. Assume that $g(\mu,\mu)=\alpha L(\mu)$, $\alpha\in\mathbb{C}$. We define the function $\Phi_p(\lambda,z)$ by the formula

$$\Phi_p(\lambda,z)=\frac{1}{2\pi p}\int_{|z|=r_p}\frac{g(\lambda,\mu)-\alpha L(\mu)}{(\lambda-\mu)L(\mu)}e^{\mu z}d\mu-\alpha e^{\lambda z}.$$

Then for a point $z\in(\operatorname{int}K)\overset{*}{-}B$ the estimate

$$|\Phi_p(\lambda,z)|\leqslant c(\varepsilon)A(\varepsilon_1)r_k e^{(\delta-2\varepsilon_1)r_p}e^{[H(\arg\lambda)+\varepsilon]|\lambda|}$$

holds, where $\delta=\rho(z,\partial(K\overset{*}{-}B))$.

This result can be obtained by complicating a little the proof of a similar lemma in [16].

Theorem 1. *Assume that the function $L(\mu)$ satisfies the estimate* (14) *and*

$$C_j=\{|z|=r_j\},\ j\in\mathbb{N}.$$

Then for each vector function $f\in\prod_{p=1}^n H(B_p)$ the relation

$$\lim_{j\to\infty}P(z,f,\varphi,C_j)=f(z) \tag{15}$$

holds in the topology of the space $\prod_{m=1}^n H((\operatorname{int}B)\overset{*}{-}B^m)$.

If $f\in\prod_{p=1}^n H(U_p)$, where domains U_p contain compact sets B_p, $p=1,\dots,n$, and the vector function f satisfies system (1), *then relation* (15) *holds true in the topology of the space $\prod_{m=1}^n H((U,\varphi,B)_m)$.*

Proof. By the linearity of the operator P, it is sufficient to prove the first part of the theorem for the functions with only one non-zero component, say, the first component.

Let $f(z)=(\exp(\lambda z),0,\dots,0)$, $1\leqslant m\leqslant n$. By the second property of the function ω,

$$P_m(z,f,\varphi,C_p)=\frac{1}{2\pi p}\int_{C_p}\frac{g_1^m(\mu,\lambda)-\delta_{m1}L(\mu)}{(\lambda-\mu)L(\mu)}e^{\mu z}d\mu.$$

The function $g_1^m(\mu,\lambda)$ satisfies estimate (10) and relation (11), and this is why by Lemma 17 and Lemma 5.1 in book [11] we obtain the needed statement.

Assume now that the vector function

$$f\in\prod_{p=1}^n H(U_p)$$

satisfies system (1) and $B^1,\dots,B^n$ is a set of convex compact sets satisfying inclusions (2).

We can assume that the function $L(\mu)$ has zeroes between neighbouring circumferences $\{|\mu|=r_p\}$ and $\{|\mu|=r_{p+1}\}$. Arguing as in monograph [17], we find entire functions of exponential type $\psi^j(\mu)$ with the growth indicator $h(-\theta,B^j)$, $j=1,\dots,n$ such that for $|\mu|=r_p$ the estimate

$$|\psi^j(\mu)|\geqslant c(\varepsilon)\exp\left\{\left[h(-\arg\mu,B^j)-\varepsilon\right]|\mu|\right\}$$

holds.

We denote by $\psi(\mu)$ the diagonal matrix with entries $\psi(\mu)$. By assumptions for compact sets B^j and property 4) of the function P we conclude:

$$P(z,f,\varphi,C_p)=P(z,f,\psi\varphi,C_p).$$

We apply the proven part of the theorem to the vector function f and the matrix $\psi(\mu)\varphi(\mu)$. The compact set B is replaced by the compact set $B + \sum_{j=1}^{m} B^j$, while the compact sets K_p^j are replaced by $K_p^j + \sum_{m\neq p} B^m$. We have the following relations:

$$\begin{aligned}\operatorname{int}\left(B+\sum_{j=1}^{m} B^j\right) \overset{*}{-} \bigcup_{p=1}^{n}\left(K_p^m+\sum_{m\neq p} B^m\right) &= \bigcap_{p=1}^{n}\left[\operatorname{int}\left(B+\sum_{j=1}^{m} B^j\right) \overset{*}{-} \left(K_p^m+\sum_{m\neq p} B^m\right)\right]\\ &\supset \bigcap_{p=1}^{n}\left[\left(B+\operatorname{int} B^p+\sum_{j=1}^{m} B^j\right) \overset{*}{-} \left(K_p^m+\sum_{m\neq p} B^m\right)\right]\\ &= \bigcap_{p=1}^{n}\left[(B+\operatorname{int} B^p) \overset{*}{-} K_p^m\right],\end{aligned}$$

and as one can see easily, for each sets A_1 and A_2 the inclusion holds:

$$\operatorname{int}(A_1 + A_2) \supset A_1 + \operatorname{int} A_2.$$

Thus, relation (15) holds in the needed topology. The proof is complete. □

Remark 1. *The convergence domain of the sequence $P(z, f, \varphi, C_p)$ is maximal if the identities*

$$B_p + K^p = B, \qquad p = 1, \dots, n$$

hold. In this case for each vector function $f \in \prod_{p=1}^{n} H(B_p)$ relation (15) *holds in the topology of the space $\prod_{p=1}^{n} H(\operatorname{int} B_p)$. In the space $\prod_{p=1}^{n} H(B_p)$, linear combinations of elementary solutions to system* (1) *are not complete, since as one can show easily, the limiting vector functions of such combinations satisfy this system.*

Proposition 1. *Suppose that under the assumptions of Theorem 1 the identities $B_p + K^p = B$, $U_p = B_p + O$ hold, where O is the domain containing the origin and a vector function $f \in \prod_{p=1}^{n} H(U_p)$ satisfies system* (1). *Then sequence* (15) *converges in the topology of the space $\prod_{p=1}^{n} H(U_p)$.*

Proof. Indeed, the first statement in Lemma 2 and first two statements in Lemma 3 imply the identity

$$(U, \varphi, B) = U_p$$

and the desired statement is implied by Theorem 1. □

Remark 2. *Conditions $B_p + K^p = B$, $p = 1, \dots, n$, hold if*

$$\sum_{p=1}^{n} B_p = B.$$

This is indeed so if $\varphi_p^p(\mu)$ are functions of completely regular growth and $\operatorname{int} B_p^p \supset B_p^j$, *$p, j = 1, \dots, n$, $p \neq j$.*

2. Let us provide the examples showing the sharpness of the theorem.

We let

$$\varphi_1^1(\mu) = \frac{\sin\mu \sin i\mu}{\mu^2}, \qquad \varphi_1^2(\mu) = \cos\mu\cos i\mu.$$

The conjugate diagram of these functions is the square M with vertices at the points $\pm 1 \pm i$.

Let U be an open square with the vertices at the points $\pm 2 \pm 2i$. As one can show easily, the relation

$$\lim_{j\to\infty} \frac{1}{2\pi p} \int\limits_{|\mu|=(j+\frac{1}{4})\pi} \frac{e^{z\mu} d\mu}{\varphi_1^1(\mu)\varphi_1^2(\mu)} = 0$$

holds for the points $z \in U$. Applying the residues, for the same z and some natural numbers $a_j^1, a_j^2, b_j^1, b_j^2 \in \mathbb{C}$, $j = 1, 2, \dots$, we obtain

$$\sum_{j=1}^{\infty}(a_j^1 e^{\lambda_j z} + a_j^2 e^{-\lambda_j z}) + \sum_{j=1}^{\infty}(b_j^1 e^{\mu_j z} + b_j^2 e^{-\mu_j z}) = 0, \tag{16}$$

where λ_j are zero of the function $\varphi_1^1(\mu)$, μ_j are the zeroes of the function $\varphi_1^2(\mu)$. We denote the sum of the first series by $f_1(z)$. Since for the exponentials in this series a biortogonal system exists [11], this function is not identically zero.

For an arbitrary entire function $L(\mu)$ of exponential type with conjugate diagram B, $B \subset \overline{U}$, there exist two functions of exponential type $\varphi_2^1(\mu)$ and $\varphi_2^2(\mu)$, whose conjugate diagrams are contained in the square M such that

$$L(\mu) = \varphi_1^1(\mu)\varphi_2^2(\mu) - \varphi_1^2(\mu)\varphi_2^1(\mu),$$

see [18].

Let $S_p^k \in H^*(\mathbb{C})$ are linear continuous functionals, whose Laplace transforms coincide with the functions φ_p^k, $p, k = 1, 2$. Relation (16) implies the identities

$$S_1^1 * f_1 = S_1^2 * f_1 = 0.$$

Letting $U_p = U$, $p = 1, 2$, we obtain the system of convolution equations with the characteristic matrix φ, for which

$$\det\varphi(\mu) = L(\mu), \qquad B_1 = M, \qquad B_2 \subset M, \qquad K^1 = B_2, \qquad K^2 = B_1,$$

and the function $f = (f_1, 0) \in H(U_1) \times H(U_2)$ satisfies this system.

Example 1. *Assume that $B \subset U$, the function $L(\mu)$ has a completely regular growth and*

$$\langle T, f_1\rangle \neq 0,$$

where $T \in H^*(\mathbb{C})$ is a linear continuous functional, whose Laplace transform coincides with the function $L(\mu)$. Estimate (14) *holds for some sequence of numbers $\{r_p \in \mathbb{R} : p \in \mathbb{N}\}$. By Lemma 3 we have*

$$(U, \varphi, B)_1 = \bigcap_{j=1}^{2}\left\{\left[B + \bigcap_{m=1}^{2}\left(U \stackrel{*}{-} B_m^j\right)\right] \stackrel{*}{-} K_j^1\right\} = (B + \operatorname{int} M) \stackrel{*}{-} M = \operatorname{int} B,$$

and according Theorem 1, the sequence $P_1(z, f, \varphi, C_k)$ for some system of circumferences C_k, $k \in \mathbb{N}$, converges to a function $f_1(z)$ in the topology of the space $H(\operatorname{int} B)$. In the space $H(B)$ this function can not be even approximated by linear combinations of elementary solutions to our system since the functional T vanishes at them (the fifth property of the function P).

For the next example we let $B = \overline{U}$. If the function $L(\mu)$ satisfies estimate (14), by Theorem 1, for each vector function $g \in \prod_{p=1}^{2} H(U_p)$ satisfying system (1), the sequence $P(z, g, \varphi, C_k)$ converges to this vector function in the topology of the space $\prod_{p=1}^{2} H(U_p)$. Without such estimate, the system of elementary solutions can be incomplete even in the class of all solutions. This is shown in the next example.

Example 2. *As the function $L(\mu)$, we choose an entire function with a conjugate diagram B, whose zeroes are part of the zeroes of a first order function of minimal type. The construction of such functions was provided in monograph* [16]. *In this case the linear combinations of the first components of the elementary solutions to our system do not approximate the function $f_1(z)$ in the topology of the space $H(V)$ for each domain $V \subset U$, since otherwise the function $f_1(z)$ would have satisfied the convolution equation with a characteristic function of minimal type. Then the representation of the function $f_1(z)$ would have been implied that the numbers λ_k, $k \in \mathbb{N}$, are zeroes of this function of minimal type, while this is impossible.*

These examples show that opposite to the scalar case, we can not obtain the results on the completeness of elementary solutions to system (1) only in terms of conjugate diagrams of entire exponential type related with the system but one also needs some conditions like lower bounds.

For the scalar case, the uniqueness theorem holds: if the function $L(\mu)$ has infinitely many zeroes and $P(z, g, L, C) = 0$ for a function g holomorphic in the neighbourhood of the conjugate diagram of the function $L(\mu)$ and each contour C not passing through the zeroes of this function, then $g \equiv 0$ [16]. In the vector case this is false even for the solutions to systems (1).

Example 3. *We let $L \equiv 1$. In this case it is obvious that the function $f(z) = (f_1(z), 0)$ satisfies identity $P(z, g, L, C) = 0$ for each contour C. If $\psi^1(\mu)$, $\psi^2(\mu)$ are entire functions of exponential type with infinitely many zeroes and with conjugate diagrams lying in the domain* $\operatorname{int} M$, *then the vector function $f(z)$ satisfies the system of convolution equations with the characteristic matrix $\psi(\mu)\varphi(\mu)$, whose determinant has infinitely many zeroes. But by property 4) of the function P, we have $P(z, f, \psi\varphi, C) = 0$ for each contour C not passing through the zeroes of the function $\psi^1(\mu)\psi^2(\mu)$.*

Let us compare Theorem 1 with Theorem 4.4 in [10]), which is the main result of the series of papers [8]–[10] for a square non-degenerate system of convolution equations.

Example 4. *We consider the system of convolution equations in the space $H(U) \times H(U)$ with the characteristic matrix*

$$\varphi(\mu) = \begin{pmatrix} e^{\varepsilon\mu}\varphi_1^1(\mu) & 0 \\ e^{-\varepsilon\mu}\varphi_2^1(\mu) & e^{-\varepsilon\mu}\varphi_2^1(\mu) \end{pmatrix},$$

where $0 < \varepsilon < 1$. The vector functions $(f_1, 0)$ and $(f_1, -f_1)$ obviously solve the homogeneous system and as one can show easily,

$$K_1^1 = B_2^2 = M + \varepsilon, \qquad K_2^1 = B_1^2 = \varnothing, \qquad K_1^2 = B_2^1 = K_2^2 = B_1^1 = M - \varepsilon,$$
$$L(\mu) = \varphi_1^1(\mu)\varphi_1^2(\mu), \qquad B = 2M, \qquad (U, \varphi, B) = (U, U),$$

and by Theorem 1, for each vector function $g \in H(U) \times H(U)$ solving the considered homogeneous system, the sequence $P(z, g, \varphi, C_k)$ for some sequence of circumferences converges to this vector function in the topology of the space $H(U) \times H(U)$.

Assume that our system admits (ω, ω')-estimate along the system of circumferences

$$\Gamma_j = \{z : |z| = r_j\}, \ r_j \to \infty,$$

where $\omega = (\omega_1, \omega_2)$ is a pair of convex domains in the complex plane and $\omega' \subset \mathbb{C}$ is a compact set [10]. *This means that inequalities hold:*

$$-\varepsilon \operatorname{Re}\theta - \ln\left|\varphi_1^1(r_j e^{i\theta})\right| \leqslant h(-\theta, \omega')r_j - h(-\theta, \omega_1)r_j + \varepsilon_j r_j,$$
$$\varepsilon \operatorname{Re}\theta - \ln\left|\varphi_2^1(r_j e^{i\theta})\right| \leqslant h(-\theta, \omega')r_j - h(-\theta, \omega_2)r_j + \varepsilon_j r_j,$$

where $\theta \in [0, 2\pi)$, $j \in \mathbb{N}$, and the sequence of positive numbers ε_j, $j \in \mathbb{N}$, tends to zero.

But

$$\ln\left|\varphi_1^1(re^{i\theta})\right| \leqslant h(-\theta, M)r, \quad r \geqslant 1, \qquad \ln\left|\varphi_1^2(re^{i\theta})\right| \leqslant h(-\theta, M)r, \qquad r \geqslant 0, \quad \theta \in [0, 2\pi),$$

and by the above inequalities we obtain

$$h(-\theta, \omega_1) \leqslant h(-\theta, M) + h(-\theta, \omega') + \varepsilon \operatorname{Re}\theta \qquad h(-\theta, \omega_2) \leqslant h(-\theta, M) + h(-\theta, \omega') - \varepsilon \operatorname{Re}\theta,$$

where $\theta \in [0, 2\pi)$, and this yields

$$\omega_1 \subset M + \omega' + \varepsilon, \qquad \omega_2 \subset M + \omega' - \varepsilon. \tag{17}$$

In our case, the set D in Theorem 4.4 coincides with the compact set $(M + [-\varepsilon, \varepsilon], M - \varepsilon)$ and $G' = (U, U)$.

Theorem 4.4 ensures the convergence of the series of elementary solutions for an arbitrary solution to the homogeneous system in the domain (Ω_1, Ω_2)*, where*

$$\Omega_1 = [U \overset{*}{-} (M + [-\varepsilon, \varepsilon] + \omega')] + \omega_1, \qquad \Omega_2 = [U \overset{*}{-} (M - \varepsilon + \omega')] + \omega_2.$$

Employing inclusions (17) *and properties 3), 4), 10) of Lemma 2, it is easy to obtain the relation* $(\Omega_1 \subset U \overset{*}{-} [0, 2\varepsilon]$*. The latter set is contained in the domain* U *and does not coincide with it.*

Thus, Theorem 1 is not implied by Theorem 4.4 in [10].

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