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BASIS IN INVARIANT SUBSPACE OF ANALYTICAL FUNCTIONS

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Abstract. In this work we study the problem on representing the functions in an invariant subspace of analytic functions on a convex domain in the complex plane. We obtain a sufficient condition for the existence of a basis in the invariant subspace consisting of linear combinations of eigenfunctions and associated functions of differentiation operator in this subspace. The linear combinations are constructed by the system of exponential monomials, whose exponents are partitioned into relatively small groups. We apply the method employing the Leontiev interpolating function. At that, we provide a complete description of the space of the coefficients of the series representing the functions in the invariant subspace. We also find necessary conditions for representing functions in an arbitrary invariant subspace admitting the spectral synthesis in an arbitrary convex domain. We employ the method of constructing special series of exponential polynomials developed by the author.

Keywords: Invariant subspace, basis, exponential monomial, entire function, series of exponentials

Mathematics Subject Classification: 30D10

1. Introduction

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and of their multiplicities n_k. We assume that $|\lambda_k|$ increases and $|\lambda_k| \to \infty$ as $k \to \infty$.

Let W be a nontrivial closed subspace in the space $H(D)$ of functions analytic in a convex domain $D \subset \mathbb{C}$ with the topology of uniform convergence on compact sets in D invariant w.r.t. the differentiation operator. Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a multiple spectrum of this operator in W and $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$ be the family of its eigenfunctions and adjoint functions in W.

The work is devoted to the existence of the basis in an invariant subspace formed by linear combinations of the functions in $\mathcal{E}(\Lambda)$.

The main problem in the theory of invariant subspaces is the problem on representing an arbitrary function in W be means of the elements in the system $\mathcal{E}(\Lambda)$. Subject to the character of such representation, the problems splits into several problems. The weakest version of the representation leads one to one of the most complicated problems. This is the spectral synthesis, that is, the approximation of an arbitrary function in W by linear combinations of the elements in $\mathcal{E}(\Lambda)$. The criterion of the admissibility of the spectral synthesis for an arbitrary invariant subspace in a convex domain was obtained by I.F. Krasichkov-Ternovskii in work [1]. In work [2] this result was applied for solving the problem on the spectral synthesis in some particular cases. For instance, it was proved that each space of solutions to a homogeneous convolution

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equation in a convex domain admits a spectral synthesis. Moreover, we establish that an invariant subspace in an unbounded convex domain always admits the spectral synthesis.

We note that invariant subspaces $W \subset H(D)$ admitting the spectral synthesis coincide with subspaces $W(\Lambda, D)$ being the closures in $H(D)$ of the linear span of the system $\mathcal{E}(\Lambda)$.

If W admits the spectral synthesis, a natural desire is to "improve" the approximation. Of course, the most desirable representation for each function $g \in W$ is as a "pure" series

$$g(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n \exp(\lambda_k z), \quad z \in D, \tag{1.1}$$

converging uniformly on compact sets in D. This problem is called the fundamental principle problem.

By means of the Laplace transform, the fundamental principle problem is reduced to the dual problem on multiple interpolation in the space of entire functions of exponential type. The studies of both problems were made first independently and they have a rich story. The main milestones were reflected in works [3] and [4]. In [4] the fundamental principle problem was solved for invariant subspaces admitting the spectral synthesis as well as the interpolation problem for an arbitrary convex domain $D \subset \mathbb{C}$ under the only restriction ($m_D(\Lambda) = 0$): $n_{k(j)}/|\lambda_{k(j)}| \to 0$, $j \to \infty$ for each subsequence $\{\lambda_{k(j)}\}$ accumulating the direction, where the support function H_D of the domain D is bounded, that is, $\lambda_{k(j)}/|\lambda_{k(j)}| \to \xi$ and $H_D(\xi) < +\infty$). In work [5], the authors succeeded to remove this restriction in the case of a bounded domain. Thus, there was found a criterion for the fundamental principle for an invariant subspace in a bounded convex domain D. It consists of two conditions. The first concerns local distribution of the spectral points and means certain "discreteness" (the concentration index satisfies $S_\Lambda = 0$; this index is introduced in the next section). The second condition is responsible for the global distribution of λ_k.

If the condition $S_\Lambda = 0$ fails, it is impossible to represent all functions $g \in W$ as series (1.1). This is why, in a natural way, there rises the problem on representing g as series (1.1) with brackets:

$$g(z) = \sum_{m=1}^{\infty} \left(\sum_{\lambda_k \in U_m} \sum_{n=0}^{n_k-1} d_{k,n} z^n \exp(\lambda_k z) \right). \tag{1.2}$$

The monograph by A.F. Leontiev [6] was devoted to studying the mentioned problems. Here a lot of results both by the author himself and its predecessors were presented.

The aim to improve representation (1.2) led us to the problem on a basis in an invariant subspace, which can be formulated as follows. Under what conditions we can partition $U = \{U_m\}_{m=1}^{\infty}$ the sequence Λ into the groups U_m and choose fixed linear combinations $e_{m,j}$, $j = \overline{1, N_m}$ of the elements $\mathcal{E}(\Lambda)$ in these groups so that the family of exponential polynomials $\mathcal{E}(\Lambda, U) = \{e_{m,j}\}$ becomes a basis in W. If such basis exists, a series of issues arises. How to make the partition U and is it possible to describe all admissible partitions? How small the diameter of the groups U_m can be? Finally, how to describe the space of the coefficients of the series over the system $\mathcal{E}(\Lambda, U)$? In the case of a bounded convex domain D, the answers to these questions were obtained in works [7]–[11]. In particular, there was found the criterion of the existence of the basis in the subspace W constructed by the partition U into relatively small groups U_m, namely, into the groups whose diameters and the number of the points are infinitesimal as $m \to \infty$ in comparison with the absolute values of these points.

Thus, in the case of a bounded convex domain, the study of representation of the functions in an invariant space can be regarded as complete. Concerning unbounded domains, here only two particular cases were studied when D is a plane or a half-plane. The complete solution of the representation problem for invariant subspaces of entire functions was obtained in work

[12]. The invariant subspaces in the half-plane we mostly studied in the case of a simple positive spectrum (see [6], [13]) and almost real spectrum [14].

We observe that in the most part of the works, see, for instance, [15], [16], the representation problem in the both cases of a single or several variables is reduced to dual problems of special interpolation in the spaces of entire functions of exponential type. The study of such problems is a rather complicated process. At that, the duality of the representation problem and interpolation problem was established in [4] only under an additional restriction for the multiplicities of the points λ_k: $m_D(\Lambda) = 0$. And the issue whether the duality is true without this restriction is still open. And if the duality is still true, it is completely unclear how to solve the corresponding interpolation problem in this case. Because of this, in work [14], another method was proposed for solving the representation problem allowing one to avoid solving the interpolation problem. This method employs the Leontiev interpolating function, see [6], [17]. Thanks to this, for the invariant subspaces with almost real spectrum, the representation problem was solved successfully in the general case without additional restriction for the multiplicities of the points in the spectrum.

The present work is devoted to studying necessary and sufficient conditions of the existence of a basis in the invariant subspace constructed by relatively small groups of the points in the spectrum.

In the second section we obtain sufficient conditions for the existence of a basis for an arbitrary convex domain (see Theorem 2.2); the domain can be also unbounded. As in work [14], we apply the method employing the Leontiev interpolating function. At that, we provide a complete description of the space of the coefficients of the series representing the functions in the invariant subspace.

In the third section we obtain necessary conditions of the existence of a basis for an arbitrary convex domain and an arbitrary invariant subspace, see Theorem 3.1. We employ the method of constructing special series of exponential polynomials developed in [18].

2. Sufficient conditions

First of all we recall some notions and mention some facts related to the Leontiev interpolating function.

Let $B(z,r)$, $S(z,r)$ be an open ball and a circumference of radius r centered at the point z. By $n(z,r,\Lambda)$ we denote the number of the points λ_k counting their multiplicities n_k located in the closed ball $\overline{B(z,r)}$, while by $\bar{n}(\Lambda)$ we denote the upper density of the sequence Λ:

$$\bar{n}(\Lambda) = \varlimsup_{r\to+\infty} \frac{n(0,r,\Lambda)}{r}.$$

If M is a convex set in $\mathbb{C}$, by the symbol $H_M(\lambda)$ we denote the support function of the set M (more precisely, of the complex conjugate set):

$$H_M(\lambda) = \sup_{w\in M} \mathrm{Re}(\lambda w), \quad \lambda \in \mathbb{C}.$$

The function H_M is convex and positively homogeneous of order one, that is, $tH_M(\lambda) = H_M(t\lambda)$, $t > 0$.

Let f be an entire function. We say that f has an exponential type if for some $A, B \geqslant 0$ the inequality holds: $\ln|f(\lambda)| \leqslant A + B|\lambda|$, $\lambda \in \mathbb{C}$. The indicator of f is the funtion

$$h_f(\lambda) = \varlimsup_{t\to\infty} \frac{\ln|f(t\lambda)|}{t}, \quad \lambda \in \mathbb{C}.$$

It is convex and positively homogeneous of order one, that is, it coincides with the support function of some compact set called the indicator diagram of f, see, for instance, [19, Ch. I, Sect. 5, Thm. 5.4]. The compact set L complex conjugate to the indicator diagram is called the complex diagram of the function f. Thus,

$$h_f(\lambda) = H_L(\lambda), \qquad \lambda \in \mathbb{C}.$$

Let D be a convex domain in $\mathbb{C}$ and $H^*(D)$ stands for the strongly dual space of $H(D)$ called the space of analytic functionals. Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ and $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$. By $W(\Lambda, D)$ we denote the closure of the linear span of the system $\mathcal{E}(\Lambda)$ in the space $H(D)$.

Let $\hat{\mu}(\lambda)$ denote the Laplace transform of the functional $\mu \in H^*(D)$: $\hat{\mu}(\lambda) = \mu(e^{\lambda z})$. The function $\hat{\mu}(\lambda)$ is entire and has an exponential type. It is known, see, for instance, [20, Ch. III, Sect. 12, Thm. 12.3] that the Laplace transform makes an algebraic and topological isomorphism between $H^*(D)$ and P_D, where P_D is the inductive limit of Banach spaces

$$P_s = \{f \in H(\mathbb{C}) : \|f\|_s = \sup_{\lambda \in \mathbb{C}} |f(\lambda)| \exp(-H_{K_s}(\lambda)) < \infty\}.$$

Here $K(D) = \{K_s\}_{s=1}^{\infty}$ is a sequence of convex compact sets exhausting D, that is, $K_s \subset \operatorname{int} K_{s+1}$, $s \geqslant 1$, (int stands for the interior of a set) and $D = \cup_{p=1}^{\infty} K_p$. The set P_D is formed by the entire functions of exponential type f whose conjugare diagrams are located in the domain D (that is, $h_f(\lambda) < H_D(\lambda)$, $\lambda \neq 0$).

Assume that the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$. By the Hahn-Banach theorem, the latter is equivalent to the existence of a non-zero functional $\mu \in H^*(D)$ vanishing on the functions in the system $\mathcal{E}(\Lambda)$, that is, to the existence of the function $f \in P_D$ ($f = \hat{\mu}$) vanishing at the points λ_k with the multiplicities at least n_k. Since f has an exponential type, according the well-known Lindelöf theorem, see, for instance, [21, Ch. I, Sect. 11, Thm. 15], in this case the upper density $\bar{n}(\Lambda)$ is finite.

Assume that there exists an entire function of exponential type $f \in P_D$ vanishing at the points λ_k with the multiplicities at least n_k. Then in the space $H^*(D)$, there exists [17, Ch. IV, Sect. 1, Subsect. 2] a system of functional biorthogonal to $\mathcal{E}(\Lambda)$ $\Xi(\Lambda, D) = \{\mu_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$: $\mu_{k,n}(z^l \exp(\lambda_j z)) = 1$ if $j = k, l = n$ and $\mu_{k,n}(z^l \exp(\lambda_j z)) = 0$ otherwise. It is constructed by means of the function f and is a part of a system $\Xi(\tilde{\Lambda}, D)$ biorthogonal to $\mathcal{E}(\tilde{\Lambda})$, where $\tilde{\Lambda}$ is the multiple zero set of f. Assume that series (1.2) converges uniformly on compact subsets of the domain D. Then, employing the continuity and linearity of the functionals $\mu_{k,n}$, we obtain $d_{k,n} = \mu_{k,n}(g)$, $k \geqslant 1$, $n = \overline{0, n_k - 1}$. Thus, if there exists the aforementioned function f, then the representation by the series (1.2) possesses the uniqueness property. At that, the coefficients of the representation are calculated by means of the biorthogonal system of the functionals.

Let D be a convex domain, $g \in H(D)$, $\alpha \in \mathbb{C}$, and f be an entire function of exponential type, whose conjugate diagram K contains the origin and its shift $K(\alpha) = K + \alpha$ lies in D ($K(\alpha)$ is the conjugate diagram $f(\lambda)\exp(\alpha\lambda)$). An interpolating function for g is called, see [6, Ch. I, Sect. 2, Subsect. 1]:

$$\omega_f(\lambda, \alpha, g) = \exp(-\alpha\lambda) \frac{1}{2\pi i} \int_{\Omega} \gamma(\xi) \left(\int_0^{\xi} g(\xi + \alpha - \eta) \exp(\lambda\eta) d\eta \right) d\xi,$$

where Ω is a contour (a simple closed continuous rectifiable curve) enveloping the compact set K and lying in the domain $D - \alpha$, $\gamma(\xi)$ is the function associated with f in the Borel sense, see [19, Ch. I, Sect. 5]. We mention some properties of $\omega_f(\lambda, \alpha, g)$ and $\Xi(\Lambda, D)$.

1. [6, Ch. I, Sect. 2, Thm. 1.2.5]. Let Ω be a boundary of a convex neighbourhood of the compact set K and $\Omega(\alpha) = \Omega + \alpha \subset D$. For each $\varepsilon > 0$ there exists $A(\varepsilon) > 0$ such that

$$|\omega_f(\lambda, \alpha, g)| \leqslant A(\varepsilon) \exp(h_f(\lambda) + \varepsilon|\lambda| - \operatorname{Re}(\alpha\lambda)) \max_{z \in \Omega(\alpha)} |g(z)|, \quad \lambda \in \mathbb{C}. \tag{2.1}$$

2. Let $\tilde{g} \in W(\Lambda, D)$ and $d_{k,n} = \mu_{k,n}(\tilde{g})$, where $\mu_{k,n} \in \Xi(\Lambda, D)$, $k \geqslant 1$, $n = \overline{0, n_k - 1}$. Then

$$\frac{1}{2\pi i}\int\limits_{S_k} \frac{\omega_f(\lambda, \alpha, \tilde{g})}{f(\lambda)} \exp(\lambda z) d\lambda = \sum_{n=0}^{n_k-1} d_{k,n} z^n \exp(\lambda_k z), \quad k \geqslant 1, \tag{2.2}$$

where S_k is a circumference inside which there are no zeroes of f different from λ_k. Moreover, if λ' is a zero of the function f not being among λ_k, $k \geqslant 1$, and S' is circumference, inside which λ' is located and there are no other zeroes of the function f, then

$$\frac{1}{2\pi i}\int\limits_{S'} \frac{\omega_f(\lambda, \alpha, \tilde{g})}{f(\lambda)} \exp(\lambda z) d\lambda = 0. \tag{2.3}$$

Indeed, let $\tilde{g}$ be the limit of the sequence

$$P_l(z) = \sum_{k=1}^{l} \sum_{n=0}^{n_k-1} d^l_{k,n} z^n \exp(\lambda_k z), \quad l \geqslant 1,$$

converging uniformly on compact sets in D. Since $\tilde{g} \in W(\Lambda, D)$, such sequence exists provided we suppose that some $d^l_{k,n}$ vanish. By Theorem 1.2.4 in [6, Ch. I, Sect. 2], we have:

$$\frac{1}{2\pi i}\int\limits_{S_k} \frac{\omega_f(\lambda, \alpha, P_l)}{f(\lambda)} \exp(\lambda z) d\lambda = \sum_{n=0}^{n_k-1} d^l_{k,n} z^n \exp(\lambda_k z), \qquad k = \overline{1, l},$$

$$\frac{1}{2\pi i}\int\limits_{S_k} \frac{\omega_f(\lambda, \alpha, P_l)}{f(\lambda)} \exp(\lambda z) d\lambda = 0, \quad k > l, \qquad \frac{1}{2\pi i}\int\limits_{S'} \frac{\omega_f(\lambda, \alpha, P_l)}{f(\lambda)} \exp(\lambda z) d\lambda = 0.$$

Employing the continuity and the linearity of the functionals in the biorthogonal system, we obtain (if $k > l$, we suppose $d^l_{k,n} = 0$):

$$d_{k,n} = \mu_{k,n}(\tilde{g}) = \lim_{l\to\infty} \mu_{k,n}(P_l) = \lim_{l\to\infty} d^l_{k,n}, \quad k \geqslant 1, n = \overline{0, n_k - 1}. \tag{2.4}$$

It follows from estimate (2.1) $\omega_f(\lambda, \alpha, P_l) \to \omega_f(\lambda, \alpha, \tilde{g})$ as $l \to \infty$ uniformly on each compact set in the plane. Together with the above facts, this give us the required identities.

3. Let $\tilde{g} \in W(\Lambda, D)$ and $d_{k,n} = \mu_{k,n}(\tilde{g})$, $k \geqslant 1$, $n = \overline{0, n_k - 1}$. Assume that series (1.2) converge uniformly on compact sets in D. Then $g \equiv \tilde{g}$.

Indeed, if $\mu' \in \Xi(\tilde{\Lambda}, D) \setminus \Xi(\Lambda, D)$, then $\mu'(g) = \mu'(\tilde{g}) = \mu'(P_l)$. In view of (2.4) and by the uniqueness theorem [6, Ch. II, Sect. 1, Thm. 2.1.2], this implies the needed identity.

Let D be an unbounded convex domain. We let

$$J(D) = \{\lambda \in \mathbb{C} : H_D(\lambda) = +\infty\}.$$

Since H_D is a convex and positively homogeneous function, the set $\mathbb{C} \setminus J(D)$ is a convex cone. Therefore, only the following four cases are possible: $\mathbb{C} \setminus J(D)$ is a point or a ray or a straight line or an angle of opening at most π. If $D = \mathbb{C}$, then $J(D) = \mathbb{C} \setminus \{0\}$. In the case, when D is the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(ze^{i\varphi}) < a\}$, the set $J(D)$ the plane with the cut along the ray $\{\lambda = te^{i\varphi} : t \geqslant 0\}$. If D is the strip $\{z \in \mathbb{C} : \mathrm{Re}(ze^{i\varphi}) < a, \mathrm{Re}(ze^{i(\varphi+\pi)}) < b\}$, then $J(D)$ are two half-plane with the common boundary straight line $\{\lambda = te^{i\varphi} : t \in \mathbb{R}\}$. In other cases the domain D contains no straight lines. However, D always contains some ray $\{z = z_0 + te^{i\varphi}, t \geqslant 0\}$. At that, the set $J(D)$ is angle of an opening strictly leas than 2π and it contains an open angle of the opening π, which is the half-plane

$$\left\{\lambda = te^{i\psi} : -\varphi - \frac{\pi}{2} < \psi < -\varphi + \frac{\pi}{2}, \, t > 0\right\}.$$

We also note that thanks to the convexity, the function H_D is continuous outside the closure of the set $J(D)$.

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$. By the symbol $U = \{U_m\}_{m=1}^{\infty}$ we denote the partition of the sequence $\{\lambda_k\}_{k=1}^{\infty}$ into the groups U_m, $m = 1, 2, \ldots$ We re-index the terms of Λ.

The points λ_k in a group U_m are denoted by $\lambda_{m,l}$ and their multiplicities are denoted by $n_{m,l}$. Here the first subscript m coincides with the index of the group, while the second subscript ranges from 1 to M_m, where M_m is the number of the points λ_k in the group U_m. Let N_m be the number of the points λ_k in a group U_m, $m = 1, 2, \ldots$, counting their multiplicities, that is, $N_m = \sum_{l=1}^{M_m} n_{m,l}$.

By the symbol $\Theta(\Lambda)$ we denote the set of all limits of all converging sequence of form $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$. The set $\Theta(\Lambda)$ is closed and lies on the unit circumference centered at the origin.

Lemma 2.1. *Let D be a convex domain in $\mathbb{C}$, $\Lambda = \{\lambda_k, n_k\}$ is partitioned into the groups $U = \{U_m\}_{m=1}^{\infty}$, where $U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$, the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and $\Theta(\Lambda)$ does not intersect the boundary of the set $J(D)$. We assume that for each convex compact set $K_0 \subset D$, each $\delta_0 > 0$ and each subsequence $\{U_{m_l}\}_{l=1}^{\infty}$ such that $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges, there exists a function $f \in P_D$, a sequence of contours $\{\gamma_l\}_{l=1}^{\infty}$ and an index l_0 possessing the following properties:*

1) f vanishes at the points λ_k, $k \geqslant 1$, with a multiplicity at least n_k;

2) for all $l \geqslant l_0$, all points of the group U_{m_l} are located inside the contour γ_l and there are no points λ_k different from $\lambda_{m_l,v}$, $v = \overline{1, M_{m_l}}$;

3) $\ln|f(\lambda)| \geqslant H_{K_0}(\lambda)$, $\lambda \in \gamma_l$, $l \geqslant l_0$;

4) $\mathrm{d}(\gamma_l) \leqslant \delta_0|\lambda_{m_l,1}|$, $l \geqslant l_0$, where $\mathrm{d}(\gamma_l)$ is the diameter of the contour γ_l;

5) $\rho(\gamma_l) \leqslant |\lambda_{m_l,1}|^2$, $l \geqslant l_0$, where $\rho(\gamma_l)$ is the length of the contour γ_l.

Then each function $g \in W(\Lambda, D)$ is expanded into the series

$$g(z) = \sum_{m=1}^{\infty} \left(\sum_{v=1}^{M_m} \sum_{n=0}^{n_{m,v}-1} c_{m,v,n} z^n \exp(\lambda_{m,v} z) \right), \quad z \in D. \tag{2.5}$$

At that, for each convex compact set $K \subset D$

$$\sum_{m=1}^{\infty} \max_{z \in K} \left| \sum_{v=1}^{M_m} \sum_{n=0}^{n_{m,v}-1} c_{m,v,n} z^n \exp(\lambda_{m,v} z) \right| < +\infty. \tag{2.6}$$

In particular, the series (w.r.t. m) converges absolutely and uniformly on compact sets in the domain D.

Proof. By the assumption, $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$. Then $\bar{n}(\Lambda) < +\infty$ and there exists a system of functionals in $H^*(D)$, $\Xi(\Lambda, D) = \{\mu_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$, biorthogonal to $\mathcal{E}(\Lambda)$.

Let $g \in W(\Lambda, D)$. We consider the series

$$\sum_{m=1}^{\infty} \left(\sum_{v=1}^{M_m} \sum_{n=0}^{n_{m,v}-1} c_{m,v,n} z^n \exp(\lambda_{m,v} z) \right), \quad z \in D,$$

where $c_{m,v,n} = \mu_{k,n}(g)$ if $\lambda_{m,v} = \lambda_k$. Let K be a convex compact set D. Assume that series (2.6) diverges. Then there exists a sequence of embedded segments $[\varphi_{1,j}, \varphi_{2,j}]$, $j \geqslant 1$, with the lengths tending to zero such that

$$\sum_{m \in Q(j)} \max_{z \in K} \left| \sum_{v=1}^{M_m} \sum_{n=0}^{n_{m,v}-1} c_{m,v,n} z^n \exp(\lambda_{m,v} z) \right| = +\infty, \quad j \geqslant 1, \tag{2.7}$$

where $Q(j)$ is the sequence of all subscripts m, for which the point $\lambda_{m,1}$ is located in the angle $\Theta_j = \{te^{i\varphi} : \varphi \in [\varphi_{1,j}, \varphi_{2,j}], t \geqslant 0\}$. This implies that there exists a sequence $\{U_{m_l}\}_{l=1}^{\infty}$ such that

$$\sum_{l=1}^{\infty} \max_{z\in K} \left| \sum_{v=1}^{M_{m_l}} \sum_{n=0}^{n_{m_l,v}-1} c_{m_l,v,n} z^n \exp(\lambda_{m_l,v} z) \right| = +\infty, \tag{2.8}$$

and for each $j \geqslant 1$ the inclusion $\lambda_{m_l,1} \in \Theta_j$, $l \geqslant l(j)$ holds for some $l(j)$. The latter means that $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ tends to the number $e^{i\varphi_0}$, where φ_0 is the common points of all segments $[\varphi_{1,j}, \varphi_{2,j},]$, $j \geqslant 1$. We consider two cases.

1) $H_D(e^{i\varphi_0}) < +\infty$ ($e^{i\varphi_0} \notin J(D)$). Since K is a compact set in D, there exists $\beta > 0$ such that

$$H_K(\lambda) + 2\beta|\lambda| \leqslant H_D(\lambda), \quad \lambda \in \mathbb{C}. \tag{2.9}$$

By assumption, $J(D) \cup \{0\}$ is a closed set. Therefore, the function H_D is continuous in the neighbourhood of the point $e^{i\varphi_0}$. This is why there exists $\delta \in (0,1)$ such that

$$|H_D(e^{i\varphi_0}) - H_D(\lambda)| < \frac{\beta}{6}, \quad \lambda \in B(e^{i\varphi_0}, \delta). \tag{2.10}$$

Moreover, we can assume that for some compact set $K_0 \subset D$ the inequality holds:

$$H_{K_0}(\lambda) + \frac{\beta|\lambda|}{6} \geqslant H_D(\lambda), \quad \lambda \in B(e^{i\varphi_0}, \delta). \tag{2.11}$$

We choose $\delta_0(0, \delta/2)$ such that

$$\max_{z\in K} \max_{\mu\in\Delta} \delta_0 |z - \mu| < \frac{\beta}{6}, \tag{2.12}$$

where Δ is the triangle defined as

$$\Delta = \{z : \mathrm{Re}(zw_1) \leqslant H_D(w_1)\} \cap \{z : \mathrm{Re}(zw_2) \leqslant H_D(w_2)\} \cap \{z : \mathrm{Re}(ze^{i\varphi_0}) \geqslant H_{K_0}(e^{i\varphi_0})\},$$

and w_1, w_2 are the points of the intersection of the circumference $S(e^{i\varphi_0}, \delta/2)$ with the unit circumference centered at the origin.

By assumption, there exists a function $f \in P_D$, a sequence of contours $\{\gamma_l\}_{l=1}^{\infty}$ and an index l_0 possessing properties 1)-4). By the symbol L we denote the conjugate diagram of function f. By the definition of the space P_D, the compact set L is located in the domain D.

Let z_0 be a point of the compact set L satisfying the identity

$$\mathrm{Re}(z_0 e^{i\varphi_0}) = H_L(e^{i\varphi_0});$$

there are the only such point or a segment in the boundary of the compact set L. Statement 3 of the lemma implies that

$$\mathrm{Re}(z_0 e^{i\varphi_0}) = H_L(e^{i\varphi_0}) \geqslant H_{K_0}(e^{i\varphi_0}). \tag{2.13}$$

Since $z_0 \in L \subset D$, then

$$\mathrm{Re}(z_0 w_1) < H_D(w_1), \quad \mathrm{Re}(z_0 w_2) < H_D(w_2).$$

Thus, the point z_0 belongs to the triangle Δ.

Consider the function $f_0(\lambda) = \exp(-z_0\lambda)$. Its conjugate diagram is the compact set $L - z_0$ containing the origin. We let $\alpha = z_0$. The shift of the compact set $L - z_0$ by the vector α coincides with L and lies in the domain D. Employing the residues, identities (2.2), (2.3) and Statements 1), 2) of the lemma, we have

$$\frac{1}{2\pi i} \int\limits_{\gamma_l} \frac{\omega_{f_0}(\lambda, \alpha, g)}{f_0(\lambda)} \exp(\lambda z) d\lambda = \sum_{v=1}^{M_{m_l}} \sum_{n=0}^{n_{m_l,v}-1} c_{m_l,v,n} z^n \exp(\lambda_{m_l,v} z), \quad l \geqslant l_0. \tag{2.14}$$

Moreover, by inequality (2.1) we have

$$\begin{aligned}|\omega_{f_0}(\lambda,\alpha,g)| \leqslant & A(\beta)\exp(h_{f_0}(\lambda)+\frac{\beta|\lambda|}{6}-\mathrm{Re}(\alpha\lambda))\max_{z\in\Omega}|g(z)|\\ =& A\exp(H_L(\lambda)-\mathrm{Re}(z_0\lambda)+\frac{\beta|\lambda|}{6}-\mathrm{Re}(\alpha\lambda)),\quad \lambda\in\mathbb{C},\end{aligned}$$

where Ω is the boundary of a convex neighbourhood of the compact set L and this neighbourhood is located inside D. In view of Statements 3) and 5) of the lemma, as $l\geqslant l_0$ we obtain

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int\limits_{\gamma_l}\frac{\omega_{f_0}(\lambda,\alpha,g)}{f_0(\lambda)}\exp(\lambda z)d\lambda\right|\\ &\leqslant |\lambda_{m_l,1}|^2A\exp\left(\max_{\lambda\in\gamma_l}\left(H_L(\lambda)-\mathrm{Re}(z_0\lambda)-H_{K_0}(\lambda)\right.\right.\\ &\qquad\left.\left.+\mathrm{Re}(z_0\lambda)+\frac{\beta|\lambda|}{6}+\mathrm{Re}((z-\alpha)\lambda)\right)\right)\\ &=\delta_0|\lambda_{m_l,1}|A\exp(\max_{\lambda\in\gamma_l}(H_L(\lambda)-H_{K_0}(\lambda)+\beta|\lambda|/6+\mathrm{Re}((z-\alpha)\lambda))),\qquad z\in K.\end{aligned}\tag{2.15}$$

According Statements 2) and 4) of the lemma, the contour γ_l lies in the ball $B(\lambda_{m_l,1},\delta_0|\lambda_{m_l,1}|)$, $l\geqslant l_0$. Since te sequence $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges to $e^{i\varphi_0}$ and $\delta_0<\delta$, there exists an index $l_1\geqslant l_0$ such that inclusions hold $\gamma_l\subset B(\lambda_{m_l,1},\delta_0|\lambda_{m_l,1}|)\subset B(|\lambda_{m_l,1}|e^{i\gamma_0},\delta_0|\lambda_{m_l,1}|)$, $l\geqslant l_1$. By inequality (2.10), the positive homogeneity of the support function and the inclusion $L\subset D$ we obtain

$$H_L(\lambda)-H_{K_0}(\lambda)<H_D(\lambda)-H_D(\lambda)+\frac{\beta|\lambda|}{6}=\frac{\beta|\lambda|}{6},\quad \lambda\in\gamma_l,\quad l\geqslant l_1.$$

Therefore, by (2.15) and (2.14) we have ($\alpha=z_0$):

$$\begin{aligned}&\left|\sum_{v=1}^{M_{m_l}}\sum_{n=0}^{n_{m_l,v}-1}c_{m_l,v,n}z^n\exp(\lambda_{m_l,v}z)\right|\leqslant\delta_0|\lambda_{m_l,1}|A\exp\left(\max_{\lambda\in\gamma_l}\left(\frac{\beta|\lambda|}{3}+\mathrm{Re}((z-\alpha)\lambda)\right)\right)\\ &\leqslant\delta_0|\lambda_{m_l,1}|A\exp\left(\max_{\lambda_{m_l,1}+\xi\in\gamma_l}\left(\frac{\beta|\lambda_{m_l,1}|}{3}+\frac{\beta|\xi|}{3}+\mathrm{Re}((z-z_0)\lambda_{m_l,1})+\mathrm{Re}((z-z_0)\xi)\right)\right)\end{aligned}$$

for all $z\in K$ and $l\geqslant l_1$. We choose an index $l_2\geqslant l_1$ such that

$$|\lambda_{m_l,1}|^2A\leqslant\exp(\beta|\lambda_{m_l,1}|/7),\qquad l\geqslant l_2.\tag{2.16}$$

According Statement 4) of the lemma, $|\xi|\leqslant\delta_0|\lambda_{m_l,1}|$, where $\lambda_{m_l,1}+\xi\in\gamma_l$, $l\geqslant l_2$. Hence, taking into consideration the previous inequality, the belonging $z_0\in\Delta$, (2.12) and the inequalities $\delta_0<\delta/2<1/2$, we obtain

$$\begin{aligned}&\max_{z\in K}\left|\sum_{v=1}^{M_{m_l}}\sum_{n=0}^{n_{m_l,v}-1}c_{m_l,v,n}z^n\exp(\lambda_{m_l,v}z)\right|\\ &\leqslant\max_{z\in K}\left(\exp\left(\frac{\beta|\lambda_{m_l,1}|(1+\delta_0)}{3}+\mathrm{Re}((z-z_0)\lambda_{m_l,1})+\frac{\beta}{7}|\lambda_{m_l,1}|+\frac{\beta}{6}|\lambda_{m_l,1}|\right)\right)\\ &\leqslant\exp\left(\frac{2\beta|\lambda_{m_l,1}|}{3}+H_K(\lambda_{m_l,1})-\mathrm{Re}(z_0\lambda_{m_l,1})+\frac{\beta}{7}|\lambda_{m_l,1}|\right),\qquad l\geqslant l_2.\end{aligned}$$

Since the sequence $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges to $e^{i\varphi_0}$, thanks to the continuity and the positive homogeneity of the support function of a compact set, we find an index $l_3 \geqslant l_2$ such that

$$H_K(\lambda_{m_l,1}) - \mathrm{Re}(z_0\lambda_{m_l,1}) + \frac{\beta}{7}|\lambda_{m_l,1}| = |\lambda_{m_l,1}| \left(H_K\left(\frac{\lambda_{m_l,1}}{|\lambda_{m_l,1}|}\right) - \mathrm{Re}\left(z_0\frac{\lambda_{m_l,1}}{|\lambda_{m_l,1}|}\right) + \frac{\beta}{7}\right)$$
$$\leqslant \leqslant |\lambda_{m_l,1}|(H_K(e^{i\varphi_0}) - \mathrm{Re}(z_0e^{i\varphi_0}) + \frac{\beta}{6}), \qquad l \geqslant l_3.$$

By the previous inequality and (2.13), (2.11), (2.9) this yields

$$\begin{aligned}\max_{z\in K}\left|\sum_{v=1}^{M_{m_l}}\sum_{n=0}^{n_{m_l,v}-1} c_{m_l,v,n}z^n\exp(\lambda_{m_l,v}z)\right| &\leqslant \exp\left(\frac{5\beta|\lambda_{m_l,1}|}{6} + |\lambda_{m_l,1}|(H_K(e^{i\varphi_0}) - \mathrm{Re}(z_0e^{i\varphi_0}))\right)\\ &\leqslant \exp\left(\frac{5\beta|\lambda_{m_l,1}|}{6} + |\lambda_{m_l,1}|(H_K(e^{i\varphi_0}) - H_{K_0}(e^{i\varphi_0}))\right)\\ &\leqslant \exp\left(\frac{5\beta|\lambda_{m_l,1}|}{6}\right.\\ &\qquad\left. + |\lambda_{m_l,1}|\left(H_D(e^{i\varphi_0}) - 2\beta - H_D(e^{i\varphi_0}) + \frac{\beta}{6}\right)\right)\\ &= \exp(-\beta|\lambda_{m_l,1}|),\end{aligned}$$

where $l \geqslant l_3$. Since Λ has a finite upper density, the series $\sum\exp(-\beta|\lambda_{m_l,1}|)$ converges. This contradicts (2.8). Thus, in the considered case (2.6) holds. Therefore, series (2.5) converges absolutely and uniformly on compact sets in the domain D. Since $c_{m,v,n} = \mu_{k,n}(g)$ if $\lambda_{m,v} = \lambda_k$, according property 3 given before the lemma, identity (2.5) holds.

2) $H_D(e^{i\varphi_0}) = +\infty$ ($e^{i\varphi_0} \in J(D)$). By assumption, $e^{i\varphi_0} \notin \partial J(D)$. Then there exists $\delta \in (0,1)$ such that

$$H_D(\lambda) = +\infty, \quad \lambda \in B(e^{i\varphi_0}, \delta). \tag{2.17}$$

Let $\delta_0 \in (0, \delta/2) \cap (0, 2/5)$ and $K_0 = K$. By assumption, there exist a function $f \in P_D$, a sequence of contours $\{\gamma_l\}_{l=1}^{\infty}$ and an index l_0 possessing properties 1)-4). By the symbol L we denote the conjugate diagram of the function f. By the definition of the space P_D, the compact set L lies in the domain D. Let $z_0 \in L$. Consider the function $f_0(\lambda) = \exp(-z_0\lambda)$. Its conjugate diagram is the compact set $L - z_0$ containing the origin.

By (2.17) and what was said before the lemma on the set $J(D)$, there exists ψ_0 such that the angle

$$\Gamma = \{\lambda = te^{i\psi} : -\psi_0 - \pi/2 < \psi < -\psi_0 + \pi/2, t > 0\}$$

lies in $J(D)$ and contains the closure of the ball $B(e^{i\varphi_0}, 2\delta_0)$. We consider the compact sets $L(t) = L + te^{i\psi_0}$, $t > 0$. For each $z \in L$ and $t > 0$ we have

$$\mathrm{Re}\left((z + te^{i\psi_0})\lambda\right) = \mathrm{Re}(z\lambda) + t\,\mathrm{Re}(e^{i\psi_0}\lambda) \leqslant \mathrm{Re}(z\lambda) \leqslant H_L(\lambda) < H_D(\lambda), \quad \lambda \notin \Gamma \cup \{0\},$$
$$\mathrm{Re}\left((z + te^{i\psi_0})\lambda\right) < +\infty = H_D(\lambda), \quad \lambda \in \Gamma.$$

Therefore, $H_{L(t)}(\lambda) < H_D(\lambda)$, $\lambda \neq 0$, that is, $L(t) \subset D$, $t > 0$.

Let $\beta > 0$. Since the closure of the ball $B(e^{i\varphi_0}, 2\delta_0)$ lies inside Γ, there exists $t_0 > 0$ such that the inequality

$$t_0\,\mathrm{Re}(e^{i\psi_0}\lambda) > H_L(\lambda) - \mathrm{Re}(z_0\lambda) + 2\beta|\lambda|, \quad \lambda \in B(e^{i\varphi_0}, 2\delta_0) \tag{2.18}$$

holds.

We let $\alpha = z_0 + t_0 e^{i\psi_0}$. Then relations (2.14) and (2.15) hold. By these relations, (2.16) and the choice of the compact set K_0 we obtain

$$\left|\sum_{v=1}^{M_{m_l}} \sum_{n=0}^{n_{m_l,v}-1} c_{m_l,v,n} z^n \exp(\lambda_{m_l,v} z)\right|$$
$$\leqslant |\lambda_{m_l,1}|^2 A \exp\left(\max_{\lambda\in\gamma_l}\left(H_L(\lambda) - H_{K_0}(\lambda) + \frac{\beta|\lambda|}{6} + \mathrm{Re}((z-\alpha)\lambda)\right)\right)$$
$$\leqslant \exp\left(\max_{\lambda\in\gamma_l}\left(H_L(\lambda) - H_K(\lambda) + \frac{\beta|\lambda|}{3} + \mathrm{Re}((z-\alpha)\lambda)\right)\right), \qquad l \geqslant l'.$$

Since the sequence $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges to $e^{i\varphi_0}$, by (2.18) and the positive homogeneity of the support function there exists $l" \geqslant l'$ such that

$$t_0 \,\mathrm{Re}(e^{i\psi_0}\lambda) > H_L(\lambda) - \mathrm{Re}(z_0\lambda) + 2\beta|\lambda|, \quad \lambda \in \gamma_l, \quad l \geqslant l".$$

By the previous inequality, Statements 2), 4) of the lemma and the inequality $\delta_0 < 2/5$ this implies

$$\max_{z\in K}\left|\sum_{v=1}^{M_{m_l}} \sum_{n=0}^{n_{m_l,v}-1} c_{m_l,v,n} z^n \exp(\lambda_{m_l,v} z)\right|$$
$$\leqslant \exp\left(\max_{\lambda\in\gamma_l}(H_L(\lambda) - H_K(\lambda) + \beta|\lambda|/3 + H_K(\lambda) - \mathrm{Re}(\alpha\lambda))\right)$$
$$\leqslant \exp\left(\max_{\lambda\in\gamma_l}(H_L(\lambda) + \beta|\lambda|/3 - \mathrm{Re}((z_0 + t_0 e^{i\psi_0})\lambda))\right)$$
$$\leqslant \exp\left(\max_{\lambda\in\gamma_l}(H_L(\lambda) + \beta|\lambda|/3 - \mathrm{Re}(z_0\lambda) - H_L(\lambda) + \mathrm{Re}(z_0\lambda) - 2\beta|\lambda|)\right)$$
$$= \exp\left(\max_{\lambda\in\gamma_l}\left(-\frac{5\beta|\lambda|}{3}\right)\right) \leqslant \exp\left(-\frac{5\beta|\lambda_{m_l,1}|}{3} + \frac{5\beta\delta_0|\lambda_{m_l,1}|}{3}\right) \leqslant \exp(-\beta|\lambda_{m_l,1}|).$$

As in the first case, this contradicts (2.8). Thus, (2.5) and (2.6) are true. The proof is complete. □

Let $\Lambda = \{\lambda_k, n_k\}$ be partitioned into the groups $U = \{U_m\}_{m=1}^{\infty}$, where $U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$. We say that U_m, $m \geqslant 1$, are groups of relatively small diameter if

$$\lim_{m\to\infty} \max_{1\leqslant j,l\leqslant M_m} \frac{|\lambda_{m,j} - \lambda_{m,l}|}{|\lambda_{m,1}|} = 0.$$

We note that the numbers $\lambda_{m,1}$ can be replaced by other representatives $\lambda_{m,j}$ of the groups U_m. This is implied immediately by the relation

$$\lim_{m\to\infty} \max_{1\leqslant j\leqslant M_m} \frac{|\lambda_{m,j}|}{|\lambda_{m,1}|} \leqslant \lim_{m\to\infty} \max_{1\leqslant j\leqslant M_m} \frac{|\lambda_{m,j} - \lambda_{m,1}|}{|\lambda_{m,1}|} + \lim_{m\to\infty} \frac{|\lambda_{m,1}|}{|\lambda_{m,1}|} = 1.$$

We also say that the groups U_m are relatively small if they are groups of relatively small diameter and the identity holds:

$$\lim_{m\to\infty} \frac{N_m}{|\lambda_{m,1}|} = 0.$$

Following work [8], by the system $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$ we construct the system of the functions $\mathcal{E}(\Lambda, U) = \{e_{m,j}(z)\}_{m=1,j=1}^{\infty,N_m}$. Let γ_m be a contour enveloping the points of the group U_m and

$$\omega_m(\lambda) = \prod_{l=1}^{M_m} (\lambda - \lambda_{m,l})^{n_{m,l}}, \quad m \geqslant 1.$$

We let

$$P_m(\lambda, z) = \frac{1}{2\pi i} \int_{\gamma_m} \frac{\exp(z\zeta)(\omega_m(\zeta) - \omega_m(\lambda))}{(\zeta - \lambda)\omega_m(\zeta)} d\zeta, \quad m \geqslant 1.$$

This formula defines the known interpolation polynomial of the degree at mots $N_m - 1$ such that at the points $\lambda_{m,l}$ its value and the values of its derivatives up to the order $n_{m,l} - 1$ coincide with that of the function $\exp(z\lambda)$:

$$P_m^{(n)}(\lambda_{m,l}, f) = z^n \exp(\lambda_{m,l} z), \quad l = 1, 2, \ldots, M_m, \quad n = 0, 1, \ldots, n_{m,l} - 1.$$

We expand $P_m(\lambda, z)$ in powers of $(\lambda - \lambda_{m,1})$. We have:

$$P_m(\lambda, z) = \sum_{j=0}^{N_m - 1} p_{m,j}(z) \frac{(\lambda - \lambda_{m,1})^j}{j!}.$$

We let $e_{m,j}(z) = p_{m,j}(z)$, $m \geqslant 1$, $j = \overline{1, N_m}$. Thus, the function $e_{m,j}(z)$ coincides with $(j-1)$th derivative of the polynomial $P_m(\lambda, z)$ at the point $\lambda_{m,1}$. According the Cauchy integral formula, we have

$$e_{m,j}(z) = \sum_{l=1}^{M_m} \sum_{n=0}^{n_{m,l}-1} c_{m,j,l,n} z^n \exp(\lambda_{m,l} z) = \frac{(j-1)!}{2\pi i} \int_{|\lambda - \lambda_{m,1}| = 1} \frac{P_m(\lambda, z) d\lambda}{(\lambda - \lambda_{m,1})^j}, \quad m \geqslant 1, \quad j = \overline{1, N_m}.$$

We consider the series over the system $\mathcal{E}(\Lambda, U)$:

$$g(z) = \sum_{m=1}^{\infty} \sum_{j=1}^{N_m} c_{m,j} e_{m,j}(z). \tag{2.19}$$

The sequence of its coefficients is denoted by $c = \{c_{m,j}\}_{m=1,j=1}^{\infty, N_m}$.

Let D be a convex domain, $K(D) = \{K_s\}_{s=1}^{\infty}$ be a sequence of convex compact sets exhausting D and Λ be partitions into the groups $U = \{U_m\}_{m=1}^{\infty}$, where $U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$. For each $s = 1, 2, \ldots$ we introduce the Banach space of complex sequences

$$Q_s(D, \Lambda, U) = \{c = \{c_{m,j}\}_{m=1,j=1}^{\infty, N_m} : \|c\|_s = \sup_{m,j} (|c_{m,j}| \exp(H_{K_s}(\lambda_{m,1}))) < \infty\}.$$

By $Q(D, \Lambda, U)$ we denote the projective limit of the spaces $Q_s(D, \Lambda, U)$.

We define an operator $\mathcal{B}$ acting on the space $Q(D, \Lambda, U)$ with values in $W(\Lambda, D)$ by the rule: a sequence $c = \{c_{m,j}\} \in Q(D, \Lambda, U)$ is mapped into the sum $g(z)$ of series (2.19) if it converges in the topology of the space $H(D)$.

For a convex domain D we let

$$D(\Theta(\Lambda)) = \{z \in \mathbb{C} : \mathrm{Re}(z\lambda) < H_D(\lambda), \lambda \in \Theta(\Lambda)\}.$$

The set $D(\Theta(\Lambda))$ is obviously a compact set and contains D. At that, the identity

$$H_D(\lambda) = H_{D(\Theta(\Lambda))}(\lambda), \quad \lambda \in \Theta(\Lambda)$$

holds. As one can see easily, this implies that the spaces $Q(D, \Lambda, U)$ and $Q(D(\Theta(\Lambda)), \Lambda, U)$ coincide.

Theorem 2.2. *Suppose that under assumption of Lemma 2.1 the groups U_m, $m \geqslant 1$, are relatively small. Then each function $g \in W(\Lambda, D)$ is expanded into series (2.19), whose sum is analytic in the domain $D(\Theta(\Lambda))$, that is, g is analytically continued into the domain $D(\Theta(\Lambda))$. At that, for each index s there exist indices p', p and numbers $A', A > 0$ independent of the function $g \in W(\Lambda, D)$ such that*

$$\sum_{m=1}^{\infty}\sum_{j=1}^{N_m}\max_{z\in K_s}|c_{m,j}e_{m,j}(z)| \leqslant A'\|c\|_{p'} \leqslant A\max_{z\in K_p}|g(z)|, \tag{2.20}$$

where $c = \{c_{m,j}\}_{m=1,j=1}^{\infty,N_m}$, $\|c\|_{p'}$ is the norm in the space $Q_{p'}(D(\Theta(\Lambda)), \Lambda, U)$ and $K_s, K_p \in K(D(\Theta(\Lambda)))$. In particular, series (2.19) converges absolutely and uniformly on the compact sets in the domain $D(\Theta(\Lambda))$. Moreover, the operator

$$\mathcal{B} : Q(D(\Theta(\Lambda)), \Lambda, U) \to W(\Lambda, D) \tag{2.21}$$

is an isomorphism of linear topological spaces.

Proof. By assumption, the sequence Λ is partitioned into relatively small groups $U = \{U_m\}_{m=1}^{\infty}$. Then according Theorem 3 in work [9], the system of the functions $\mathcal{E}(\Lambda, U)$ is an almost exponential sequence in the domain D with exponents $\lambda_{m,1}$ in the sense of the definition in work [7]. Moreover, by Theorem 5 in work [9], system $\mathcal{E}(\Lambda, U)$ possess the Köthe group property. This means that for each compact set $K \subset D$ there exists a compact set $K' \subset D$ and a number $A'' > 0$ satisfying the condition: for each $m \geqslant 1$ and each function h_m of form

$$h_m(z) = \sum_{j=1}^{N_m} a_{m,j}e_{m,j}(z) \tag{2.22}$$

the identity

$$\sum_{j=1}^{N_m}|a_{m,j}|\sup_{z\in K}|e_{m,j}(z)| \leqslant A\text{"}\sup_{z\in K'}|h_m(z)| \tag{2.23}$$

holds.

Given $g \in W(\Lambda, D)$, by Lemma 2.1 this function is expanded into series (2.5) and (2.6) holds. We let

$$h_m(z) = \sum_{v=1}^{M_m}\sum_{n=0}^{n_{m,v}-1} c_{m,v,n}z^n \exp(\lambda_{m,v}z).$$

Then by the definition of $e_{m,j}(z)$, the function $h_m(z)$ is of form (2.22). Therefore, according (2.23) and (2.6), for each compact set $K \subset D$, the series converges

$$\sum_{m=1}^{\infty}\sum_{j=1}^{N_m}\max_{z\in K}|c_{m,j}e_{m,j}(z)|. \tag{2.24}$$

This means that series (2.19) converges absolutely and uniformly on compact sets in the domain D. By Lemma 2.1, some subsequence of its partial sums converges to the function g. This is why identity (2.19) holds.

Since series (2.24) converges for each $K \subset D$, by Theorem 3.1 in work [22] being an analogue of Abel theorem for the series of exponential polynomials, for each compact set $K_s \in K(D(\Theta(\Lambda)))$ there exist an index p' and a number $A' > 0$ independent of the function $g \in W(\Lambda, D)$ such that

$$\sum_{m=1}^{\infty}\sum_{j=1}^{N_m}\max_{z\in K_s}|c_{m,j}e_{m,j}(z)| \leqslant A'\|c\|_{p'}, \tag{2.25}$$

where $c = \{c_{m,j}\}_{m=1,j=1}^{\infty,N_m}$ and $\|c\|_{p'}$ is the norm in the space $Q_{p'}(D(\Theta(\Lambda)),\Lambda,U)$. This implies that the function g is continued analytically into the domain $D(\Theta(\Lambda))$ and is represented there by series (2.19), which converges absolutely and uniformly on compact sets in the domain $D(\Theta(\Lambda))$. This means that the spaces $W(\Lambda,D)$ and $W(\Lambda,D(\Theta(\Lambda)))$ coincide and operator (2.21) is a surjection.

By Lemma 2.3 in work [22], operator (2.21) is defined on the entire space $Q(D(\Theta(\Lambda)),\Lambda,U)$. If the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$, as it was mentioned above, the representation by series (1.2) possesses the uniqueness property. Therefore, operator (2.21) is injective. Thus, $\mathcal{B}$ is an isomorphism of linear spaces. By (2.25), the operator $\mathcal{B}$ is continuous. As one can see easily, the spaces $Q(D(\Theta(\Lambda)),\Lambda,U)$ and $W(\Lambda,D)$ are Fréchet spaces. Then by the Banach theorem on the inverse operator for the Fréchet spaces, the operator $\mathcal{B}$ is an isomorphism of topological spaces and the inequality

$$A'\|c\|_{p'} \leqslant A \max_{z\in K_p} |g(z)|$$

holds, where $K_p \in K(D(\Theta(\Lambda)))$ and the index p and the number $A>0$ are independent of the function g. The latter estimate and (2.25) lead us to (2.20). The proof is complete. □

Remark. *In Theorem 2.2, both with representing the functions in $W(\Lambda,D)$, a continuation of these functions is made to a wider convex domain. The problem on such continuation of the functions in the invariant spaces has a rich story. The most general results on this problem both for the case of one and several variables were obtained in works [23]–[25]. These works provide also a historical survey on the continuation problem.*

The partition $U=\{U_m\}_{m=1}^{\infty}$ of a sequence Λ is called trivial if each group U_m of the only point $\lambda_{m,1}$, $m\geqslant 1$. In this case the functions of the system $\mathcal{E}(\Lambda,U)$ are easily calculated. We have

$$e_{m,j}(z) = z^{j-1}\exp(\lambda_{m,1}z), \quad j=1,\dots,N_m(=n_{m,1}), \quad m\geqslant 1.$$

In this case, the relative smallness of the groups is equivalent to the identity

$$\sigma(\Lambda) = \lim_{m\to\infty} n_{m,1}/|\lambda_{m,1}| = \lim_{k\to\infty} n_k/|\lambda_k| = 0.$$

Thus, in a particular case, by Theorem 2.2 we obtain the solution for the fundamental principle problem.

Corollary 2.3. *Suppose that under the assumptions of Lemma 2.1 the partition U is trivial and $\sigma(\Lambda)=0$. Then each function $g\in W(\Lambda,D)$ is expanded into series (1.1), whose sum is analytic in the domain $D(\Theta(\Lambda))$, that is, g is analytically continued in the domain $D(\Theta(\Lambda))$. At that, for each index s there exist indices p', p and numbers A', $A>0$ independent of $g\in W(\Lambda,D)$ such that*

$$\sum_{k=1,n=0}^{\infty,n_k-1} \max_{z\in K_s} |d_{k,n}z^n\exp(\lambda_k z)| \leqslant A'\|d\|_{p'} \leqslant A\max_{z\in K_p}|g(z)|,$$

where $d=\{d_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$, $\|d\|_{p'}$ is the norm in the space $Q_{p'}(D(\Theta(\Lambda)),\Lambda,U)$ and $K_s,K_p\in K(D(\Theta(\Lambda)))$. In particular, series (1.1) converges absolutely and uniformly on compact sets in the domain $D(\Theta(\Lambda))$. Moreover, the operator $\mathcal{B}: Q(D(\Theta(\Lambda)),\Lambda,U)\to W(\Lambda,D)$ is an isomorphism of linear topological spaces.

Now we consider the case, when the functions $f\in P_D$ whose existence is needed in Lemma 2.1 are constructed in advance. Here we need one known characteristics of the sequence $\Lambda =$

$\{\lambda_k, n_k\}_{k=1}^{\infty}$. Let $U = \{U_m\}_{m=1}^{\infty}$, $U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$, be a partition of Λ into the groups. Following work [4], we let

$$q_\Lambda(\lambda, w, \delta) = \prod_{\lambda_k \in B(w,\delta|w|)} \left(\frac{\lambda - \lambda_k}{3\delta|\lambda_k|}\right)^{n_k} = \prod_{\lambda_{m,v} \in B(w,\delta|w|)} \left(\frac{z - \lambda_{m,v}}{3\delta|\lambda_{m,v}|}\right)^{n_{m,v}}.$$

In the case when the ball $B(w, \delta|w|)$ contains none of λ_k, we let $q_\Lambda(\lambda, w, \delta) \equiv 1$. The absolute value of $q_\Lambda(\lambda, w, \delta)$ can be interpreted as a measure of accumulation of the points $\lambda_k \in B(w, \delta|w|)$ at λ. The quantity $\ln |q_\Lambda(\lambda, w, \delta)|/|w|$ is similar by the meaning to the logarithm of the geometric mean (arithmetic mean of the logarithms) of the normed distances from the points $\lambda_k \in B(w, \delta|w|)$ to the point λ.

We introduce also the functions (see [10]): we let

$$q^m_{\Lambda,U}(z, \delta) = \prod_{\lambda_{k,v} \in B(\lambda_{m,1},\delta|\lambda_{m,1}|), k \neq m} \left(\frac{z - \lambda_{k,v}}{3\delta|\lambda_{k,v}|}\right)^{n_{k,v}}, \quad m \geqslant 1.$$

If the ball $B(\lambda_{m,1}, \delta|\lambda_{m,1}|)$ contains no points $\lambda_{k,v}$, $k \neq m$, then $q^m_{\Lambda,U}(z, \delta)$. We note that in distinction of $q_\Lambda(z, w, \delta)$, the function $q^{m(z,\delta)}_{\Lambda,U}$ depends on the partition U of the sequence Λ. If $\delta \in (0, 1)$, then the absolute value of each factor $q^m_{\Lambda,U}$ is estimated from above by $2(3(1 - \delta))^{-1}$ in the ball $B(\lambda_{m,1}, \delta|\lambda_{m,1}|)$. This is why for $\delta \in (0, 1/3)$ it does not exceed one. Therefore,

$$|q^m_{\Lambda,U}(z, \delta_1)| \geqslant |q^m_{\Lambda,U}(z, \delta_2)|, \quad z \in B(\lambda_{m,1}, \delta_2|\lambda_{m,1}|), \tag{2.26}$$

if $0 < \delta_1 \leqslant \delta_2 < 1/3$. We let

$$S_\Lambda(U) = \lim_{\delta \to 0} \varliminf_{m \to \infty} \min_{\lambda_{m,v} \in B(\lambda_{m,1},\delta|\lambda_{m,1}|)} \ln \frac{|q^m_{\Lambda,U}(\lambda_{m,v}, \delta)|}{|\lambda_{m,v}|}.$$

This definition is well-defined since according (2.26) the limit in δ exists. We also note that the inequality $S_\Lambda(U) \leqslant 0$ is true. It is implied by the non-positiveness of $\ln |q^m_{\Lambda,U}(\lambda_{m,v}, \delta)|$ as $\lambda_{m,l} \in B(\lambda_{m,1}, \delta|\lambda_{m,1}|)$ and $\delta \in (0, 1/3)$. If the groups U_m, $m \geqslant 1$, are relatively small, for each $\delta > 0$, beginning with some index $m(\delta)$, all points of the group U_m lie in the ball $B(\lambda_{m,1}, \delta|\lambda_{m,1}|)$. This is why as $m \geqslant m(\delta)$, the minimum of quantity in the definition of $S_\Lambda(U)$ can be taken not over the points $\lambda_{m,v}$ in the ball $B(\lambda_{m,1}, \delta|\lambda_{m,1}|)$ but over all $v = 1, \ldots, M_m$. In the case of the trivial partition U, the quantity $S_\Lambda(U)$ coincides with the quantity S_Λ introduced in work [4].

Corollary 2.4. *Let $\Lambda = \{\lambda_k, n_k\}$ be partitioned into relatively small groups $U = \{U_m\}_{m=1}^{\infty}$, where $U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$ and $\bar{n}(\Lambda) < \infty$. Assume that $S_\Lambda(U) > -\infty$. Then each function $g \in W(\Lambda, \mathbb{C})$ is expanded into series (2.19) over the system $\mathcal{E}(\Lambda, U)$. At that, for each index s there exist indices p', p and numbers $A', A > 0$ independent of $g \in W(\Lambda, \mathbb{C})$) such that*

$$\sum_{m=1}^{\infty} \sum_{j=1}^{N_m} \max_{z \in K_s} |c_{m,j} e_{m,j}(z)| \leqslant A' \|c\|_{p'} \leqslant A \max_{z \in K_p} |g(z)|,$$

where $c = \{c_{m,j}\}_{m=1,j=1}^{\infty,N_m}$, $\|c\|_{p'}$ is the norm in the space $Q_{p'}(\mathbb{C}, \Lambda, U)$ and $K_s, K_p \in K(\mathbb{C})$. In particular, series (2.19) converges absolutely and uniformly on compact sets in the plane. Moreover, the operator $\mathcal{B} : Q(\mathbb{C}, \Lambda, U) \to W(\Lambda, \mathbb{C})$ is an isomorphism of linear topological spaces.

Proof. Let us show that all assumptions of Theorem 2.2 are satisfied. Since the groups U_m are relatively small, it is sufficient to check the assumptions of Lemma 2.1 for the domain $D = \mathbb{C}$.

By assumption, the upper density of the sequence Λ is finite. Then by the aforementioned Lindelöf theorem, the function

$$\sum_{k=1}^{\infty}\left(1-\frac{\lambda^2}{\lambda_k^2}\right)^{n_k}$$

is entire and has an exponential type. It vanishes at the points λ_k with the multiplicities n_k. Therefore, the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(\mathbb{C})$. The set $\Theta(\Lambda)$ does not intersect the boundary of the set $J(D)$, that is, $J(D)=\mathbb{C}\setminus\{0\}$.

Let K_0 be a convex compact set, $\delta_0>0$, and $\{U_(m_l)\}_{l=1}^{\infty}$ be subsequence of the groups such that $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges.

By Theorem 4.1 in work [12], there exists a sequence $\Lambda'=\{\xi_l,1\}_{l=1}^{\infty}$ having no common points with Λ and such that

1) $\tilde{\Lambda}=\Lambda\cup\Lambda'$ is the zero set (counting multiplicities) of an entire function $\tilde{f}$ of an exponential type, that is, $\tilde{f}\in P_{\mathbb{C}}$;

2) The partition $\tilde{U}=U\cup U'$ of the sequence $\tilde{\Lambda}$, where U' is the trivial partition of Λ', satisfies the inequality $S_{\tilde{\Lambda}}(\tilde{U})>-\infty$ and $\tilde{U}$ and is a partition into relatively small groups.

Then Theorem 5.1 in work [12] implies that there exist positive numbers $\{\alpha_{m,j}\}_{j=1,m=1}^{M_m,\infty}$ satisfying the conditions

$$\lim_{m\to\infty}\max_{1\leqslant s,j\leqslant M_m}\frac{\alpha_{m,j}}{|\lambda_{m,s}|}=0, \tag{2.27}$$

the sets $B_m=\bigcup_{j=1}^{M_m}B(\lambda_{m,j},\alpha_{m,j},\ m\geqslant 1$, are mutually disjoint and the diameters d_m of the sets B_m obey the relation

$$\lim_{m\to\infty}\max_{1\leqslant j\leqslant M_m}\frac{d_m}{|\lambda_{m,j}|}=0. \tag{2.28}$$

There exist b, $b_1>0$ such that

$$\ln|\tilde{f}(\lambda)|\geqslant -b_1-b|\lambda|,\quad \lambda\in\partial B_m,\quad m\geqslant 1. \tag{2.29}$$

We let $f(\lambda)=\tilde{f}(z)e^{\tau\lambda}$ and $\gamma_l=\partial B_{m_l}$, $l\geqslant 1$. By property 1) of the sequence $\tilde{\Lambda}$ we obtain Statement 1) of Lemma 2.1 for each $\tau\in\mathbb{C}$. Statement 2) of this lemma is implied by the definition of the sets B_m and the fact that they are mutually disjoint. Relation (2.28) gives us Statement 4) of Lemma 2.1. According the definition of sets B_m, the length of the contour γ_l satisfies the estimate:

$$\rho(\gamma_l)\leqslant 2\pi\sum_{j=1}^{N_{m_l}}\alpha_{m_l,j}.$$

Since the groups U_m are relatively small, then

$$\lim_{m\to\infty}\frac{M_m}{|\lambda_{m,1}|}=0.$$

By (2.27) it leads us to Statement 5) of Lemma 2.1. It remains to show that for an appropriate number $\tau\in\mathbb{C}$ Statement 3) of this lemma holds.

Let $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converge to $e^{i\varphi_0}$. We let $\tau=t_0e^{-i\varphi_0}$. Them

$$\ln|f(\lambda)|=\ln|\tilde{f}(\lambda)|+t_0\operatorname{Re}(e^{-i\varphi_0}\lambda).$$

By the continuity and the positive homogeneity of the support function there exist a, δ, $t_0>0$ such that

$$t_0\operatorname{Re}(e^{-i\varphi_0}\lambda)-b_1-b|\lambda|>H_{K_0}(\lambda),\quad \lambda/|\lambda|\in B(e^{i\varphi_0},\delta),\quad |\lambda|\geqslant a.$$

Since $\{\lambda_{m_l,1}/|\lambda_{m_l,1}|\}_{l=1}^{\infty}$ converges to $e^{i\varphi_0}$, by the definition of the sets B_m and relations (2.28), (2.29), the latter identity implies Statement 3 of Lemma 2.1. The proof is complete. □

Remark. *Earlier the result of Corollart 2.4 was obtained by A.S. Krivosheev in [12, Sect. 9] via solving a special rather complicated interpolation problem in the space of entire functions of exponential type.*

3. Necessary conditions

Let us show that a condition for the condensation index $S_\Lambda(U)$ similar to one presented in the latter statement is necessary for the existence of the basis in the most general case, for an arbitrary invariant subspace admitting the spectral synthesis in an arbitrary convex domain.

Let D be a convex domain and a sequence $\Lambda = \{\lambda_k, n_k\}$ is partitioned into the groups $U = \{U_m\}_{m=1}^{\infty}$ ($U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$) of relatively small diameter. For a set E in the circumference $S(0,1)$, by the symbol $\Lambda(E)$ we denote the subsequence Λ consisting of all groups U_m such that $\lambda_{m,1}/|\lambda_{m,1}| \in E$. Following [4], we let

$$S_\Lambda(U,F) = \sup_{E\supset F} S_{\Lambda(E)}(U), \qquad S_\Lambda(U,D) = \inf_{F\supset(S(0,1)\setminus J(D))} S_\Lambda(U,F),$$

where the supremum is taken over all sets $E \supset F$ open in $S(0,1)$, while the infimum is taken over all compact subsets $F \supset (S(0,1) \setminus J(D))$.

Lemma 3.1. *Let D be a convex domain and a sequence Λ is partitioned into the groups $U = \{U_m\}_{m=1}^{\infty}$ $\left(U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}\right)$ of a relatively small diameter. Assume that the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and each function $g \in W(\Lambda, D)$ is expanded into series (2.5) converging w.r.t. m uniformly on compact sets in D. Then $S_\Lambda(U,D) = 0$.*

Proof. Assume that $S_\Lambda(U,D) \leqslant -3\beta < 0$. Then there exists a compact set $F \subset (S(0,1)\setminus J(D))$ such that $S_{\Lambda(E)}(U) \leqslant -2\beta$ for each set $E \supset F$ open in $S(0,1)$. This is why according the definition of $S_{\Lambda(E)}(U)$, for each $p \geqslant 1$ there exist indices $m(p)$, $v(p)$ and a number $\delta_p \in (0, 1/4p)$ satisfying the conditions

$$\min_{\lambda\in F}\left|\frac{\lambda_{m(p),1}}{|\lambda_{m(p),1}|} - \lambda\right|, \tag{3.1}$$

$$\frac{\ln|q_{\Lambda,U}^{m(p)}(\lambda_{m(p),v(p)}, \delta_p)|}{|\lambda_{m(p),v(p)}|} \leqslant -\beta, \tag{3.2}$$

$$|\lambda_{m(p+1),v(p+1)}| \geqslant 2|\lambda_{m(p),v(p)}|. \tag{3.3}$$

Passing to a subsequence, we can assume that $\{\lambda_{m(p),1}/|\lambda_{m(p),1}|\}_{p=1}^{\infty}$ converges to $e^{i\varphi_0}$. By (3.1), the inclusion holds: $e^{i\varphi_0} \in F$. In particular, $H_D(e^{i\varphi_0}) < +\infty$.

We consider the functions

$$g_p(z) = \frac{1}{2\pi i}\int\limits_{S(\lambda_{m(p),v(p)},5\delta_p|\lambda_{m(p),v(p)}|)} \frac{\exp(\lambda z)d\lambda}{(\lambda - \lambda_{m(p),v(p)})q_{\Lambda,U}^{m(p)}(\lambda,\delta_p)}, \quad p \geqslant 1. \tag{3.4}$$

Let us find upper bounds for $|g_p|$. Taking into consideration that $\delta_p < 1/4$, we have

$$\begin{aligned} |q_{\Lambda,U}^{m(p)}(\lambda,\delta_p)| &= \prod_{\lambda_{m,v}\in B(\lambda_{m(p),v(p)},\delta_p|\lambda_{m(p),v(p)}|), m\neq m(p)} \left|\frac{\lambda - \lambda_{m,v}}{3\delta|\lambda_{m,v}|}\right|^{n_{k,v}} \\ &\geqslant \left(\frac{4\delta_p|\lambda_{m(p),v(p)}|}{(3\delta_p(1+\delta_p)|\lambda_{m(p),v(p)}|}\right)^{s(p)} > 1, \qquad \lambda \in S(\lambda_{m(p),v(p)}, 5\delta_p|\lambda_{m(p),v(p)}|), \end{aligned} \tag{3.5}$$

where $s(p)$ is the number of the points $\lambda_{k,v}$, $k \neq m(p)$, counting the multiplicities located in the ball $B(\lambda_{m(p),v(p)}, \delta_p|\lambda_{m(p),v(p)})$. By (3.5) we have

$$\begin{aligned} |g_p(z)| \leqslant & 5\delta_p|\lambda_{m(p),v(p)}| \sup_{\lambda \in S(\lambda_{m(p),v(p)}, 5\delta_p|\lambda_{m(p),v(p)}|)} \left| \frac{\exp(\lambda z)}{(\lambda - \lambda_{m(p),v(p)}} \right| \\ \leqslant & \exp(\mathrm{Re}(\lambda_{m(p),v(p)} z) + 5\delta_p|\lambda_{m(p),v(p)})||z|). \end{aligned} \tag{3.6}$$

Let K be an arbitrary compact set in the domain D. Then

$$\mathrm{Re}(ze^{i\varphi_0}) \leqslant H_D(e^{i\varphi_0}) - 2\tau, \quad z \in K, \tag{3.7}$$

for some number $\tau > 0$. Since U_m are the groups of relatively small diameters, the sequence $\{\lambda_{m(p),v(p)}/|\lambda_{m(p),v(p)}|\}_{p=1}^{\infty}$, as $\{\lambda_{m(p),1}/|\lambda_{m(p),1}|\}_{p=1}^{\infty}$, converges to $e^{i\varphi_0}$. This is why in view of (3.6), (3.7) and $\delta_p < 1/4p \to 0$ we obtain

$$\begin{aligned} |g_p(z)| \leqslant & \exp\left(|\lambda_{m(p),v(p)}| \left(\mathrm{Re}\, \frac{\lambda_{m(p),v(p)} z}{|\lambda_{m(p),v(p)}|} + 5\delta_p \right) \right) \\ \leqslant & \exp(|\lambda_{m(p),v(p)}|(\mathrm{Re}(ze^{i\varphi_0}) + \tau)) \leqslant \exp(|\lambda_{m(p),v(p)}|(H_D e^(i\varphi_0) - \tau)), \end{aligned} \tag{3.8}$$

where $z \in K$ and $p \geqslant p(K)$.

We consider the function

$$g(z) = \sum_{p=1}^{\infty} c_p g_p(z), \tag{3.9}$$

where $c_p = \exp(-|\lambda_{m(p),v(p)}|H_D(e^{i\varphi_0}))$, $p \geqslant 1$. By (3.8) and (3.3)

$$\sum_{p=p(K)}^{\infty} |c_p g_p(z)| \leqslant \sum_{p=p(K)}^{\infty} \exp(-\tau|\lambda_{m(p),v(p)}|) < \infty, \quad z \in K.$$

Therefore, series (3.9) converges uniformly on compact sets in the domain D. Employing the residues, by definition of g_p we have

$$g_p(z) = d_p \exp(\lambda_{m(p),v(p)} z) + \sum_{\lambda_{m,v} \in B(\lambda_{m(p),v(p)}, \delta_p|\lambda_{m(p),v(p)}|), m \neq m(p)} \sum_{n=0}^{n_{m,v}-1} d_{m,v,n} z^n \exp(\lambda_{m,v} z), \tag{3.10}$$

$$d_p = \left(q_{\Lambda,U}^{m(p)}(\lambda_{m(p),v(p)}, \delta_p) \right)^{-1}. \tag{3.11}$$

By (3.9) this implies $g \in W(\Lambda, D)$. Let us show that opposed to the assumption of the lemma, the function g is not expanded into series (2.5) uniformly on compact sets in the domain D.

By assumption, the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$. Therefore, as it has been mentioned above, there exists a biorthogonal to $\mathcal{E}(\Lambda)$ system of functionals $\Xi(\Lambda, D)$. Since $\delta_p < 1/4$, and U_m are groups of relatively small diameter, according (3.3), the points $\lambda_{m(p),v}$, $v = \overline{1, M_{m(p)}}$, do not lie in the ball $B(\lambda_{m(j),v(j)}, \delta_p|\lambda_{m(j),v(j)}|)$ if $j \neq p$. This is why by (3.10)

$$\mu_{m(p),v(p),0}(g) = d_p c_p, \quad \mu_{m(p),v(p),n}(g) = 0, \quad n = \overline{1, n_{m(p),v(p)} - 1}, \quad p \geqslant 1, \tag{3.12}$$

$$\mu_{m(p),v,n}(g) = 0, \quad n = \overline{0, n_{m(p),v} - 1}, \quad v = \overline{1, M_{m(p)}}, \quad v \neq v(p), \quad p \geqslant 1, \tag{3.13}$$

where $\{\mu_{m(p),v,n}\} \in \Xi(\Lambda, D)$ is a system biorthogonal to the system $\{z^n \exp(\lambda_{m(p),v} z)\}$.

By assumption, the function g is expanded into series (2.5) converging uniformly on compact sets in D. Then

$$c_{m(p),v,n} = \mu_{m(p),v,n}(g), \quad n = \overline{0, n_{m(p),v} - 1}, \quad v = \overline{1, M_{m(p)}}, \quad p \geqslant 1. \tag{3.14}$$

In view of (3.12) and (3.13) this implies that the term of series (2.5) with the index $m = m(p)$ reads as

$$d_p c_p \exp(\lambda_{m(p),v(p)} z), \quad p \geqslant 1. \tag{3.15}$$

According the definition of the support function, there exists a point $z_0 \in D$ such that

$$\mathrm{Re}(z_0 e^{i\varphi_0}) \geqslant H_D(e^{i\varphi_0}) - \beta/2.$$

Since $\{\frac{\lambda_{m(p),v(p)}}{|\lambda_{m(p),v(p)}|}\}_{p=1}^{\infty}$ converges to $e^{i\varphi_0}$, then

$$\mathrm{Re}\, \frac{\lambda_{m(p),v(p)} z_0}{|\lambda_{m(p),v(p)}|} \geqslant H_D(e^{i\varphi_0}) - \beta, \quad p \geqslant p_0.$$

Therefore, in view of (3.11), (3.2) and the definition of c_p we have:

$$\begin{aligned}|d_p c_p \exp(\lambda_{m(p),v(p)} z_0)| =& \frac{\exp(-|\lambda_{m(p),v(p)}| H_D(e^{i\varphi_0})}{|q^{m(p)}_{\Lambda,U}(\lambda_{m(p),v(p)}, \delta_p)|} \exp(\mathrm{Re}(\lambda_{m(p),v(p)} z_0)) \\ \geqslant& \exp(|\lambda_{m(p),v(p)}|(\beta - H_D(e^{i\varphi_0}))) \exp\left(|\lambda_{m(p),v(p)}|\, \mathrm{Re}\, \frac{\lambda_{m(p),v(p)} z_0}{|\lambda_{m(p),v(p)}|}\right) \geqslant 1\end{aligned}$$

as $p \geqslant p_0$. This contradicts the convergence of series (2.5) at the point $z_0 \in D$. Hence, our initial assumption is wrong and $S_\Lambda(U, D) \geqslant 0$. As it has been mentioned above, the inequality $S_\Lambda(U, D) \leqslant 0$ is true as well. Therefore, $S_\Lambda(U, D) = 0$. The proof is complete. □

Lemma 3.2. *Let D be a convex domain and a sequence Λ is partitioned into groups $U = \{U_m\}_{m=1}^{\infty}$ ($U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}$) of relatively small diameter. Assume that the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and $S_\Lambda(U) = -\infty$. Then there exists $g \in W(\Lambda, \mathbb{C})$, which can not be represented as series (2.5) converging uniformly on compact sets in D.*

Proof. By assumption, $S_\Lambda(U) = -\infty$. This is why there exist numbers $\delta_p \in (0, 1/4p)$, $p \geqslant 1$, and a sequence $\{\lambda_{m(p),v(p)}\}$ such that

$$\lim_{p\to\infty} \frac{\ln |q^{m(p)}_{\Lambda,U}(\lambda_{m(p),v(p)}, \delta_p)|}{|\lambda_{m(p),v(p)}|} = -\infty. \tag{3.16}$$

We can assume that (3.3) holds for all $p \geqslant 1$.

We define $g_p(z)$, $p \geqslant 1$, by formula (3.4). Then, as in Lemma 3.1, relations (3.6), (3.10) and (3.11) hold. We consider the function $g(z)$ defined by formula (3.9), where we let

$$c_p = \sqrt{|q^{m(p)}_{\Lambda,U}(\lambda_{m(p),v(p)}, \delta_p)|}, \quad p \geqslant 1.$$

Let $R > 0$. By (3.16), there exists an index p_0 such that

$$c_p = |c_p| \leqslant \exp(-2R|\lambda_{m(p),l(p)}|), \quad p \geqslant p_0.$$

Then in view of (3.6) we get

$$\sum_{p=1}^{\infty} |c_p| \max_{|z|\leqslant R} |g_p(z)| \leqslant A + \sum_{p=p_0}^{\infty} \exp\left((-2R + R + 5\delta_p R)|\lambda_{m(p),l(p)}|\right).$$

Since $\delta_p \to 0$ and (3.3) holds, the latter series converges. Therefore, the latter series converges. Therefore, by (3.10), the belonging $g \in W(\Lambda, \mathbb{C})$ holds, and as in Lemma 3.1, identities (3.12), (3.13) hold.

Assume that the function g can be represented by series (2.5) converging uniformly on compact sets in the domain D. Then (3.14) holds. This is why, as in Lemma 3.1, the term of series

(2.5) with an index $m = m(p)$ is of form (3.15). It follows from (3.11), the definition of the numbers c_p and convergence of series (2.5) that

$$|d_p c_p \exp(\lambda_{m(p),v(p)} z)| = \frac{|\exp(\lambda_{m(p),v(p)} z)|}{\sqrt{|q^{m(p)}_{\Lambda,U}(\lambda_{m(p),v(p)}, \delta_p)|}} \to 0, \quad p \to \infty, \quad z \in D.$$

By (3.16), it is impossible. Thus, our assumption is wrong. The proof is complete. □

Lemmata 3.1 and 3.2 imply immediately the following result.

Theorem 3.3. *Let D be a convex domain and a sequence Λ is partitioned into groups $U = \{U_m\}_{m=1}^{\infty}$ $\left(U_m = \{\lambda_{m,v}\}_{v=1}^{M_m}\right)$ of relatively small diameter. Assume that the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and each function $g \in W(\Lambda, D)$ is expanded into series (2.5) converging w.r.t. m uniformly on compact sets D. Then $S_\Lambda(U, D) = 0$ and $S_\Lambda(U) > -\infty$.*

Remark. *1. Earlier the result of Theorem 3.3 was obtained in Theorem 5.1 in work [4]. In this theorem there was considered a case, when $U = \{U_m\}_{m=1}^{\infty}$ is the trivial partition; here U_m are immediately groups of relatively small diameter. At that, an additional condition was imposed: $m_D(\Lambda) = 0$, that is, $n_{k(j)}/|\lambda_{k(j)}| \to 0$, $j \to \infty$ for an arbitrary subsequence $\{\lambda_{k(j)}\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}| \to \xi$ and $H_D(\xi) < +\infty$. Under this condition, the groups U_m become relatively small. We also observe that in work [4], this result was obtained via solving a rather complicated interpolation problem in the space of entire functions of exponential type.*

2. Theorem 3.3 and Corollary 2.4 imply Theorem 9.1 in work [12].

BIBLIOGRAPHY

1. I.F. Krasichkov-Ternovskii. *Invariant subspaces of analytic functions. I. Spectral analysis on convex regions* // Matem. Sborn. **88(130)**:1, 459–489 (1972). [Math. USSR-Sb. **16**:4, 471–500 (1972).]
2. I.F. Krasichkov-Ternovskii. *Invariant subspaces of analytic functions. II. Spectral synthesis of convex domains* // Matem. Sborn. **88(130)**:1, 3–30 (1972). [Math. USSR-Sb. **17**:1, 1–29 (1972).]
3. A. A. Gol'dberg, B. Ya. Levin, I. V. Ostrovskii. *Entire and meromorphic functions* // Itogi Nauki Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr. **85**, 5–185 (1991). [Complex Analysis I. Encycl. Math. Sci. **85**, 1–193 (1995).]
4. A.S. Krivosheev. *A fundamental principle for invariant subspaces in convex domains* // Izv. RAN. Ser. Matem. **68**:2, 71–136 (2004). [Izv. Math. **68**:2, 291–353 (2004).]
5. O.A. Krivosheeva, A.S. Krivosheev. *A criterion for the fundamental principle to hold for invariant subspaces on bounded convex domains in the complex plane* // Funkts. Anal. Pril. **46**:4, 14–30 (2012). [Funct. Anal. Appl. **46**:4, 249–261 (2012).]
6. A.F. Leont'ev. *Sequences of exponential polynomials.* Nauka, Moscow (1980). (in Russian).
7. A.S. Krivosheyev. *An almost exponential basis* // Ufimskij Matem. Zhurn. **2**:1, 87–96 (2010). (in Russian).
8. A.S. Krivosheyev. *Bases "in relatively small groups"* // Ufimskij Matem. Zhurn. **2**:2, 67–89 (2010). (in Russian).
9. A.S. Krivosheyev. *An almost exponential sequence of exponential polynomials* // Ufimskij Matem. Zhurn. **4**:1, 88–106 (2012). [Ufa Math. J. **4**:1, 82–100 (2012).]
10. A.S. Krivosheev, O.A. Krivosheeva. *A basis in an invariant subspace of analytic functions* // Matem. Sborn. **204**:12, 49–104 (2013). [Sb. Math. **204**:12, 1745–1796 (2013).]
11. A.S. Krivosheev, O.A. Krivosheeva. *Fundamental principle and a basis in invariant subspaces* // Matem. Zamet. **99**:5, 684–697 (2016). [Math. Notes. **99**:5, 685–696 (2016).]
12. A.S. Krivosheev, O.A. Krivosheeva. *A basis in invariant subspace of entire functions* // Alg. Anal. **27**:2, 132–195 (2015). [St. Petersburg Math. J. **27**:2, 273–316 (2016).]
13. A.S. Krivosheyev, O.A. Krivosheyeva. *A closedness of set of Dirichlet series sums* // Ufimskij Matem. Zhurn. **5**:3, 96–120 (2013). [Ufa Math. J. **5**:3, 94–117 (2013).]

14. O.A. Krivosheyeva, A.S. Krivosheyev. *A representation of functions from an invariant subspace with almost real spectrum* // Alg. Anal. **29**:4, 82–139 (2017). [St. Petersburg Math. J. **29**:4, (2018), to appear.]
15. A.S. Krivosheev. *Representation of solutions of a nomegeneous convolution equation in convex domains of the space* $\mathbb{C}^n$ // Izv. RAN. Ser. Mat. **58**:1, 71–91 (1994). [Russ. Acad. Sci. Izv. Math. **44**:1, 69–89 (1995).]
16. A.S. Krivosheev. *Interpolation with estimates in* $\mathbb{C}^n$ *and its applications* // Matem. Sborn. **192**:9, 39–84 (2001). [Sb. Math. **192**:9, 1297–1340 (2001).]
17. A.F. Leont'ev. *Exponential series.* Nauka, Moscow (1976). (in Russian).
18. O.A. Krivosheyeva. *Singular points of the sum of a series of exponential monomials on the boundary of the convergence domain* // Alg. Anal. **23**:2, 162–205 (2011). [St.-Petersburg Math. J. **23**:2, 321–350 (2012).]
19. A.F. Leont'ev. *Entire functions. Exponential series.* Nauka, Moscow (1983). (in Russian).
20. V.V. Napalkov. *Convolution equations in multi-dimensional spaces.* Nauka, Moscow (1982). (in Russian).
21. B.Ya. Levin. *Distribution of zeros of entire functions.* Gostekhizdat, Moscow (1956). [Amer. Math. Soc., Providence, RI (1980).]
22. O.A. Krivosheyeva. *Convergence domain for series of exponential polynomials* // Ufimskij Matem. Zhurn. **5**:4, 84–90 (2013). [Ufa Math. J. **5**:4, 82–87 (2013).]
23. A.S. Krivosheyev. *The invariant subspaces in convex domains in* $\mathbb{C}^n$ // Ufimskij Matem. Zhurn. **1**:2, 53–74 (2009). (in Russian).
24. A.S. Krivosheyev. *The invariant subspaces in convex domains in* $\mathbb{C}^n$ // Ufimskij Matem. Zhurn. **1**:3, 65–86 (2009). (in Russian).
25. A.S. Krivosheev. *Criterion of analytic continuability of functions in principal invariant subspaces on convex domains in* $\mathbb{C}^n$ // Alg. Anal. **22**:4, 137–197 (2010). [St. Petersburg Math. J. **22**:4, 615–655 (2011).]

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