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PERTURBATION OF SECOND ORDER NONLINEAR EQUATION BY DELTA-LIKE POTENTIAL

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Abstract. We consider boundary value problems for one-dimensional second order quasi-linear equation on bounded and unbounded intervals I of the real axis. The equation perturbed by the delta-shaped potential $\varepsilon^{-1}Q\left(\varepsilon^{-1}x\right)$, where $Q(\xi)$ is a compactly supported function, $0<\varepsilon\ll 1$. The mean value of $\langle Q\rangle$ can be negative, but it is assumed to be bounded from below $\langle Q\rangle \geqslant -m_0$. The number m_0 is defined in terms of coefficients of the equation. We study the convergence rate of the solution of the perturbed problem u^ε to the solution of the limit problem u_0 as the parameter ε tends to zero. In the case of a bounded interval I, the estimate of the form $|u^\varepsilon(x)-u_0(x)|<C\varepsilon$ is established. As the interval I is unbounded, we prove a weaker estimate $|u^\varepsilon(x)-u_0(x)/<C\varepsilon^{1/2}$. The estimates are proved by using original cut-off functions as trial functions. For simplicity, the proof of the existence of solutions to perturbed and limiting problems are made by the method of contracting mappings. The disadvantage of this approach, as it is known, is the smallness of the nonlinearities in the equation. We consider the cases of the Dirichlet, Neumann and Robin condition.

Keywords: second order nonlinear equation, delta-like potential, small parameter.

Mathematics Subject Classification: 34E15

1. Introduction

Let I be either a bounded interval (a,b) or a semi-axis (a,∞) or $(-\infty,b)$ or the entire axis $(-\infty,\infty)$, $\{0\}\in I$, $a<-1$, $b>1$, $\Omega:=I\times(-\infty,\infty)$, $0<\varepsilon\ll 1$. We denote by $\mathcal{L}$ and $\mathcal{L}_\varepsilon$ the mappings from the space $W_2^1(I)$ into the space of generalized functions $D'(I)$ of the form:

$$\begin{aligned}
&\mathcal{L}u=-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(x,u\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\frac{\mathrm{d}}{\mathrm{d}x}p(x,u)+q_1\left(x,u\right)+q_2(x)u,\\
&\mathcal{L}_\varepsilon=\mathcal{L}+\varepsilon^{-1}Q\left(\frac{x}{\varepsilon}\right).
\end{aligned}$$

For the functions involved in $\mathcal{L}$, the following conditions are imposed:

$$\begin{aligned}
&k,p,q_1\in C^1(\overline{\Omega}), \qquad q_2,Q\in C(\overline{I}),\\
&0<q_0\leqslant q_2(x)\leqslant \overline{q}_2, \qquad |Q(x)|\leqslant\overline{Q}, \quad x\in I,\\
&0<k_0\leqslant k(x,s), \quad (x,s)\in\Omega,\\
&|k(x,s)|\leqslant\overline{k}(M), \quad x\in I, \qquad |s|\leqslant M,
\end{aligned}\tag{1}$$

for each $M>0$.

T.R. Gadylshin, F.Kh. Mukminov, Perturbation of second order nonlinear equation by delta-like potential.

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Without loss of generality we assume that $p(x,0)=0$, $q_1(x,0)=0$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}p(x,u)+q_1(x,u)=\frac{\mathrm{d}}{\mathrm{d}x}(p(x,u)-p(x,0))+p(x,0)'+(q_1(x,u)-q_1(x,0))+q_1(x,0),$$

and the terms $p(x,0)'+q_1(x,0)$ are moved to the right hand side of the equation $\mathcal{L}u=f$.

We suppose that $\operatorname{supp} Q\subset[-1,1]$ and we impose the following restriction for the mean of the function Q:

$$\langle Q\rangle:=\int\limits_{-1}^{1}Q(\tau)\mathrm{d}\tau>-\frac{\min\{k_0;q_0\}}{4}. \tag{2}$$

That is, the mean $\langle Q\rangle$ can be negative.

The nonlinearities involved in the operator $\mathcal{L}$ are assumed to be small in the following sense. We denote

$$m_{kq}=\min\{k_0;q_0\},\qquad \gamma=\frac{3}{4}m_{kq},\qquad K_g(M)=\sup_{x\in\overline{I},|s|\leqslant M}|g_s(x,u)|,$$

where $g(x,s)$ is an arbitrary smooth function. For some M we let

$$A(M)=k_0^{-1}\left(2(\overline{k}(M)+\overline{q}_2+m_{kq})+3\overline{Q}\right)M. \tag{3}$$

We assume that there exist constants M and $\gamma_1\in(0,\gamma)$ such that the inequalities hold:

$$6K_p(2M)+2K_k(2M)A+2K_{q_1}(2M)<\gamma_1, \tag{4}$$

$$(p_a(M)-h_a)H_a\leqslant 0,\qquad p_a(M)=\sup_{|s|\leqslant M}|p_s(a,s)|, \tag{5}$$

$$(p_b(M)+h_b)H_b\geqslant 0,\qquad p_b(M)=\inf_{|s|\leqslant M}|p_s(b,s)|. \tag{6}$$

The class of nonlinearities satisfying the above conditions is rather wide. For instance, let the nonlinearities be proportional to a small parameter μ:

$$k(x,s)=\mu k_1(x,s)+k(x,0),\qquad k_1(x,s)\geqslant 0,\qquad p(x,s)=\mu\overline{p}(x,s),\qquad q_1(x,s)=\mu\overline{q}_1(x,s).$$

Then the Lipschitz constants satisfy the formulae $K_p(M)=\mu K_{\overline{p}}(M)$ and the smallness conditions are satisfied for sufficiently large M provided μ is small enough. In particular, in the case of a linear operator $\mathcal{L}$ we have $\mu=0$ and the number M can be arbitrary.

In the case when I is a bounded interval (a,b), we consider the boundary value problem

$$\mathcal{L}_\varepsilon u^\varepsilon=f,\ x\in I,\quad l_a u^\varepsilon=0,\ l_b u^\varepsilon=0; \tag{7}$$

$$l_a u^\varepsilon:=h_a u^\varepsilon(a)-H_a k(a,u^\varepsilon(a))\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}(a),\quad l_b u^\varepsilon:=h_b u^\varepsilon(b)+H_b k(b,u^\varepsilon(b))\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}(b),$$

where h_a, $h_b\geqslant 0$, H_a, H_b are either 0 or 1, $h_a+H_a>0$, $h_b+H_b>0$. If I is either a semi-axis (a,∞) or $(-\infty,b)$ or the entire axis $(-\infty,\infty)$, the boundary conditions at the infinities are formally imposed as

$$\lim_{x\to\pm\infty}u(x)=0,$$

but in fact, they are ensured by a choice of the spaces in which the solution to the problem is sought. In what follows we consider all four types of the interval I and for the sake of brevity, in all case we shall employ writing (7).

In the same way we treat the boundary value problem

$$\mathcal{L}u_0=f,\quad x\in I\backslash\{0\}, \tag{8}$$

$$l_a u_0=0,\quad l_b u_0=0,\qquad k(0,u_0(0))\{u_0'\}(0)=\langle Q\rangle\,u_0(0), \tag{9}$$

where the notation has been employed:

$$\{h\}(0):=h(+0)-h(-0).$$

The main aim of the work is to prove the following statement.

Theorem 1. *Let $I = (a, b)$ be a bounded interval and conditions (1), (2), (4)–(6) are satisfied. Then for each $f \in L_2(I)$ such that $\|f\|_{L_2(I)} \leqslant \gamma_1 M/2$, the solution u^ε to boundary value problem (7) satisfies the inequality*

$$\|u^\varepsilon - u_0\|_{C(\bar{I})} \leqslant C\varepsilon,$$

where u_0 is the solution to boundary value problem (8), (9).

In the case of an unbounded interval I we establish a weaker statement.

Theorem 2. *Assume that conditions (1), (2), (4)–(6) hold. Then for each $f \in L_2(I)$ such that $\|f\|_{L_2(I)} \leqslant \gamma_1 M/2$, the solution u^ε to boundary value problem (7) satisfies the inequality*

$$\|u^\varepsilon - u_0\|_{C(\bar{I})} \leqslant C\varepsilon^{\frac{1}{2}},$$

where u_0 is the solution to boundary value problem (8), (9).

Earlier, by employing another technique, similar results were established in work [1] by the authors for a semi-linear equation with a coefficient $k = k(x)$ independent of u.

Problem (8), (9) in fact involves the operator $\mathcal{L}u + \delta(x)u$ being a singular perturbation of a nonlinear operator $\mathcal{L}u$. In book [2], there was considered the self-adjoint operator generated by the differential expression

$$H_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \varepsilon^{-1}V(\varepsilon^{-1}x), \qquad V \in L_1(\mathbb{R}),$$

on the real axis and the existence of the limit of the resolvent was proved [2, Thm 3.2.3]

$$n - \lim_{\varepsilon\to 0}(H_\varepsilon - k^2)^{-1} = (\Delta_\alpha - k^2)^{-1},$$

where

$$\Delta_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha\delta(x), \qquad \alpha = \int_{\mathbb{R}} V(x)\mathrm{d}x.$$

In work [3], there was considered the self-adjoint operator $H_{\mu,\varepsilon}$ generated by the differential expression

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W(x) + \mu^{-1}V(\varepsilon^{-1}x), \qquad V, W \in L_\infty(\mathbb{R}),$$

on the real axis, where V is compactly supported. The complete asymptotics was constructed for a simple eigenvalue of the operator $H_{\mu,\varepsilon}$ as $\lambda, \mu \to 0$.

In work [4], there were considered boundary value problems on a segment $[a, b]$ for the Schrödinger equation with the potential being the sum $q(x, \mu^{-1}x) + \varepsilon^{-1}Q(\varepsilon^{-1}x)$, where $q(x, y)$ is a 1-periodic in y function, $Q(x)$ is a compactly supported function, $0 \in (a, b)$, μ, ε are small positive parameters. By a combination of the multi-scale method and the method of matching asymptotic expansions, the solutions to these boundary value problems were constructed up to an error $O(\mu + \varepsilon)$.

There are a lot of known results on linear operators with singular coefficients. In work [5], a well-defined Sturm-Liouville operator

$$l_\varepsilon y = -y''(x) + u'_\varepsilon(x)y(x)$$

is provided and there was proved the existence of the limit of its resolvent in the case $u_\varepsilon \to u \in L_2(0, 1)$. In work [6] these results were extended for linear operators of higher even order.

In work [7], in the space $L_2([0, \infty) \setminus X)$, $X = \{x_j\}_{j=1}^\infty$, the operator $H_{X,\alpha}$ generated by the differential expression

$$l_{x,\alpha} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{j=1}^{\infty} \alpha_j\delta(x - x_j)$$

was considered. The self-adjoint and lower-semibounded properties and the discreteness of the spectrum were studied for the operator $H_{X,\alpha}$ in the case $\inf\{x_j - x_{j-1}\}_{j=1}^{\infty} = 0$.

In works [8, 9] there were considered perturbations of a non-stationary Schrödinger equation by potentials with small supported. In particular, it was proved in [8] that if non-negative potentials $V_m(x) \in L_2(\mathbb{R}^n)$ have compact supports S_m with capacities tending to zero as $m \to \infty$, the associated semi-groups tend to the semi-group of the non-perturbed equation. In [9] this result was extended to a wider class of potentials.

The fact that the delta-like potential approximates delta interaction is essentially one-dimensional. The mathematical study of the multi-dimensional operator $-\Delta + \delta(x)$ was made in [10]. It implied that this operator can not be uniquely well-defined. In particular, it was proved in work [11] that the operators$(-\Delta)^s + V_m(x)$ converge to the operator $(-\Delta)^s + \delta_{S_p}$ in the resolvent sense, where S_p is a manifold of a dimension $1 \leqslant p \leqslant n-1$ provided $s > (n-p)/2$ and the potentials $V_m(x)$ converge to δ_{S_p} in the distribution sense.

In work [12], there were studied the operators $-\Delta + V_m(x)$ with the potentials $V_m(x)$ in some space of multipliers. In particular, it was proved that the convergence of the potentials in the space of multipliers implies the norm resolvent convergence of the operators. This result was generalized in work [13] for some class of strongly elliptic operators of higher order.

We also mention work [14] in which the criterion of the boundedness in a Sobolev space was found for the operator, $a_{ij}\partial_i\partial_j + b_j\partial_j + c$ with the coefficients a_{ij}, b_j, c in the space of distributions.

The solvability of problem (7) in the case of smooth coefficients was established in work [15]. If the right hand side $f \in L_1(I)$ is only summable, one needs another technique, see, for instance, [16] and the references therein. Unfortunately, we failed to find the works on solvability of problem (8), (9). This is why in the next section we prove the solvability of problem (8), (9) by the contracting mappings method. This made us to restrict the consideration by small nonlinearities. For the completeness of the presentation, we also discuss the solvability of problem (7).

2. Solvability of boundary value problems (7) and (8), (9)

For a function $w \in C(\overline{I})$ we consider the following bilinear forms on $W_2^1(I)$:

$$(u,v)_w = \int\limits_I \left(k(x,w)u'v' + q_2(x)uv\right) \mathrm{d}x,$$

$$(u,v)'_{w,r} = \int\limits_I \Big(k(x,w)u'v' + \Big(q_2(x) + r\varepsilon^{-1}Q(\frac{x}{\varepsilon})\Big)\, uv\Big) \mathrm{d}x + (1-r)\,\langle Q\rangle\, u(0)v(0),$$

where either $r=1$ or $r=0$.

By the conditions $k(x,u) \geqslant k_0 > 0$, $q_2(x) \geqslant q_0 > 0$ and (1) we have the obvious inequalities

$$m_{kq}\|u\|^2_{W_2^1(a,b)} \leqslant (u,u)_w \leqslant C_1\|u\|^2_{W_2^1(a,b)}.$$

Therefore, the bilinear form $(u,v)_w$ is a scalar product in $W_2^1(I)$ equivalent to the classical one.

The linear normed space $W_2^1(c,d)$ is embedded into $C[c,d]$ (see, for instance, [17, Ch. III, Sect. 6]). In particular, $\|u\|_{C[0,1]} \leqslant \|u\|_{W_2^1(0,1)}$. Therefore,

$$\|u\|_{C[c,d]} \leqslant \|u\|_{W_2^1(c,d)}, \qquad u \in W_2^1(c,d), \quad d-c \geqslant 1. \tag{10}$$

It is obvious that

$$u^2(0) \leqslant \|u\|^2_{C(I)} \leqslant \|u\|^2_{W_2^1(I)} \leqslant (m_{kq})^{-1}(u,u)_w, \quad u \in W_2^1(I). \tag{11}$$

We recall that the smooth functions satisfy the Steklov inequality:

$$\int_c^d v^2 \mathrm{d}x \leqslant (d-c)^2 \int_c^d (v')^2 \mathrm{d}x, \quad v(c)=0.$$

We denote by V the Hilbert space

$$\begin{aligned} V =&\{u \in W_2^1(I) | u(a)=0\}, &&\text{if } I=(a,\infty) \text{ and } H_a=0,\\ V =&\{u \in W_2^1(I) | u(b)=0\}, &&\text{if } I=(-\infty,b) \text{ and } H_b=0,\\ V =&\{u \in W_2^1(I) | u(a)=u(b)=0\}, &&\text{if } I=(a,b) \text{ and } H_a=H_b=0,\\ V =&\{u \in W_2^1(I) | u(a)=0\}, &&\text{if } I=(a,b) \text{ and } H_a=0, \text{ but } H_b\neq 0,\\ V =&\{u \in W_2^1(I) | u(b)=0\}, &&\text{if } I=(a,b) \text{ and } H_b=0, \text{ but } H_a\neq 0, \end{aligned}$$

$V=W_2^1(I)$ in other cases; $\|u\|_V := \|u\|_{W_2^1(I)}$.

Lemma 1. *Assume that inequality (2) holds. Then for sufficiently small ε the bilinear form $(u,v)'_{w,r}$ is a scalar product in V equivalent to the scalar product $(u,v)_w$. At that, the inequality*

$$\gamma_1 \|u\|_V^2 \leqslant (u,u)'_{w,r} \tag{12}$$

holds.

Proof. Let us prove the inequality

$$d(u) = \left| \int_I \varepsilon^{-1} Q(\frac{x}{\varepsilon}) u^2 \mathrm{d}x - \langle Q\rangle u^2(0) \right| \leqslant 4\varepsilon^{1/2}\overline{Q}\|u'\|_{L_2(I_\varepsilon)}\|u\|_V, \quad u\in V. \tag{13}$$

Let $u_1,u_2,v\in V$. We estimate the difference

$$\begin{aligned} &\int_I \varepsilon^{-1} Q(\frac{x}{\varepsilon}) u_1 v \mathrm{d}x - \langle Q\rangle u_2(0)v(0)\\ &\quad= \int_I \varepsilon^{-1} Q(\frac{x}{\varepsilon})(u_1 v - u_1(0)v(0))\mathrm{d}x + \langle Q\rangle (u_1(0)-u_2(0))v(0). \end{aligned} \tag{14}$$

Let $I_\varepsilon=(-\varepsilon,\varepsilon)$. The following estimates are obvious:

$$\begin{aligned} \left|\int_I \varepsilon^{-1} Q(\frac{x}{\varepsilon})(u_1 v - u_1(0)v(0))\mathrm{d}x\right| &\leqslant \int_{I_\varepsilon} \varepsilon^{-1}\overline{Q}|(u_1-u_1(0))v + u_1(0)(v-v(0))|\mathrm{d}x\\ &\leqslant \varepsilon^{-1}\overline{Q}(\|u_1-u_1(0)\|_{L_2(I_\varepsilon)}\|v\|_{L_2(I_\varepsilon)} + \|v-v(0)\|_{L_2(I_\varepsilon)}\|u_1(0)\|_{L_2(I_\varepsilon)}) = J. \end{aligned}$$

By the Steklov inequality and (10),

$$\begin{aligned} J \leqslant& \overline{Q}(\|u_1'\|_{L_2(I_\varepsilon)}\|v\|_{L_2(I_\varepsilon)} + \|v'\|_{L_2(I_\varepsilon)}\|u_1(0)\|_{L_2(I_\varepsilon)})\\ \leqslant& 2\varepsilon^{1/2}\overline{Q}(\|u_1'\|_{L_2(I_\varepsilon)}\|v\|_V + \|v'\|_{L_2(I_\varepsilon)}\|u_1\|_V). \end{aligned}$$

Now, by (14), as $u_1=u_2=v=u$, we get inequality (13), while as $v=u_1-u_2$, we arrive at the inequality

$$\int_I \varepsilon^{-1} Q(\frac{x}{\varepsilon}) u_1 v \mathrm{d}x - \langle Q\rangle u_2(0)v(0) \geqslant \langle Q\rangle v^2(0) - 2\varepsilon^{1/2}\overline{Q}(\|u_1'\|_{L_2(I_\varepsilon)}\|v\|_V + \|v'\|_{L_2(I_\varepsilon)}\|u_1\|_V). \tag{15}$$

Let us prove (12). By (2), (11), (13), we have

$$(u,u)'_{w,r} \geqslant (u,u)_w + \langle Q\rangle u^2(0) - r d(u) \geqslant m_{kq}\|u\|_V^2 - \frac{m_{kq}u^2(0)}{4} - 4\varepsilon^{1/2}\overline{Q}\|u'\|_{L_2(I_\varepsilon)}\|u\|_V \geqslant \gamma_1\|u\|_V^2.$$

The latter inequality is true for sufficiently small $\varepsilon > 0$.

By (11) and (13) we obtain the estimate $(u,u)'_{w,r} \leqslant C(u,u)_w$, which complete the proof of the equivalence of the scalar products $(u,v)'_{w,r}$ and $(u,v)_w$. $\square$

We let $\widetilde{h}_a = h_a H_a$, $\widetilde{h}_b = h_b H_b$ and

$$
\begin{aligned}
(u,v)_{w,r} :=& (u,v)'_{w,r}, \qquad I = (-\infty,\infty),\\
(u,v)_{w,r} :=& (u,v)'_{w,r} + \widetilde{h}_a u_0(a)v(a), \qquad I = (a,\infty),\\
(u,v)_{w,r} :=& (u,v)'_{w,r} + \widetilde{h}_b u_0(b)v(b), \qquad I = (-\infty,b),\\
(u,v)_{w,r} :=& (u,v)'_{w,r} + \widetilde{h}_a u_0(a)v(a) + \widetilde{h}_b u_0(b)v(b), \qquad I = (a,b).
\end{aligned}
$$

Lemma 2. *Assume that inequality* (2) *is true. Then for sufficiently small* ε *the bilinear form* $(u,v)_{w,r}$ *is a scalar product in* V *equivalent to the original scalar product in* V. *At that, the inequality*

$$\gamma_1 \|u\|_V^2 \leqslant (u,u)_{w,r}$$

is true.

The proof is implied immediately by Lemma 1 and inequality (11).

By Riesz theorem (see, for instance, [17, Ch. II, Sect. 3, Subsect. 2]), the formula

$$(u,v)_{w,r} = F(v), \qquad v \in V,$$

where $F \in V'$ is a linear continuous functional, defines a linear mapping $u = S_{w,r}F$, $S_{w,r} : V' \to V$. Since

$$\gamma_1 \|u\|_V^2 \leqslant (u,u)_{w,r} \leqslant \|F\|_{V'} \|u\|_V,$$

the estimate

$$\|u\|_V = \|S_{w,r}F\|_V \leqslant \gamma_1^{-1} \|F\|_{V'} \tag{16}$$

is true.

The generalized solutions to nonlinear boundary value problems (7) and (8), (9) are introduced as functions $u^\varepsilon \in V$ and $u_0 \in V$ satisfying the integral identities

$$(u^\varepsilon, v)_{u^\varepsilon,1} = \int_I f v \mathrm{d}x - \int_I ((p(x,u^\varepsilon))' + q_1(x,u^\varepsilon)) v \mathrm{d}x \tag{17}$$

$$(u_0, v)_{u_0,0} = \int_I f v \mathrm{d}x - \int_I ((p(x,u_0))' + q_1(x,u_0)) v \mathrm{d}x, \tag{18}$$

for each $v \in V$, respectively.

We observe that by (17), (18) and (16) we get the following statement.

Lemma 3. *Assume that inequality* (2) *is true. Then in the linear case, that is, as* $k(x,u) \equiv k(x,0)$, $p(x,u) = q_1(x,u) \equiv 0$, *boundary value problem (8), (9), as well as boundary value problem (7) for sufficiently small* ε *are uniquely solvable in* $W_2^1(I)$ *and their solutions satisfy the estimate*

$$\|u\|_{W_2^1(I)} \leqslant C \|f\|_{L_2(I)}.$$

The proof of solvability of nonlinear boundary value problems (7) and (8), (9) is based on the contracting mappings method.

Lemma 4. *Assume that conditions* (2), (4) *hold. Then for each fixed* $f \in L_2(I)$ *such that*

$$\|f\|_{L_2(I)} \leqslant \frac{\gamma_1 M}{2}$$

boundary value problems (7) and (8), (9) are uniquely solvable in the ball of radius M in the space V. They satisfy the inequalities

$$\|u^{\epsilon}(x)\|_{C(I)} \leqslant M, \qquad \|u_0(x)\|_{C(I)} \leqslant M. \tag{19}$$

Proof. We fix f so that $\|f\|_{L_2(I)} \leqslant \gamma_1 M/2$. In the space V we consider the ball

$$B_M := \{v : \quad \|v\|_V \leqslant M\},$$

where M is the constant in smallness condition (4).

We define operator $D:\ B_M \to V'$ acting as follows:

$$\begin{aligned} Dw(v) &= \int\limits_I (f - \frac{\mathrm{d}}{\mathrm{d}x} p(x,w) - q_1(x,w)) v \mathrm{d}x \\ &= \int\limits_I (fv + p(x,w)v' - q_1(x,w)v)\mathrm{d}x - p(b,w(b))v(b) + p(a,w(a))v(a). \end{aligned}$$

It follows from (10) that

$$\|Dw\|_{V'} \leqslant \|f\|_{L_2(I)} + \|p(x,w)\|_{L_2(I)} + \|q_1(x,w)\|_{L_2(I)} + |p(a,w(a))| + |p(b,w(b))|. \tag{20}$$

Let us estimate the terms in the right hand side. Since $p \in C^1(\overline{\Omega})$, and $w \in C(\overline{I})$ and by (10), $\|w(x)\|_{C(I)} \leqslant \|w(x)\|_V \leqslant M$, the Lagrange formula implies ($p(x,0)=0$):

$$\|p(x,w)\|_{L_2(I)} = \|p_u(x,\theta(x)w)w\|_{L_2(I)} \leqslant K_p(M)M.$$

In the same way,

$$\|q_1(x,w)\|_{L_2(I)} \leqslant K_{q_1}(M)M.$$

Then

$$|p(a,w(a))| = |p_u(a,\theta(a)w(a))w(a)| \leqslant K_p(M)M.$$

Taking into consideration the choice of f and condition (4), by (20) we obtain

$$\|Dw\|_{V'} \leqslant \frac{\gamma_1 M}{2} + 3K_p(M)M + K_{q_1}(M)M \leqslant \gamma_1 M. \tag{21}$$

We consider the operators $A_r:\ B_M \to V$, $r=0,1$, defined by the formula $u = A_r w = S_{w,r} Dw$. Then it follows from (16) that $\|u\|_V \leqslant \gamma_1^{-1}\|Dw\|_{V'} \leqslant M$, that is, $A_r:\ B_M \to B_M$. It is obvious that

$$(u,v)_{w,r} = Dw(v), \qquad v \in V. \tag{22}$$

In terms of these notations, boundary value problems (7), (8) become $u_\varepsilon = A_1 u_\varepsilon$, $u_0 = A_0 u_0$, respectively. Therefore, to prove the theorem, it is sufficient to show that the operator A_r is contracting in B_M.

Let $u_1 = A_r v_1$, $u_2 = A_r v_2$. We write down relation (22) for u_1, u_2 and deduct the latter from the former:

$$(u_1,v)_{v_1,r} - (u_2,v)_{v_2,r} = Dv_1(v) - Dv_2(v).$$

For the sake of definiteness, the further proof is made in the case $I=(a,\infty)$, $H_a=1$. We consider the expanded writing for this identity:

$$\begin{aligned} &\int\limits_I \big(v'(k(x,v_1)u_1' - k(x,v_2)u_2') + (q_2(x) + r\varepsilon^{-1}Q(x/\varepsilon))(u_1-u_2)v\big)\,\mathrm{d}x \\ &\quad + (1-r)\left\langle Q\right\rangle(u_1(0)-u_2(0))v(0) + h_a(u_1(a)-u_2(a))v(a) \\ &= \int\limits_I v((\frac{\mathrm{d}}{\mathrm{d}x}p(x,v_2) + q_1(x,v_2)) - (\frac{\mathrm{d}}{\mathrm{d}x}p(x,v_1) + q_1(x,v_1)))\mathrm{d}x. \end{aligned}$$

We substitute $v = u_1 - u_2$ into this identity and integrate by parts:

$$\begin{aligned}&\int\limits_I [v'\left(v'k(x,v_1) + (k(x,v_1) - k(x,v_2))u_2'\right) + (q_2(x) + r\varepsilon^{-1}Q(x/\varepsilon))v^2]\mathrm{d}x\\ &\quad + (1-r)\left\langle Q\right\rangle v^2(0) + h_a v^2(a)\\ &= \int\limits_I (v(q_1(x,v_2) - q_1(x,v_1)) + (p(x,v_1) - p(x,v_2))\, v')\mathrm{d}x\\ &\quad + (p(a,v_1(a)) - p(x,v_2(a)))\, v(a).\end{aligned}$$

Taking into consideration (12), we arrive at the inequality

$$\begin{aligned}\gamma_1\|v\|_V^2 \leqslant & K_p(2M)\int\limits_I |v'|\cdot|v_1 - v_2|\mathrm{d}x + K_p(2M)|v(a)|\cdot|v_1(a) - v_2(a)|\\ & + K_{q_1}(2M)\int\limits_I |v|\cdot|v_1 - v_2|\mathrm{d}x + \sup_{x\in[a,\infty)}|k(x,v_1) - k(x,v_2)|\,\|u_2\|_V\|v\|_V,\end{aligned}$$

Since

$$|k(x,v_1) - k(x,v_2)| \leqslant |k_u'(v_1 - v_2)| \leqslant K_k(2M)\|v_1 - v_2\|_V,$$

we arrive at the inequality

$$\gamma_1\|v\|_V \leqslant (2K_p(2M) + K_k(2M)M + K_{q_1}(2M))\|v_1 - v_2\|_V.$$

This implies that under condition (4), the operator A_r is contracting and therefore, boundary value problems (7), (8) are uniquely solvable in the ball B_M. $\square$

Lemma 5. *The solution to problem (7) belongs to $C^1(\overline{I})$. The solution to boundary value problem (8), (9) belongs to $C(\overline{I})\cap C^1(\overline{I}\setminus\{0\})$.*

Proof. In order to prove the second statement, we introduce the notation $I_- = (a,0)$ and we write equation (18) as $r = 0$ for the function $v \in C_0^\infty(I_-)$:

$$\int\limits_{I_-} (k(x,u_0)u_0'v' + q_2(x)u_0 v)\,\mathrm{d}x = \int\limits_{I_-} vF(u_0)\mathrm{d}x,$$

where

$$F(u) = f - \frac{\mathrm{d}}{\mathrm{d}x}p(x,u) - q_1(x,u), \qquad F(u_0) \in L_2(I_-).$$

This means that the function $z = k(x,u_0)u_0'$ has the distributional derivative

$$z' = q_2(x)u_0 - F(u_0) \in L_2(I_-),$$

that is, the function z is absolutely continuous on $\overline{I}_-$ and equation (8) holds almost everywhere, $u_0 \in C^1(\overline{I}_-)$.

Let us estimate the norms of the functions $F(u_0) = Du_0$, $F(u_\varepsilon)$. We employ inequality (21):

$$\|F(u_0)\|_{L_2(I)} \leqslant \gamma_1 M,\ \|F(u_\varepsilon)\|_{L_2(I)} \leqslant \gamma_1 M.$$

This implies the estimate

$$\|z\|_{C(I_-)} \leqslant \|z\|_{W_2^1(I_-)} \leqslant \overline{k}(M)M + \overline{q}_2 M + \gamma_1 M.$$

This is why

$$\|u_0'\|_{C(I_-)} \leqslant k_0^{-1}\|z\|_{C(I_-)} \leqslant k_0^{-1}(\overline{k}(M) + \overline{q}_2 + \gamma_1)M = c(M). \tag{23}$$

In the same we establish the inequality $\|u_0'\|_{C(0,b)} \leqslant c(M)$.

We write equation (17) as $r=1$ for the function $v \in C_0^\infty(I)$:

$$\int\limits_I \left(k(x,u^\varepsilon)(u^\varepsilon)'v' + \left(q_2(x)+\varepsilon^{-1}Q(\frac{x}{\varepsilon})\right)u^\varepsilon v\right)\mathrm{d}x = \int\limits_I vF(u^\varepsilon)\mathrm{d}x.$$

This means that the function $z = k(x,u^\varepsilon)(u^\varepsilon)'$ has the distributional derivative

$$z' = q_2(x)u^\varepsilon + \varepsilon^{-1}Q(x/\varepsilon)u^\varepsilon - F(u^\varepsilon) \in L_2(I),$$

that is, the function z is absolutely continuous on $\overline{I}$ and equation (7) is satisfied almost everywhere. The function $(u^\varepsilon)' = z/k(x,u^\varepsilon)$ is also absolutely continuous on $\overline{I}$. As above, we establish the estimate

$$\|z\|_{C(a,-\varepsilon)} \leqslant (\overline{k}(M)+\overline{q}_2+\gamma_1)M,$$

and this yields that $\|(u^\varepsilon)'\|_{C(a,-\varepsilon)} \leqslant c(M)$. In the same way, $\|(u^\varepsilon)'\|_{C(\varepsilon,b)} \leqslant c(M)$.

Let us show that

$$\|(u^\varepsilon)'\|_{C(I_\varepsilon)} \leqslant c(M) + 3k_0^{-1}\overline{Q}M = c_1(M), \quad \|(u^\varepsilon)'\|_{L_2(I_\varepsilon)} \leqslant c_1(M)\sqrt{2\varepsilon}. \tag{24}$$

This is implied by the inequality

$$\|\varepsilon^{-1}Q(x/\varepsilon)u^\varepsilon\|_{L_1(I_\varepsilon)} \leqslant \overline{Q}\|u^\varepsilon\|_{C(I_\varepsilon)}\|\varepsilon^{-1}\|_{L_1(I_\varepsilon)} \leqslant 2\overline{Q}M.$$

Indeed,

$$\|z'\|_{L_1(I_\varepsilon)} \leqslant \sqrt{2\varepsilon}(\overline{q}_2+\gamma_1)M + 2\overline{Q}M < 3\overline{Q}M$$

for small ε and

$$z(x) = z(-\varepsilon) + \int_{-\varepsilon}^x z'\mathrm{d}x.$$

This implies the inequalities

$$\|z\|_{C(I_\varepsilon)} \leqslant (\overline{k}(M)+\overline{q}_2+\gamma_1)M + 3\overline{Q}M$$

and (24). The proof is complete. □

3. Proof of Theorems 1, 2.

By (3) we obtain the identity $c(M)+c_1(M)=A$, then by (23), (24) we have the inequalities $|u_0'| \leqslant A$, $|(u^\varepsilon)'| \leqslant A$ and $|v'| \leqslant A$.

We let $v = u^\varepsilon - u_0$. We deduct identity (18) with the test function $\widetilde{v}$ instead of v from (17):

$$(u^\varepsilon, \widetilde{v})_{u^\varepsilon,1} - (u_0,\widetilde{v})_{u_0,0} = Du^\varepsilon(\widetilde{v}) - Du_0(\widetilde{v}).$$

We write the latter in the expanded form for the case $I=(a,b)$:

$$\begin{aligned}&\int\limits_I \left(\widetilde{v}'(k(x,u^\varepsilon)(u^\varepsilon)' - k(x,u_0)u_0') + (q_2(x)v + \varepsilon^{-1}Q(x/\varepsilon)u^\varepsilon)\widetilde{v}\right)\mathrm{d}x \\ &\quad - \langle Q\rangle\, u_0(0)\widetilde{v}(0) + \widetilde{h}_a v(a)\widetilde{v}(a) + \widetilde{h}_b v(b)\widetilde{v}(b) \\ &= \int\limits_I \widetilde{v}((\frac{\mathrm{d}}{\mathrm{d}x}p(x,u_0) + q_1(x,u_0)) - (\frac{\mathrm{d}}{\mathrm{d}x}p(x,u^\varepsilon) + q_1(x,u^\varepsilon)))\mathrm{d}x.\end{aligned}$$

Integrating by the parts in the integral in the right hand side, we get

$$\begin{aligned}&\int\limits_I [\widetilde{v}'\,(v'k(x,u^\varepsilon) + (k(x,u^\varepsilon)-k(x,u_0))u_0') + (q_2(x)v + \varepsilon^{-1}Q(x/\varepsilon)u^\varepsilon)\widetilde{v}]\mathrm{d}x \\ &= \int\limits_I ((q_1(x,u_0)-q_1(x,u^\varepsilon))\widetilde{v} + (p(x,u^\varepsilon)-p(x,u_0))\widetilde{v}')\mathrm{d}x + \langle Q\rangle\, u_0(0)\widetilde{v}(0) + \\ &\quad + (p(a,u^\varepsilon(a)) - p(a,u_0(a)))\,\widetilde{v}(a) - \widetilde{h}_a v(a)\widetilde{v}(a) - (p(b,u^\varepsilon(b)) - p(b,u_0(b)))\,\widetilde{v}(b) - \widetilde{h}_b v(b)\widetilde{v}(b).\end{aligned}$$

Let us make some estimates. In what follows, the test function $\widetilde{v}$ will be chosen to satisfy the inequality $v(x)\widetilde{v}(x) \geqslant 0$, $x \in I$. Then

$$P_a = (p(a, u^\varepsilon(a)) - p(a, u_0(a)))\,\widetilde{v}(a) - \widetilde{h}_a v(a)\widetilde{v}(a) = (p_u(a,\nu) - \widetilde{h}_a)v(a)\widetilde{v}(a),$$

$\nu \in [u_0(a), u^\varepsilon(a)]$. Hence, by (5) we obtain the inequality $P_a \leqslant 0$. In the same way,

$$-\,(p(b, u^\varepsilon(b)) - p(b, u_0(b)))\,\widetilde{v}(b) - \widetilde{h}_b v(b)\widetilde{v}(b) \leqslant 0.$$

Then,

$$\int\limits_I |\widetilde{v}'\,((k(x,u^\varepsilon) - k(x,u_0))u_0')\,| \leqslant K_k(2M)A \int\limits_I |\widetilde{v}'v|\mathrm{d}x,$$
$$\int\limits_I |(p(x,u^\varepsilon) - p(x,u_0))\widetilde{v}'|\mathrm{d}x \leqslant K_p(2M) \int\limits_I |\widetilde{v}'v|\mathrm{d}x,$$
$$\int\limits_I |q_1(x,u_0) - q_1(x,u^\varepsilon)|\widetilde{v}\mathrm{d}x \leqslant K_{q_1}(2M) \int\limits_I v\widetilde{v}\mathrm{d}x.$$

Now by (25) we get the inequality

$$\begin{aligned}&\int\limits_I [k_0\widetilde{v}'v' + ((q_0 - K_{q_1}(2M))v + \varepsilon^{-1}Q(x/\varepsilon)u^\varepsilon)\widetilde{v}]\mathrm{d}x \\ &\qquad \leqslant (K_k(2M)A + K_p(2M)) \int\limits_I |\widetilde{v}'v|\mathrm{d}x + \langle Q\rangle\, u_0(0)\widetilde{v}(0).\end{aligned} \tag{25}$$

To prove Theorem 2, we let $\widetilde{v} = v$ in the above identity and employ the inequality (15). We obtain

$$\begin{aligned}&\int\limits_I (k_0 - (K_k(2M)A + K_p(2M))/2)v'^2\mathrm{d}x \\ &\quad + \int\limits_I ((q_0 - K_{q_1}(2M) - (K_k(2M)A + K_p(2M))/2)v^2\mathrm{d}x \\ &\leqslant -\,\langle Q\rangle v^2(0) + 2\varepsilon^{1/2}\overline{Q}(\|(u^\varepsilon)'\|_{L_2(I_\varepsilon)}\|v\|_V + \|v'\|_{L_2(I_\varepsilon)}\|u^\varepsilon\|_V).\end{aligned} \tag{26}$$

By (2) and (11) we have the inequality

$$-\langle Q\rangle v^2(0) \leqslant \frac{m_{kq}}{4}\|v\|_V.$$

Employing (2), (19), by (26) we obtain

$$\frac{m_{kq}}{4}\|v\|_V \leqslant 2\varepsilon^{1/2}\overline{Q}(\|A\|_{L_2(I_\varepsilon)} + \|M\|_V)\|v\|_V.$$

This implies the statement of Theorem 2.

We proceed to proving Theorem 1. For the sake of definiteness, let $u^\varepsilon(0) \geqslant u_0(0)$, $v(0) \geqslant 0$.

We let $\widetilde{v} = \max(0, v - v(0) - A\varepsilon)$. It is obvious that

$$v(x) = v(0) + \int_0^x v'\mathrm{d}x \leqslant v(0) + A\varepsilon$$

as $x \in (-\varepsilon, \varepsilon)$. This is why $\widetilde{v}(x) = 0$ as $x \in (-\varepsilon, \varepsilon)$. We redefine $\widetilde{v}(x) = 0$ as $x > 0$. By (25) we get the inequality

$$\begin{aligned} &\int\limits_{I_-} [k_0 \widetilde{v}' v' + (q_0 - K_{q_1}(2M)) v \widetilde{v}] \mathrm{d}x \\ &\qquad \leqslant (K_k(2M)A + K_p(2M)) \int\limits_{I_-} |\widetilde{v}' v| \mathrm{d}x. \end{aligned} \tag{27}$$

It is obvious that

$$\int\limits_{I_-} |\widetilde{v}' v| \leqslant \int\limits_{v > v(0)+A\varepsilon} ((v')^2/2 + v^2/2) \mathrm{d}x \leqslant \int\limits_{v > v(0)+A\varepsilon} (\widetilde{v}' v'/2 + \widetilde{v}^2 + (v(0) + A\varepsilon)^2) \mathrm{d}x.$$

Now by (27) we get the inequality

$$\begin{aligned} &\left(k_0 - \frac{K_k(2M)A + K_p(2M)}{2}\right) \int\limits_{v > v(0)+A\varepsilon} (v')^2 \mathrm{d}x \\ &\qquad + (q_0 - K_k(2M)A - K_p(2M) - K_{q_1}(2M)) \int\limits_{I_-} \widetilde{v}^2 \mathrm{d}x \\ &\qquad \leqslant \left(K_k(2M)A + K_p(2M) + \frac{K_{q_1}(2M)}{2}\right) \int\limits_{I_-} (v(0) + A\varepsilon)^2 \mathrm{d}x. \end{aligned}$$

Since

$$m_{kq} > 2K_k(2M)A + 2K_{q_1}(2M) + 2K_p(2M),$$

this implies the inequality

$$\frac{m_{kq}}{2} \|\widetilde{v}\|^2_{C(I_-)} \leqslant \frac{m_{kq}}{2} \|\widetilde{v}\|^2_{W_2^1(I_-)} \leqslant C(v(0) + A\varepsilon)^2.$$

This is why

$$v(x) \leqslant C|v(0) + A\varepsilon|, \qquad x \in I_-.$$

In the same way, letting $\widetilde{v} = \max(0, -v - A\varepsilon)$, we establish the same inequality for $-v(x)$ and then

$$|v(x)| \leqslant C|v(0) + A\varepsilon|, \qquad x \in I_-. \tag{28}$$

Of course, these inequalities are also true on the segment $[0, b]$.

If $0 \leqslant v(0) \leqslant 2A\varepsilon$, the latter inequality implies the estimate $|v(x)| < C\varepsilon$. This is why hereafter we assume that $v(0) > 2A\varepsilon$.

In order to estimate $v(0)$, we let $\widetilde{v} = \min(1, \max(0, \theta v))$, $\theta = (v(0) - A\varepsilon)^{-1}$. We note that

$$v(x) = v(0) + \int_0^x v' \mathrm{d}x \geqslant v(0) - A\varepsilon$$

as $x \in (-\varepsilon, \varepsilon)$. This is why $\widetilde{v}(x) = 1$ as $x \in (-\varepsilon, \varepsilon)$. It is obvious that $\widetilde{v}' = \theta v'$ as $0 < v < v(0) - A\varepsilon$.

Let us establish an important inequality:

$$\begin{aligned} \int\limits_{I} \varepsilon^{-1} Q(x/\varepsilon) u^\varepsilon \widetilde{v} \mathrm{d}x \, \langle Q \rangle \, u_0(0) \widetilde{v}(0) &= \int\limits_{I_\varepsilon} \varepsilon^{-1} Q(x/\varepsilon)(u^\varepsilon - u_0(0)) \mathrm{d}x \\ &= \int\limits_{I_\varepsilon} \varepsilon^{-1} Q(x/\varepsilon)(u^\varepsilon - u_\varepsilon(0) + v(0)) \mathrm{d}x \geqslant v(0) \, \langle Q \rangle - 2\varepsilon A \overline{Q}. \end{aligned}$$

Hence, by (25) we obtain

$$\begin{aligned}&\int\limits_I [k_0\widetilde{v}'v' + (q_0 - K_{q_1}(2M))v\widetilde{v}]\mathrm{d}x \\ &\qquad\leqslant (K_k(2M)A + K_p(2M))\int\limits_I |\widetilde{v}'v|\mathrm{d}x - v(0)\left\langle Q\right\rangle + 2\varepsilon A\overline{Q}.\end{aligned} \tag{29}$$

As above, we establish the estimate

$$\int\limits_I |\widetilde{v}'v|\mathrm{d}x \leqslant \int\limits_{0<v<v(0)-A\varepsilon} \theta((v')^2/2 + v^2/2)\mathrm{d}x.$$

Now by (29) we obtain

$$\begin{aligned}&\left(k_0 - \frac{K_k(2M)A}{2} - \frac{K_p(2M)}{2}\right)\int\limits_{0<v<v(0)-A\varepsilon} (\widetilde{v}')^2/\theta\mathrm{d}x \\ &\qquad + \left(q_0 - \frac{K_k(2M)A}{2} - K_{q_1}(2M) - \frac{K_p(2M)}{2}\right)\int\limits_I (\widetilde{v})^2/\theta\mathrm{d}x \\ &\qquad\leqslant - v(0)\left\langle Q\right\rangle + 2\varepsilon A\overline{Q}.\end{aligned}$$

Therefore,

$$\frac{m_{kq}}{2\theta}\|\widetilde{v}\|^2_{C(I)} \leqslant \frac{m_{kq}}{2\theta}\|\widetilde{v}\|^2_{W_2^1(I_-)} \leqslant -v(0)\left\langle Q\right\rangle + 2\varepsilon A\overline{Q}.$$

Thus, by (2),

$$\frac{m_{kq}(v(0) - A\varepsilon)}{2} = \frac{m_{kq}}{2\theta}\|\widetilde{v}\|^2_{C(I)} \leqslant v(0)(\frac{m_{kq}}{4} + K_p(2M)) + 2\varepsilon A\overline{Q},$$

which implies inequality $v(0) < C\varepsilon$. Combining this with (28), we find that $|v(x)| < C\varepsilon$.

In the case $u^\varepsilon(0) < u_0(0)$, one should let $v = -u^\varepsilon + u_0$ and reproduce the above arguing. The proof is complete.

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