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APPROXIMATION OF SOLUTIONS TO SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS BY HERMITE-FEJER POLYNOMIALS

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Abstract. Singular integral and integro-differential equations have a lot of applications and thus were thoroughly studied by domestic and foreign mathematicians since the beginning of 20th century, and by the 70th years the theory of such equations was finally completed. It is known from this theory that the exact solutions to such equations exist only in rarely particular cases, so since that time the approximate methods for solving these equations as well as the techniques of the justification of these methods were developed. Justification of the approximate method means the proof of the existence and the uniqueness of the approximate solution, estimation of its error and the proof of the convergence of the approximate solutions to the exact solution. Moreover, to compare the approximate methods in different aspects, the optimization theory for approximate methods was created.

However, sometimes, depending on the particular problem, an important role is also played by the form of an approximate solution. For instance, sometimes it is desirable to have an approximate solution as a spline, sometimes, as a polynomial, sometimes it is enough to have just the approximate values of the solution at the nodes. It is quite obvious that depending on the kind of the approximate solution the technique of the justification of the method should be chosen. Unfortunately, there are very few of such techniques, that is why the theory of justification of the approximate methods is now intensively studied.

In the present work we justify an approximate method for solving singular integro-differential equations in the periodic case. An approximate solution is sought as a trigonometric interpolation Hermite-Fejer polynomials. For justification of this approximate method, the technique developed by B.G. Gabdulkhaev and his pupils is used. The convergence of the method is proved and the errors of the approximate solutions are estimated.

Keywords: singular integro-differential equations, justification of the approximate methods.

Mathematics Subject Classification: 65R20

1. Introduction

Algebraic interpolation polynomials with multiple nodes called Hermite polynomials are well studied and are employed successfully for a wide range of applied problems. Their trigonometric analogue is substantially less and many issues on existence, uniqueness and approximative properties of such polynomials are still open.

The first studies of trigonometric interpolation polynomials with multiple nodes seemed to be initiated from the ends of 30s in the last century. S.M. Lozinsky [1] considered issues on approximating the functions of one complex variable regular inside the unit circle and continuous up to the boundary by trigonometric interpolation polynomials with multiple nodes located at the unit circumference. Also he first called such polynomials Hermite-Fejer polynomials.

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Generalizing the results of the predecessors [4], [5], [6], [7], È.O. Zeel' proved the existence of trigonometric interpolation polynomials over the system of equidistributed nodes of an arbitrary multiplicity $m > 0$ for real-valued 2π-periodic functions and he proposed a method for constructing corresponding fundamental polynomials. Moreover, he obtained the conditions of uniform convergence of such polynomials to an interpolated function subject on its smoothness and on the parity of m.

B.G. Gabdulkhaev obtained order sharp estimates in a convenient form for the convergence rate of trigonometric interpolation polynomials of first multiplicity in the space of continuously differentiable functions [8]. Moreover, in this work he first studied the properties of quadrature formulae for singular integrals with the Hilbert kernel obtained in multiple interpolation of the density. Basing on the results of work [3] and employing the approach by B.G. Gabdulkhaev [8], Yu.S. Soliev studied systematically the quadrature formulae with nodes of various multiplicity for singular integrals with the Cauchy and Hilbert kernels [9], [10], [11].

Up to now, the Hermite-Fejer polynomials were employed for approximate solving operator equations only in works by the author [12] , [13].

In the present work we construct a numerical scheme and justify the collocation method for a complete singular integro-differential equation in a periodic case. We prove the convergence of the method and obtain effective estimates for the error of the approximate solution.

2. Formulation of problem

We consider a singular integro-differential equation

$$\sum_{\nu=0}^{1}(a_\nu(t)x^{(\nu)}(t) + b_\nu(t)(Jx^{(\nu)})(t) + (J_0 h_\nu x^{(\nu)})(t)) = y(t), \quad t \in [0, 2\pi], \tag{2.1}$$

where x is the unknown, a_ν, b_ν, h_ν (w.r.t. the both variables), $\nu = 0, 1$, and y are known continuous 2π-periodic functions, the singular integrals

$$(Jx^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} x^{(\nu)}(\tau)\cot\frac{\tau - t}{2}d\tau, \qquad \nu = 0, 1, \quad t \in [0, 2\pi],$$

are treated in the sense of the Cauchy-Lebesgue principal value, while

$$(J_0 h_\nu x^{(\nu)})(t) = \frac{1}{2\pi}\int_0^{2\pi} h_\nu(t, \tau)x^{(\nu)}(\tau)d\tau, \qquad \nu = 0, 1, \quad t \in [0, 2\pi],$$

are regular integrals.

3. Numerical scheme

As usually, by $\mathbb{N}$ we denote the set of natural numbers, the symbol $\mathbb{N}_0$ stands for the set of natural numbers and zero, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

We fix $n \in \mathbb{N}$. We seek an approximate solution to problem (2.1) as the Hermite-Fejer polynomial

$$x_n(t) = \frac{1}{n^2}\sum_{k=0}^{n-1}(x_{2k} + x'_{2k}\sin(t - t_{2k}))\frac{\sin^2\frac{n}{2}(t - t_{2k})}{\sin^2\frac{t - t_{2k}}{2}}, \quad t \in [0, 2\pi], \tag{3.1}$$

where t_{2k}, $k = 0, 1, \ldots, n-1$, are the nodes with the even indices of the grid

$$t_k = \frac{\pi k}{n}, \quad k = 0, 1, \ldots, 2n-1. \tag{3.2}$$

We determine the unknown coefficients x_{2k}, x'_{2k}, $k = 0, 1, \ldots, n-1$, of polynomial (3.1) by the system of linear algebraic equations

$$\sum_{\nu=0}^{1}(a_\nu(t_k)x_n^{(\nu)}(t_k) + b_\nu(t_k)(Jx_n^{(\nu)})(t_k) + (J^0P_{2n}^\tau(h_\nu x_n^{(\nu)}))(t_k)) = y(t_k), \quad k = 0, 1, \ldots, 2n-1, \tag{3.3}$$

where

$$P_{2n}^\tau(h_\nu x_n^{(\nu)})(t,\tau) = \frac{1}{2n}\sum_{k=0}^{2n-1} h_\nu(t,t_k)x_n^{(\nu)}(t_k)\frac{\sin n(\tau - t_k)\cos\frac{\tau-t_k}{2}}{\sin\frac{\tau-t_k}{2}}, \qquad \nu = 0, 1, \quad t,\tau \in [0, 2\pi],$$

is a Lagrange operator P_{2n} over nodes (3.2) applied in the variable τ to the functions $h_\nu x_n^{(\nu)}$, $\nu = 0, 1$. At that[1]

$$(Jx_n)(t_k) = \frac{1}{n}\sum_{j=0}^{n-1}(\alpha_{0,k-2j}^0 x_{2j} + \alpha_{0,k-2j}^1 x'_{2j}), \qquad k = 0, 1, \ldots, 2n-1,$$

$$\alpha_{0,r}^0 = \left\{-\cot\frac{r\pi}{2n} \quad \text{as} \quad r \neq 0, \quad 0 \quad \text{as} \quad r = 0\right\},$$

$$\alpha_{0,r}^1 = \left\{-\frac{1}{n} \quad \text{as} \quad r \neq 0, \quad 2 - \frac{1}{n} \quad \text{as} \quad r = 0\right\};$$

$$(Jx'_n)(t_{2k}) = \frac{1}{n}\sum_{j=0}^{n-1}(\alpha_{1,2k-2j}^0 x_{2j} + \alpha_{1,2k-2j}^1 x'_{2j}), \quad k = 0, 1, \ldots, n-1,$$

$$(Jx'_n)(t_{2k+1}) = \frac{1}{n}\sum_{j=0}^{n-1}\alpha_{1,2k-2j+1}^0 x_{2j}, \quad k = 0, 1, \ldots, n-1,$$

$$\alpha_{1,r}^0 = \left\{\csc^2\frac{r\pi}{2n} \quad \text{as} \quad r \neq 0, \quad -\frac{n^2-1}{3} \quad \text{as} \quad r = 0\right\},$$

$$\alpha_{1,r}^1 = \{(-1)^r\csc\frac{r\pi}{2n} \quad \text{as} \quad r \neq 0, \quad 0 \quad \text{as} \quad r = 0\};$$

$$(J^0P_{2n}^\tau(h_\nu x_n^{(\nu)}))(t_k) = \frac{1}{2n}\sum_{j=0}^{2n-1} h_\nu(t_k,t_j)x_n^{(\nu)}(t_j), \quad \nu = 0, 1, \quad k = 0, 1, \ldots, 2n-1.$$

4. Auxiliary results

We denote by C the set of continuous 2π-periodic functions with the usual norm

$$\|f\|_{\mathrm{C}} = \sup_{t\in\mathbb{R}} |\ f(t)\ |, \qquad f \in \mathrm{C}.$$

For a fixed $m \in \mathbb{N}_0$ we denote by $\mathrm{C}^m \subset \mathrm{C}$ the set of the functions having on $\mathbb{R}$ a bounded mth derivative ($\mathrm{C}^0 = \mathrm{C}$). The norm on the set C^m is introduced by the identity

$$\|f\|_{\mathrm{C}^m} = \max_{0\leqslant\nu\leqslant m}\|f^{(\nu)}\|_{\mathrm{C}}, \quad f \in \mathrm{C}^m.$$

[1]We observe that in works [8], formulae (4), (5) and (6) being the trigonometric interpolation polynomial with nodes of first multiplicity, the quadrature formula constructed on the base of this polynomial and the corresponding quadrature sum were given with misprints. The quadrature formulae free of the misprints can be found, for instance, in dissertation [11], while the formulae for the polynomial and quadrature sum are given in this work.

The set of the functions satisfying the Hölder condition with the exponent $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$, is denoted by H_α. For the functions in H_α we introduce the quantity

$$H(f;\alpha) = \sup_{\substack{t\neq\tau\\ t,\tau\in\mathbb{R}}} \frac{| f(t) - f(\tau) |}{| t-\tau |^\alpha}.$$

This is the smallest constant in the Hölder condition of a function $f \in \mathrm{H}_\alpha$. The introduced quantity allows us to define the norm on the set H_α, namely,

$$\|f\|_{\mathrm{H}_\alpha} = \max\{\|f\|_{\mathrm{C}}, H(f;\alpha)\}.$$

Given a fixed constant $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$, in the set of functions C^m, we select a set of functions H^m_α, whose derivatives of order m satisfy the Hölder condition

$$| f^{(m)}(t) - f^{(m)}(\tau) | \leqslant H(f^{(m)};\alpha) \, | t-\tau |^\alpha, \quad t,\tau\in\mathbb{R}.$$

The norm on the set H^m_α ($\mathrm{H}^0_\alpha = \mathrm{H}_\alpha$) is defined by the relation

$$\|f\|_{\mathrm{H}^m_\alpha} = \max\{\|f\|_{\mathrm{C}^m}, H(f^{(m)};\alpha)\}.$$

By $\mathcal{T}_n$ we denote the set of all trigonometrical polynomials of degree at most n. In what follows we make use of two lemmata implying the results of work [14].

Lemma 4.1. *Let numbers $\alpha,\beta \in \mathbb{R}$, $0 < \alpha \leqslant 1$, $0 < \beta \leqslant 1$, and $m, r \in \mathbb{N}_0$, $m \leqslant r$, be such that $m + \beta \leqslant r + \alpha$. Then for each $n \in \mathbb{N}$ and each function $x \in \mathrm{H}^r_\alpha$ the estimate holds* [1]

$$\|x - T_n\|_{\mathrm{H}^m_\beta} \leqslant c n^{m-r-\alpha+\beta} H(x^{(r)};\alpha),$$

where $T_n \in \mathcal{T}_n$ is the polynomial of the best uniform approximation of the function x.

Lemma 4.2. *For each $n \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < \beta \leqslant 1$ and arbitrary trigonometrical polynomial $T_n \in \mathcal{T}_n$ the following estimate holds*

$$\|T_n\|_{\mathrm{H}_\beta} \leqslant (1 + 2^{1-\beta} n^\beta)\|T_n\|_{\mathrm{C}}.$$

The operator P_{2n} is exact for each trigonometrical polynomial of degree $n-1$ and, as it was shown in [15, 16], it possesses the following properties:

$$\|P_{2n}\|_{\mathrm{H}^m_\beta \to \mathrm{H}^m_\beta} \leqslant c\|P_{2n}\|_{\mathrm{C}\to\mathrm{C}} \leqslant c \ln n \qquad (4.1)$$

for each $n \in \mathbb{N}$, $n \geqslant 2$, $\beta \in \mathbb{R}$, $0 < \beta \leqslant 1$ and arbitrary fixed $m \in \mathbb{N}$.

5. Justification of the method

For numerical scheme (3.1)–(3.3) for equation (2.1) the following theorem holds.

Theorem 5.1. *Assume that equation (2.1) satisfies the following conditions:*

A1 *The functions a_ν, b_ν, $\nu = 0, 1$, and y satisfy the Hölder condition with some exponent $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$; the functions h_ν, $\nu = 0, 1$, satisfy the Hölder with the same α w.r.t. each variable uniformly in the other variable;*

A2 $a_1^2(t) + b_1^2(t) \neq 0, \quad t \in [0, 2\pi]$,

A3 $\kappa = \mathrm{ind}(a_1 + ib_1) = 0$,

A4 *Equation (2.1) has the unique solution $x^* \in \mathrm{H}^1_\beta$ for each right hand side $y \in \mathrm{H}_\beta$, $0 < \beta < \alpha \leqslant 1$.*

Then for sufficiently large n system of equations (3.3) is uniquely solvable and solutions x^*_n converge to the exact solution x^* of equation (2.1) in the norm of the space H^1_β as $n \to \infty$ with the rate

$$\|x^* - x^*_n\|_{\mathrm{H}^1_\beta} \leqslant c n^{-\alpha+\beta} \ln n, \quad 0 < \beta < \alpha \leqslant 1.$$

[1]Hereinafter c denotes certain constants independent of n, generally speaking, different in different inequalities.

Proof. Let us show that Condition **A4** is non-empty that in the considered class there exist equations obeying this condition. Indeed, we consider the equation

$$a_1(t)(x'(t) + x(t)) + b_1(t)((Jx')(t) + (Jx)(t)) = y(t), \quad t \in [0, 2\pi]. \tag{5.1}$$

It is known [17] that the characteristic operator

$$Bx \equiv a_1(t)x(t) + b_1(t)(Jx)(t), \quad B : H_\beta \to H_\beta,$$

of equation (5.1) is invertible and the inverse operator $B^{-1} : H_\beta \to H_\beta$ can be written explicitly. We apply the operator B^{-1} to equation (5.1). This leads us to the equivalent equation

$$x'(t) + x(t) = (B^{-1}y)(t), \quad t \in [0, 2\pi]. \tag{5.2}$$

In the pair of the spaces (H^1_β, H_β), equation (5.2) is Fredholm. The homogeneous equation

$$x'(t) + x(t) = 0, \quad t \in [0, 2\pi],$$

in space of real-valued functions has the solution $x(t) = ce^{-t}$, $t \in [0, 2\pi]$, but this solution is not periodic as $c \neq 0$ and this is why the only appropriate value is $c = 0$, that is, the homogeneous equation has only the trivial solution in the space of periodic functions H^1_β. This means that equation (5.2), and hence, equation (5.1), are uniquely solvable for each right hand side $y \in H_\beta$, $0 < \beta < \alpha \leqslant 1$.

In the rest of the proof we employ the methods of works [18], [19].

We fix $\beta \in \mathbb{R}$, $0 < \beta < \alpha \leqslant 1$, and let $\mathrm{X} = \mathrm{H}^1_\beta$, $\mathrm{Y} = \mathrm{H}_\beta$. Then problem (2.1) can be written as the operator equation

$$Qx = y, \quad Q : \mathrm{X} \to \mathrm{Y}. \tag{5.3}$$

To each function $x \in \mathrm{X}$, we associate the Cauchy type integral of form

$$\Phi(z) = \Phi(x; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x(\tau)d\tau}{1 - z\exp(-i\tau)}, \quad z \in \mathbb{C}.$$

We denote by $x^+(t)$ and $x^-(t)$ the limiting values of the function $\Phi(z)$ as z tends to the point $\exp(it)$ respectively inside and outside the unit circumference. The functions x^+ and x^- satisfy Sokhotski formulae[1]

$$x^\pm(t) = \frac{1}{2}((\pm I - iJ)x)(t) + \frac{1}{2}J_0 x, \quad t \in \mathbb{R}. \tag{5.4}$$

Differentiating (5.4) and employing the known formulae

$$(x'(t))^\pm = (x^\pm(t))', \quad (Jx)'(t) = (Jx')(t),$$

we have

$$x'(t) = x'^+(t) - x'^-(t), \quad (Jx')(t) = i(x'^+(t) + x'^-(t)). \tag{5.5}$$

By Conditions **A2**, **A3** and according [20],

$$\frac{a_1 - ib_1}{a_1 + ib_1} = \frac{\psi^+}{\psi^-},$$

where

$$\psi(z) = e^{\theta(z)}, \quad \theta(z) = \Phi(u; z), \quad u = \ln \frac{a_1 - ib_1}{a_1 + ib_1}, \quad z \in \mathbb{C}.$$

Then, employing (5.5), we can represent the characteristic part of equation (2.1) as [17], [20]

$$a_1(t)x'(t) + b_1(t)(Jx')(t) = \frac{(a_1(t) + ib_1(t))}{\psi^-(t)}(\psi^-(t)x'^+(t) - \psi^+(t)x'^-(t)).$$

we write equation (2.1) or, the same, equation (5.3) as the equivalent operator equation

$$Kx \equiv Ux + Vx = f, \quad K : \mathrm{X} \to \mathrm{Y}, \tag{5.6}$$

[1] I stands for the identity mapping.

where

$$Ux = \psi^- x'^+ - \psi^+ x'^-, \qquad Vx = Ax + Bx + Wx,$$

$$Ax = v^{-1} a_0 x, \qquad Bx = v^{-1} b_0 Jx, \qquad Wx = v^{-1} \sum_{\nu=0}^{1} J^0 h_\nu x^{(\nu)},$$

$$f = v^{-1} y, \qquad v = \frac{a_1 + ib_1}{\psi^-};$$

at that, by Condition **A2** of Theorem 1, $v(t) \neq 0$, $t \in [0, 2\pi]$. The equivalence is understood in the sense that equations (2.1) and (5.6) are solvable simultaneously and their solutions coincide.

Let $\mathrm{X}_n \subset \mathcal{T}_n$ be the set of trigonometrical polynomials of form (3.1), and $\mathrm{Y}_n = P_{2n}\mathrm{Y} \subset \mathcal{T}_n$. Then system of equations (3.3) is equivalent to the operator equation

$$K_n x_n \equiv U_n x_n + V_n x_n = f_n, \quad K_n : \mathrm{X}_n \to \mathrm{Y}_n, \tag{5.7}$$

where

$$U_n = P_{2n} U, \qquad V_n x_n = P_{2n} A x_n + P_{2n} B x_n + W_n x_n,$$

$$W_n x_n = P_{2n} \sum_{\nu=0}^{1} J_0 (P_{2n}^\tau (h_\nu x_n^{(\nu)})), \qquad f_n = P_{2n} f.$$

Here the equivalence is understood in the sense that if system of equations (3.3) has solution $x_{2k}^*, x_{2k}'^*, \quad k = 0, 1, \dots, n-1$, then equation (5.7) also has the polynomial solution

$$x_n^*(t) = \frac{1}{n^2} \sum_{k=0}^{n-1} (x_{2k}^* + x_{2k}'^* \sin(t - t_{2k})) \frac{\sin^2 \frac{n}{2}(t - t_{2k})}{\sin^2 \frac{t - t_{2k}}{2}}, \qquad t \in \mathbb{R}.$$

Let us prove that the operators K and K_n are closed on X_n.

For each $x_n \in \mathrm{X}_n$, employing the polynomial of the best uniform approximation $T_{n-1} \in \mathcal{T}_{n-1}$ for the function Ax_n, we obtain

$$\|Ax_n - P_{2n}Ax_n\|_{\mathrm{Y}} \leqslant (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}}) \|Ax_n - T_{n-1}\|_{\mathrm{Y}}. \tag{5.8}$$

Taking into consideration the structural properties of the function Ax_n, we estimate

$$H(Ax_n; \alpha) \leqslant c(\|x_n\|_C + \|x_n'\|_{\mathrm{C}}) \leqslant c\|x_n\|_{\mathrm{X}}. \tag{5.9}$$

By (5.8), employing Lemma 4.1, estimate (4.1) and taking into consideration (5.9), we find

$$\|Ax_n - P_{2n}Ax_n\|_{\mathrm{Y}} \leqslant c(n^{-\alpha+\beta} \ln n) \|x_n\|_{\mathrm{X}}. \tag{5.10}$$

Arguing as above, we obtain

$$\|Bx_n - P_{2n}Bx_n\|_{\mathrm{Y}} \leqslant c(n^{-\alpha+\beta} \ln n) \|x_n\|_{\mathrm{X}}. \tag{5.11}$$

Since the quadrature formulae for the regular integrals employed in (3.3) are exact for trigonometric polynomials, we can write

$$\begin{aligned} \|Wx_n - W_n x_n\|_{\mathrm{Y}} &\leqslant \| \sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n} \sum_{\nu=0}^{1} J^0 P_{2n}^\tau (h_\nu x_n^{(\nu)}) \|_{\mathrm{Y}} \\ &\leqslant \| \sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n} \sum_{\nu=0}^{1} J^0 (h_\nu x_n^{(\nu)}) \|_{\mathrm{Y}} \\ &\quad + \| P_{2n} \sum_{\nu=0}^{1} J^0 (x_n^{(\nu)} (h_\nu - P_{2n}^\tau h_\nu)) \|_{\mathrm{Y}}. \end{aligned} \tag{5.12}$$

Employing now the polynomial of the best uniform approximation $T_{n-1} \in \mathcal{T}_{n-1}$ for the function $\sum\limits_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}$, we obtain

$$\|\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}) - P_{2n}\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)})\|_{\mathrm{Y}} \leqslant (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - T_{n-1}\|_{\mathrm{Y}}. \quad (5.13)$$

Taking into consideration the structural properties of the function $h_\nu(t,\tau)$ w.r.t. the variable t, it is easy to show that

$$H(\sum_{\nu=0}^{1} J^0(h_\nu x_n^{(\nu)}); \alpha) \leqslant c\sum_{\nu=0}^{1} \|x_n^{(\nu)}\|_{\mathrm{C}} \leqslant c\|x_n\|_{\mathrm{X}}. \quad (5.14)$$

Employing Lemma 4.1 and estimate (4.1), by (5.13) and (5.14) we find

$$\|\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)} - P_{2n}\sum_{\nu=0}^{1} J^0 h_\nu x_n^{(\nu)}\|_{\mathrm{Y}} \leqslant c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \quad (5.15)$$

Taking into consideration the structural properties of the functions $h_\nu(t,\tau)$ w.r.t. the variable τ, the estimate for the error term of the employed quadrature formulae and Lemma 4.2, for the second term in the right hand side of estimate (5.12) we obtain

$$\begin{aligned}\|P_{2n}\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{Y}} &\leqslant c(n^\beta \ln n)\|\sum_{\nu=0}^{1} J^0(x_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{C}} \\ &\leqslant c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}.\end{aligned} \quad (5.16)$$

Finally, employing estimates (5.12), (5.15) and (5.16), we find

$$\|Wx_n - W_n x_n\|_{\mathrm{Y}} \leqslant c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \quad (5.17)$$

We denote by $\psi_{n-1}(t) \in \mathcal{T}_{n-1}$ the polynomial of the best uniform approximation of the function $\psi(t)$. Employing the auxiliary operator,

$$\bar{U}_n : \mathrm{X}_n \to \mathrm{Y}_n, \quad \bar{U}_n x_n = \psi_{n-1}^- x_n'^+ - \psi_{n-1}^+ x_n'^-,$$

we find:

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \leqslant (1 + \|P_{2n}\|_{\mathrm{Y}\to\mathrm{Y}})\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}}. \quad (5.18)$$

Then we have:

$$\|Ux_n - \bar{U}_n x_n\|_{\mathrm{Y}} \leqslant \|(\psi^- - \psi_{n-1}^-)x_n'^+\|_{\mathrm{Y}} + \|(\psi^+ - \psi_{n-1}^+)x_n'^-\|_{\mathrm{Y}}. \quad (5.19)$$

We estimate each term in the right hand side in (5.19) by applying Lemma 4.1 as follows:

$$\|(\psi^\mp - \psi_{n-1}^\mp)x_n'^\pm\|_{\mathrm{Y}} \leqslant \|\psi^\mp - \psi_{n-1}^\mp\|_{\mathrm{Y}}\|x_n'^\pm\|_{\mathrm{Y}} \leqslant cn^{-\alpha+\beta}\|x_n\|_{\mathrm{X}}. \quad (5.20)$$

Now, in view of (5.19), (5.20) and (4.1), inequality (5.18) becomes

$$\|Ux_n - U_n x_n\|_{\mathrm{Y}} \leqslant c(n^{-\alpha+\beta}\ln n)\|x_n\|_{\mathrm{X}}. \quad (5.21)$$

Finally, employing estimates (5.10), (5.11), (5.17) and (5.21), we obtain

$$\|K - K_n\|_{\mathrm{X}_n\to\mathrm{Y}} \leqslant cn^{-\alpha+\beta}\ln n.$$

Since the operators Q and K are invertible simultaneously and

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}} \leqslant \|v\|_{\mathrm{Y}}\|Q^{-1}\|_{\mathrm{Y}\to\mathrm{X}}, \quad (5.22)$$

for sufficiently large n we have:

$$\|K^{-1}\|_{\mathrm{Y}\to\mathrm{X}}\|K - K_n\|_{\mathrm{X}_n\to\mathrm{Y}} \leqslant cn^{-\alpha+\beta}\ln n \leqslant \frac{1}{2}.$$

By Theorem 1.1 in work [19], for such n there exist the operators $K_n^{-1}: \mathrm{Y}_n \to \mathrm{X}_n$ and they are bounded. Moreover, similar to (5.8), we employ Condition **A1** of Theorem 1, Lemma 1 and estimate (4.1) and for the right hand sides of equation (5.6), (5.7) we obtain

$$\|y - y_n\|_{\mathrm{Y}} = \|y - P_{2n}y\|_{\mathrm{Y}} \leqslant cn^{-\alpha+\beta}\ln n. \tag{5.23}$$

Employing Corollary of Theorem 1.2 in work [19] for solutions x^* and x_n^* of equations (5.6), (5.7), taking into consideration estimates (5.22), (5.23), we find

$$\|x^* - x_n^*\|_{\mathrm{X}} \leqslant cn^{-\alpha+\beta}\ln n.$$

The proof is complete. □

Corollary 1. *Suppose that under the assumptions of Theorem 1, the functions a_ν, b_ν, h_ν (w.r.t. the both variables), $\nu = 0, 1$, and y belong to H_α^r, $r \in \mathbb{N}$. Then the approximate solutions x_n^* converge to the exact solution x^* of equation (2.1) in the norm of the space H_β^1 as $n \to \infty$ with the rate*

$$\|x^* - x_n^*\|_{\mathrm{H}_\beta^1} \leqslant cn^{-r-\alpha+\beta}\ln n, \quad r + \alpha > \beta. \tag{5.24}$$

Proof. Employing Theorem 6 in [18], we write

$$\|x^* - x_n^*\|_{\mathrm{X}} \leqslant (1 + \|K_n^{-1}P_{2n}K\|)\|x^* - \bar{x}_n\|_{\mathrm{X}} + \|K_n^{-1}\|\|K_n\bar{x}_n - P_{2n}K\bar{x}_n\|_{\mathrm{Y}}, \tag{5.25}$$

where $\bar{x}_n$ is arbitrary element in X_n. Under the assumptions of Corollary 1, solution x^* of equation (2.1) is such that $x^{*\prime} \in \mathrm{H}_\alpha^r$ as $0 < \alpha < 1$ and $x^{*(r+1)} \in Z$ as $\alpha = 1$'; Z is the Zygmund class. Choosing the polynomial of the best uniform approximation for the function x^* as $\bar{x}_n \in \mathcal{T}_n$ and employing Lemma 4.1, for the first term in the right hand side in (5.25) we find

$$(1 + \|K_n^{-1}P_{2n}K\|)\|x^* - \bar{x}_n\|_{\mathrm{X}} \leqslant cn^{-r-\alpha+\beta}\ln n. \tag{5.26}$$

Taking into consideration the structural properties of the functions $h_\nu(t,\tau)$, $\nu = 0, 1$, in the variable τ, the estimate for the error in quadrature formulae, employing Lemma 4.2 and estimate (4.1) for the second term in the right hand side in inequality (5.25), we get:

$$\begin{aligned}\|K_n\bar{x}_n - P_{2n}K\bar{x}_n\|_{\mathrm{Y}} =& \|W_n\bar{x}_n - P_{2n}W\bar{x}_n\|_{\mathrm{Y}} \leqslant \|P_{2n}\sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{Y}} \\ \leqslant& c(n^\beta \ln n)\|\sum_{\nu=0}^{1} J_0(\bar{x}_n^{(\nu)}(h_\nu - P_{2n}^\tau h_\nu))\|_{\mathrm{C}} \leqslant c(n^{-r-\alpha+\beta})\ln n\|\bar{x}_n\|_{\mathrm{X}}.\end{aligned} \tag{5.27}$$

Substituting estimates (5.26) and (5.27) into (5.25) and taking into consideration that

$$\|\bar{x}_n\|_{\mathrm{X}} \leqslant \|x^*\|_{\mathrm{X}} + \|x^* - \bar{x}_n\|_{\mathrm{X}} \leqslant \|x^*\|_{\mathrm{X}} + cn^{-r-\alpha+\beta},$$

we arrive at estimate (5.24). The proof is complete. □

BIBLIOGRAPHY

1. S.M. Lozinsky. *On Fejer interpolation process* // Dokl. Akad. Nauk SSSR. **24**:4, 318–321 (1939). (in Russian).
2. È. O. Zeel'. *Trigonometric* $(0, p, q)$*-interpolation* // Izv. VUZov. Matem. 3, 27–35 (1970). (in Russian).
3. È. O. Zeel'. *Multiple trigonometric interpolation* // Izv. VUZov. Matem. 3, 43–51 (1974). (in Russian).
4. O. Kish. *On trigonometric* $(0, r)$*-interpolation* // Acta math. Acad. scient. Hung. **11**:3–4, 243–276 (1960).
5. A. Sharma, A.K. Varma. *Trigonometric interpolation* // Duke Math. J. **32**:2, 341–357 (1965).
6. A.K. Varma. *Trigonometric interpolation* // J. Math. Anal. Appl. **28**:3, 652–659 (1969).
7. H.E. Salzer. *New formulas for trigonometric interpolation* // J. Math. Phys. **39**:1, 83–96 (1960).

8. B.G. Gabdulkhaev. *Quadrature formulas with multiple nodes for singular integrals* // Dokl. Akad. Nauk. **227**:3, 531–534 (1976). [Sov. Math. Dokl. **17**, 427–431 (1976).]
9. Yu. Soliev. *Quadrature and cubature formulas for singular integrals with kernels of Cauchy type* // Izv. VUZov. Matem. 3, 108–122 (1977). [Soviet Math. Izv. VUZ. Matem. **21**:3, 85–89 (1977).]
10. Yu. Soliev. *Interpolational quadrature formulas with multiple nodes for singular integrals* // Izv. VUZov. Matem. 9, 122–126 (1977). [Soviet Math. Izv. VUZ. Matem. **21**:9, 98–101 (1977).]
11. Yu. Soliev. *Quadrature and cubature formulae with multiple nodes for singular integrals.* PhD thesis. Kazan State Univ., Kazan (1978). (in Russian).
12. A.I. Fedotov. *Approximation of solutions to one class of singular integro-differential equations by trigonometrical polynomials with multiple nodes* // Preprint: VINITI no. 2483-B86, 12 pp. (1986).
13. A. I. Fedotov. *An approah to the construction of a quadrature-difference method for solving singular integrodifferential equations* // Zhurn. Vychisl. Matem. Matem. Fiz. **29**:7, 978–986 (1989). [USSR Comp. Math. Math. Phys. **29**:4, 11–17 (1989).]
14. B.G. Gabdulkhaev. *Approximation in H-spaces and applications* // Dokl. Akad. Nauk SSSR. **223**:6, 1293–1296 (1975). [Sov. Math., Dokl. **16**, 1067–1071 (1976).]
15. B.G. Gabdulkhaev. *Finite-dimensional approximations of singular integrals and direct methods for solving singular integral and integro-differential equations* // Itogi Nauki Tekhn. Ser. Mat. Anal. **18**, 251–307 (1980). [J. Soviet Math. **18**:4, 593–627 (1982).]
16. B.G. Gabdulkhaev. *Optimal approximations of solutions to linear problems.* Kazan State Univ. Publ., Kazan (1980). (in Russian).
17. F.D. Gakhov. *Boundary value problems.* Nauka, Moscow (1977). [Int. Ser. Monog. Pure Appl. Math. **85**. Pergamon Press, Oxford (1966).]
18. B.G. Gabdulkhaev. *Certain questions of the theory of approximate methods. I-IV* // Izv. VUZov. Matem. 6, 15–23 (1971). (in Russian).
19. B.G. Gabdulkhaev. *Direct methods of solution of certain operator equations. I-IV* // Izv. VUZov. Matem. 11, 33–44 (1971); 12, 28–38 (1971); 3, 18–31 (1972); 4, 32–43 (1972). (in Russian).
20. N.I. Muskhelishvili. *Singular integral equations. Boundary problems of function theory and their application to mathematical physics.* Nauka, Moscow (1968). [P. Noordhoff, Groningen (1953).]

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