ISSN 2304-0122 Ufa Mathematical Journal. Vol. 10. No 1 (2018). P. 76-79.

doi:10.13108/2018-10-1-76 UDC 517.5

ON COMPLETENESS OF EXPONENTIAL SYSTEMS IN CONVEX DOMAIN

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Abstract. The work is devoted to studying the completeness of the systems of exponentials in the space of functions analytic in a convex domain. The problem on the completeness of the systems of the exponentials in various functional spaces is classical and was studied by many mathematicians, for instance, by B.Ya. Levin, A.F. Leontiev, A.M. Sedletskii, B.N. Khabibullin, R.S. Yulmukhametov, and others.

We prove that the completeness of the system of exponentials in the space of functions analytic in a convex domain is equivalent to the completeness of the system of exponentials in the space of functions analytic in a circle with the radius depending on the properties of a given convex domain. We also consider an example by choosing an ellipse as the convex domain. Here we find the values of the support function and the radius of the corresponding circle.

Keywords: completeness of a system, convex domain, entire function, Fourier-Laplace transform.

Mathematics Subject Classification: 30D20

Let D be a bounded convex domain in the complex plane and $H(D)$ be the space of analytic in D functions with the topology of uniform convergence on compact sets. For each set of complex numbers $\Lambda = \{\lambda_k\}$, in which the numbers λ_k can appear with some multiplicity $n_k \in \mathbb{N}\bigcup\{0\}$, $k = 1, 2, \ldots$, we associate the system of exponential monomials

$$\exp\Lambda = \{\lambda^j e^{\lambda_k \lambda},\ k = 1, 2, \ldots,\ j = 0, \ldots, n_k\}.$$

The multiplicity $n_k = 0$ means that the number λ_k appears in Λ exactly one time.

The problem on the completeness of such system in the space $H(D)$ and in other functional spaces is classical and was an object of studying by many authors [1]–[9]. We show that if the boundary of the domain is smooth enough, then the problem on the completeness of the system $\exp\Lambda$ in the space $H(D)$ is equivalent to the problem on the completenss of some system $\exp\Lambda'$ in the space $H(K)$, where K is a circle, whose radius depends on the properties of the domain D.

For each linear continuous functional S on the space $H(D)$, by $\widehat{S}(\lambda) = S(e^{\lambda z})$ we denote the Fourier-Laplace transform of this functional. As it is known, the mapping $L : S \longrightarrow \widehat{S}$ makes a one-to-one correspondence between the dual space $H^*(D)$ and the space of entire functions F satisfying the estimate

$$|F(re^{i\varphi})| \leqslant \text{Const.} e^{(h(\varphi)-\varepsilon)r}, \quad re^{i\varphi} \in \mathbb{C}. \tag{1}$$

for some $\varepsilon = \varepsilon(F) > 0$. Here

$$h(\varphi) = \max_{z\in\overline{D}} \operatorname{Re} e^{i\varphi} z$$

A.A. Makhota, On completeness of exponential systems in convex domain.

Submitted October 12, 2017.

is the support function of the domain D. It is known that $\widehat{S}^{(k)}(\lambda) = S(z^k e^{\lambda z})$. Thus, by the Banach completeness theorem, the system $\exp\Lambda$ is incomplete in the space $H(D)$ if and only if there exists an entire function satisfying condition (1) for some $\varepsilon > 0$ and satisfying the zeroes of multiplicities n_k at the points λ_k, $k = 1, 2, \dots$

In the work we consider the domain with twice continuously differentiable support function. We let

$$M = \max_{\varphi\in[0;2\pi]} (h''(\varphi) + h(\varphi)). \tag{2}$$

It follows from the geometric interpretation of the support functions that M is the maximal curvature radius at the points in the boundary of D (see [1]). The sum $\Lambda' + \Lambda''$ of two sets $\Lambda' = \{(\lambda'_k, n_k)\}$ and $\Lambda'' = \{(\lambda''_k, m_k)\}$ is the union of the sets λ'_k, λ''_k, $k = 1, 2, \dots$, with the multiplicity n_k or m_k if a point belongs just to one of the set and with the total multiplicity if it is in both sets.

In what follows we make use of Theorem A in [2].

Theorem A. *Let u be subharmonic in the entire plane and has a finite growth ρ. Then there exists an entire function f such that for each $\gamma \geqslant \rho$*

$$|u(z) - \ln|f(z)|| \leqslant c_\gamma \ln|z|, \quad z \notin E_\gamma,$$

and the exceptional set E_γ can be covered by the circles $\{z : |z - z_j| < r_j\}$ so that

$$\sum_{|z_j|>R} r_j = o(R^{\rho-\gamma}), \quad R \to \infty.$$

If the number M is defined by formula (2), then the function

$$u(re^{i\varphi}) = Mr - h(\varphi)r \tag{3}$$

is subharmonic in the entire plane. One can easily confirm that by employing the expression of the Laplace operator in polar coordinates:

$$\Delta u(re^{i\varphi}) = \frac{1}{r}\left(M - h(\varphi)\right) - \frac{1}{r^2} r h''(\varphi) = \frac{1}{r}(M - h(\varphi) - h''(\varphi)) \geqslant 0.$$

In what follows, by L we denote some fixed entire function satisfying the assumptions of Theorem A with the function u defined in (3) (as $\rho = 1$). By Λ_0 we denote the set of the zeroes of the function L counting the multiplicities.

Theorem 1. *The system of the exponentials $\exp\Lambda$ is incomplete in the space $H(D)$ if and only if the system $\exp(\Lambda + \Lambda_0)$ is incomplete in the space $H(K_M)$, where K_M is a circle of radius M centered at the origin.*

Proof. If the system $\exp\Lambda$ is incomplete in the space $H(D)$, then, as it has been mentioned above, there exists an entire function F satisfying condition (1) and vanishing at the points $\lambda_k \in \Lambda$ with the multiplicity n_k.

Consider the function $G(z) = F(z)L(z)$. It vanishes on the set $\Lambda + \Lambda_0$ with corresponding multiplicities and outside the set E_1, it satisfies the estimate

$$\left|G(re^{i\varphi})\right| \leqslant |F(z)L(z)| \leqslant Ce^{(h(\varphi)-\varepsilon)r} e^{\left(M - h(\varphi) + \frac{\varepsilon}{2}\right)r} = Ce^{(M-\epsilon')r}, \tag{4}$$

where $\varepsilon' = \frac{\varepsilon}{2}$. By Theorem A, the set E_1 is covered by circles with a convergent sum of radii. Let the sum of the radii for some covering is equal to A. For an arbitrary point $z \in \mathbb{C}$, we make the projection of the circles in the covering along the circumferences

$$C(z,t) = \{w : w = z + te^{i\varphi}, \quad \varphi \in [0; 2\pi]\}$$

into the ray $z + \tau$, $\tau > 0$.

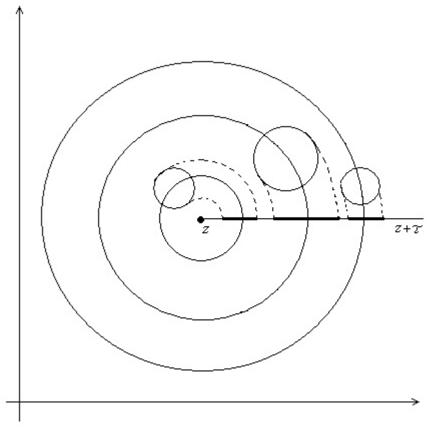

FIGURE 1. Projections of circumferences

Such geometric construction shows that we can find a circumference $C(z,t)$ with $t \in [0;3A]$ disjoint with the circles of the covering and therefore, disjoint with the exceptional set E_1. Then on this circumference, estimate (4) holds. By the maximum principle we have

$$|G(z)| \leqslant C \exp(\max_{w \in C(z,t)} (M - \epsilon')|w|) \leqslant C' e^{(M-\epsilon')|z|}.$$

Since the function $h_1(\varphi) \equiv M$ is the support function of the circle K_M, we obtain that the system $\exp(\Lambda + \Lambda_0)$ is incomplete in the space $H(K_M)$.

And vice versa: assume that the system $\exp(\Lambda+\Lambda_0)$ is incomplete in the space $H(K_M)$. This means that there exists an entire function G satisfying the estimate

$$|G(z)| \leqslant Ce^{(M-\varepsilon)|z|}$$

for some $\varepsilon > 0$ and vanishing on the set $\Lambda + \Lambda_0$. Then the function

$$F(z) = \frac{G(z)}{L(z)}$$

is entire.

By the estimates for the functions L and G we obtain that outside the set E_1, the estimate

$$|F(z)| \leqslant \text{Const.} r^C e^{(M-\varepsilon)r} e^{(-M+h(\varphi))r} \leqslant \text{Const.} e^{(h(\varphi)-\frac{\varepsilon}{2})r}$$

holds. As above, this estimate can be continued to the entire plane. At that, one should employ the property of the support functions

$$|h(\varphi)r - h(\vartheta)t| \leqslant \max_{z \in \overline{D}} \left| re^{i\varphi} - te^{i\vartheta} \right|, \quad re^{i\varphi}, te^{i\vartheta} \in \mathbb{C}.$$

Thus, the system $\exp \Lambda$ is incomplete in the space $H(D)$. The proof is complete. □

Let us provide an example. As a convex domain D, we consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this case, the support function $h(\varphi)$ of the domain D is of the form:

$$h(\varphi) = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}.$$

We need to find

$$M = \max_{\varphi \in [0;2\pi]} (h''(\varphi) + h(\varphi)) < \infty.$$

In order to do this, we find the first, second and third derivative of the support function $h(\varphi)$.

We make the change

$$h(\varphi) =\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi} = \sqrt{\frac{a^2(1+\cos 2\varphi)}{2}+\frac{b^2(1-\cos 2\varphi)}{2}}$$
$$=\sqrt{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2\varphi}.$$

We denote

$$A:=\frac{a^2+b^2}{2}; B:=\frac{a^2-b^2}{2}.$$

This implies

$$h(\varphi)=\sqrt{A+B\cos 2\varphi}.$$

We find

$$h''(\varphi)=\frac{-\left(A+B\cos 2\varphi\right)^2+A^2-B^2}{\left(A+B\cos 2\varphi\right)^{3/2}}.$$

We obtain that

$$M=\max_{\varphi\in[0;2\pi]}(h''(\varphi)+h(\varphi))=\frac{a^2}{b}.$$

Thus, the maximal curvature radius at the points in the boundary of the domain D is equal to $M=\frac{a^2}{b}$. The system of exponentials $\exp\Lambda$ is incomplete in the space of the functions analytic in the ellipse if and only if the system $\exp(\Lambda+\Lambda_0)$ is incomplete in the space of functions analytic in the circle of the radius $M=\frac{a^2}{b}$ centered at the origin.

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