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ON THE GROWTH OF SOLUTIONS OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. In this paper, by using the value distribution theory, we study the growth and the oscillation of meromorphic solutions of the linear differential equation

$$f^{(k)} + \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right) f^{(k-1)}$$
$$+ \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = F(z),$$

where $A_{j,i}(z)\,(\not\equiv 0)$ $(j = 0, \ldots, k-1)$, $F(z)$ are meromorphic functions of a finite order, and $P_j(z), Q_j(z)$ $(j = 0, 1, \ldots, k-1; i = 1, 2)$ are polynomials with degree $n \geqslant 1$. Under some conditions, we prove that as $F \equiv 0$, each meromorphic solution $f \not\equiv 0$ with poles of uniformly bounded multiplicity is of infinite order and satisfies $\rho_2(f) = n$ and as $F \not\equiv 0$, there exists at most one exceptional solution f_0 of a finite order, and all other transcendental meromorphic solutions f with poles of uniformly bounded multiplicities satisfy $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \max\{n, \rho(F)\}$. Our results extend the previous results due Zhan and Xiao [19].

Keywords: Order of growth, hyper-order, exponent of convergence of zero sequence, differential equation, meromorphic function.

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1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory, see [12], [18]. Let $\rho(f)$ stands for the order of growth of a meromorphic function f and the hyper-order of f is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f , see [12], [14], [18].

Definition 1.1. *([15], [17]) Let f be a meromorphic function. The convergence exponent of the zero-sequence of a meromorphic function f is defined by*

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log N\left(r,\frac{1}{f}\right)}{\log r},$$

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where $N\left(r,\frac{1}{f}\right)$ *is the integrated counting function of zeros of* f *in* $\{z:|z|\leqslant r\}$*, and the exponent of convergence the sequence of distinct zeros of* f *is defined by*

$$\overline{\lambda}\left(f\right)=\limsup_{r\to+\infty}\frac{\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ *is the integrated counting function of distinct zeros of* f *in* $\{z:|z|\leqslant r\}$*. The hyper convergence exponents of the zero-sequence and the distinct zeros of* f *are defined respectively by*

$$\lambda_2\left(f\right)=\limsup_{r\to+\infty}\frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},\qquad \overline{\lambda}_2\left(f\right)=\limsup_{r\to+\infty}\frac{\log\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

Several authors [3], [9], [14] have study the growth of solutions of the second order linear differential equation

$$f''+A_1(z)e^{P(z)}f'+A_2(z)e^{Q(z)}f=0, \tag{1.1}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_2(z)$ ($\not\equiv 0$) are entire functions such that $\rho\left(A_1\right)<\deg P(z)$, $\rho\left(A_2\right)<\deg Q(z)$. Gundersen showed in [9] that if $\deg P(z)\neq\deg Q(z)$, then each nonconstant solution of (1.1) is of infinite order. If $\deg P(z)=\deg Q(z)$, then (1.1) may have nonconstant solutions of a finite order. For instance $f(z)=e^z+1$ satisfies $f''+e^zf'-e^zf=0$.

In [10], Habib and Belaïdi studied the order and hyper-order of solutions of some higher order linear differential equations and they proved the following result.

Theorem 1.1. *(*[10]*) Let* $A_j(z)$ $(\not\equiv 0)$, $(j=1,2)$, $B_l(z)\,(\not\equiv\ 0)$ $(l=1,\ldots,k-1)$, D_m $(m=0,\ldots,k-1)$ *be entire functions with*

$$\max\left\{\rho\left(A_j\right),\rho\left(B_l\right),\rho\left(D_m\right)\right\}<1,$$

b_l $(l=1,\ldots,k-1)$ *be complex constants such that* (i) $\arg b_l=\arg a_1$ *and* $b_l=c_la_1$ $(0<c_l<1)$ $(l\in I_1)$ *and* (ii) b_l *is a real constant such that* $b_l\leqslant 0$ $(l\in I_2)$*, where* $I_1\neq\varnothing$, $I_2\neq\varnothing$, $I_1\cap I_2=\varnothing$, $I_1\cup I_2=\{1,2,\ldots,k-1\}$*, and* a_1, a_2 *are complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<\frac{b}{1-c}$*, where* $c=\max\{c_l:l\in I_1\}$ *and* $b=\min\{b_l:l\in I_2\}$*, then each solution* $f\not\equiv 0$ *of the equation*

$$\begin{aligned} f^{(k)}&+\left(D_{k-1}+B_{k-1}e^{b_{k-1}z}\right)f^{(k-1)}+\ldots+\left(D_1+B_1e^{b_1z}\right)f'\\ &+\left(D_0+A_1e^{a_1z}+A_2e^{a_2z}\right)f=0 \end{aligned}\tag{1.2}$$

satisfies $\rho\left(f\right)=+\infty$ *and* $\rho_2\left(f\right)=1$.

And in [2], they studied the order and hyper-order of solutions of some higher order linear differential equations with meromorphic coefficient and they proved the following result.

Theorem 1.2. *(*[2]*) Let* $A_j(z)$ $(\not\equiv 0)$ $(j=1,2\ )$*,* $B_l(z)$ $(\not\equiv 0)$ $(l=1,\ldots,k-1)$ *be meromorphic functions with*

$$\max\left\{\rho\left(A_j\right)\quad(j=1,2)\,,\rho\left(B_l\right)\quad(l=1,\ldots,k-1)\right\}<1,$$

b_l $(l=1,\ldots,k-1\ )$ *be complex constants such that* (i) $b_l=c_la_1$ $(0<c_l<1)$ $(l\in I_1)$ *and* (ii) b_l *is a real constant such that* $b_l<0$ $(l\in I_2)$*, where* $I_1\neq\emptyset$, $I_2\neq\emptyset$, $I_1\cap I_2=\emptyset$, $I_1\cup I_2=\{1,2,\ldots,k-1\}$*, and* a_1, a_2 *are complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<\frac{b}{1-c}$*, where*

$c = \max\{c_l, l \in I_1\}$ *and* $b = \min\{b_l, l \in I_2\}$, *then each meromorphic solution* $f\,(\not\equiv 0)$ *with poles of uniformly bounded multiplicities of the equation*

$$f^{(k)} + B_{k-1}e^{b_{k-1}z}f^{(k-1)} + \cdots + B_1e^{b_1z}f' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)f = 0 \tag{1.3}$$

satisfies $\rho(f) = +\infty$ *and* $\rho_2(f) = 1$.

In [19], Zhan and Xiao studied the homogeneous and nonhomogeneous higher order differential equations and obtained the following results.

Theorem 1.3. *([19]) Let* $A_{ji}(z)$ $(\not\equiv 0)$ *be entire functions with* $\rho(A_{ji}) < n$, $n \geqslant 1$ *is a positive integer,* $j = 0, 1, \ldots, k-1$; $i = 1, 2$. *Let* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ *and* $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials, where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ *and* $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ *are distinct numbers. Then each solution* $f\,(\not\equiv 0)$ *of the equation*

$$\begin{aligned} f^{(k)} &+ \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right)f^{(k-1)} \\ &+ \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right)f = 0 \end{aligned} \tag{1.4}$$

of a finite order.

Theorem 1.4. *([19]) Let* $A_{ji}(z)$ $(\not\equiv 0)$ *be entire functions with* $\rho(A_{ji}) < n$, *where* $n \geqslant 1$ *is a positive integer,* $j = 0, 1, \ldots, k-1; i = 1, 2$. *Let* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ *and* $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials, where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ *and* $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ *are distinct numbers.* $F(z)(\not\equiv 0)$ *is an entire function of a finite order. Then the equation*

$$\begin{aligned} f^{(k)} &+ \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right)f^{(k-1)} \\ &+ \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right)f = F(z) \end{aligned} \tag{1.5}$$

satisfies the following statements:
(*i*) *There exists at most one exceptional solution* f_0 *of a finite order, and all other solutions satisfy* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ *and* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \max\{n, \rho(F)\}$.
(*ii*) *If there exists* f_0 *of a finite order, then* $\rho(f_0) \leqslant \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}$.
(*iii*) *If* $F(z)$ *is an entire function of order less than* n *and* $\arg a_{0,n} \neq \arg b_{0,n}$, *then each solution of (1.5) is of infinite order.*

In this paper, we are concerned with a more general problem. We extend and improve Theorem 1.3 and Theorem 1.4. In fact, we will prove the following theorems.

Theorem 1.5. *Let* $A_{ji}(z)$ $(\not\equiv 0)$ *be meromorphic functions of a finite order such that* $\max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1; i = 1, 2\} < n$, *where* $n \geqslant 1$ *is a positive integer. Let* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ *and* $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials, where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ *and* $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ *are distinct numbers. Then each meromorphic solution* $f\,(\not\equiv 0)$ *of equation (1.4) with poles of uniformly bounded multiplicity is of infinite order and satisfies* $\rho_2(f) = n$.

Theorem 1.6. *Let* $A_{ji}(z)$ $(\not\equiv 0)$, $F(z)(\not\equiv 0)$ *be meromorphic functions of a finite order with* $\max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1;\ i = 1, 2\} < n$, *where* $n \geqslant 1$ *is a positive integer. Let* $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ *and* $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials, where* $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ *are complex numbers such that* $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ *and* $a_{j,n} = c_ja_{0,n}$, $b_{j,n} = c_jb_{0,n}$, $c_j > 1$, $j = 1, \ldots, k-1$ *are distinct numbers. Then the equation (1.5) satisfies:*

(i) There exists at most one exceptional meromorphic solution f_0 with finite order, and all other transcendental meromorphic solutions f with poles of uniformly bounded multiplicities satisfy

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \max\{n, \rho(F)\}.$$

(ii) If there exists f_0 of a finite order, then $\rho(f_0) \leqslant \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}$.
(iii) If $F(z)$ is a meromorphic function of order less than n and $\arg a_{0,n} \neq \arg b_{0,n}$, then each meromorphic solution f of (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

2. Auxiliary lemmata

First, we recall the following definitions. The linear measure of a set $E \subset [0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt$$

and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by

$$lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt,$$

where $\chi_H(t)$ is the characteristic function of a set H.

Lemma 2.1. *([1]) Let $P_j(z)\,(j = 0, 1, \ldots, k)$ be polynomials with $\deg P_0 = n$ $(n \geqslant 1)$ and $\deg P_j \leqslant n$ $(j = 1, \ldots, k)$. Let $A_j(z)$ $(j = 0, 1, \ldots, k)$ be meromorphic functions of a finite order and $\max\{\rho(A_j), j = 0, 1, \ldots, k\} < n$ such that $A_0(z) \not\equiv 0$. We denote*

$$F(z) = A_k e^{P_k(z)} + A_{k-1} e^{P_{k-1}(z)} + \cdots + A_1 e^{P_1(z)} + A_0 e^{P_0(z)}.$$

If $\deg(P_0(z) - P_j(z)) = n$ for all $j = 1, \ldots, k$, then F is a nontrivial meromophic function with finite order satisfying $\rho(F) = n$.

Lemma 2.2. *([8]) Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset (1, +\infty)$ of a finite logarithmic measure and a constant $B > 0$ that depends only on α and positive integers (n, m) obeying $n > m \geqslant 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leqslant B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f)\right]^{n-m}.$$

Lemma 2.3. *([11]) Let $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) be a polynomial with degree $n \geqslant 1$ and $A(z)$ be a meromorphic function with $\rho(A) < n$. Let*

$$f(z) = A(z) e^{P(z)}, \quad z = re^{i\theta}, \quad \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta.$$

Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for each $\theta \in [0, 2\pi) \setminus H$ $(H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\})$ and for $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$, we have
(i) if $\delta(P, \theta) > 0$, then

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leqslant \left|f\left(re^{i\theta}\right)\right| \leqslant \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\},$$

(ii) if $\delta(P, \theta) < 0$, then

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leqslant \left|f\left(re^{i\theta}\right)\right| \leqslant \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\}.$$

Lemma 2.4. *([5]) Let $f(z)$ be a meromorphic function of order $\rho(f) = \rho < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that as $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$, we have $|f(z)| \leqslant \exp\left(r^{\rho+\varepsilon}\right)$.*

It is well known that due to the Wiman-Valiron theory [13], [15], it is important to studyt the properties of entire solutions of differential equations. In [4], Chen extended the Wiman-Valiron theory from entire functions to meromorphic functions. Here we give a special form of the result given by Wang and Yi in [17], when meromorphic function has infinite order.

Let $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. By $\mu(r) = \max\{|a_n| r^n;\ n = 0, 1, \ldots\}$ we denote the maximum term of g and by $\nu_g(r) = \max\{m :\ \mu(r) = |a_m| r^m\}$ we denote the central index of g.

Lemma 2.5. *([17]) Let $f(z) = g(z)/d(z)$ be a meromorphic function of infinite order obeying $\rho_2(f) = \sigma$, $g(z)$ and $d(z)$ are entire functions, where $\rho(d) < +\infty$. Then there exists a sequence of complex numbers $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfying*

$$r_m \to +\infty, \quad \theta_m \in [0, 2\pi)\,; \quad m \in \mathbb{N}, \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0, 2\pi)\,, \quad |g(z_m)| = M(r_m, g)$$

and for sufficiently large m we have

$$\frac{f^{(n)}(z_m)}{f(z_m)} = \left(\frac{\nu_g(r_m)}{z_m}\right)^n (1 + o(1)) \quad (n \in \mathbb{N}),$$
$$\limsup_{r_m\to+\infty} \frac{\log\log \nu_g(r_m)}{\log r_m} = \rho_2(g) = \sigma.$$

Lemma 2.6. *([9]) Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be a monotone nondecreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin (E_4 \cup [0, 1])$, where E_4 is a set of a finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \leqslant \psi(\alpha r)$ for all $r > r_1$.*

Lemma 2.7. *Suppose that $k \geqslant 2$ and F, A_0, $A_1, \ldots, A_{k-1}$ are meromorphic functions such that $\rho = \max\{\rho(A_j)\ j = 0, 1, 2, \ldots, k-1,\ \rho(F)\} < +\infty$. Let $f(z)$ be a transcendental meromorphic solution with all poles of f are of uniformly bounded multiplicity, of equation*

$$f^{(k)} + A_{k-1}\, f^{(k-1)} + \cdots + A_1 f' + A_0\, f = F. \tag{2.1}$$

Then $\rho_2(f) \leqslant \rho$.

Proof. We assume that f is a transcendental meromorphic solution of equation (2.1). If $\rho(f) < +\infty$, then $\rho_2(f) = 0 \leqslant \rho$. Assume that f is a meromorphic solution to equation (2.1) of infinite order with poles of uniformly bounded multiplicity. By (2.1) we have

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right| + |A_0(z)|\,. \tag{2.2}$$

By (2.1) it follows that the poles of f can locate only at the poles of A_j $(j = 0, \ldots, k-1)$ and F. Note that the poles of f are of uniformly bounded multiplicity. Hence, $\lambda(1/f) \leqslant \rho$. By the Hadamard factorization theorem, we know that f can be expressed as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda(1/f) \leqslant \rho < \rho(f) = \rho(g) = +\infty$$

and $\rho_2(f) = \rho_2(g)$. By Lemma 2.5, there exists a sequence $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfying

$$r_m \to +\infty, \quad \theta_m \in [0, 2\pi), \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0, 2\pi), \quad |g(z_m)| = M(r_m, g)$$

such that for m sufficiently large we have

$$\frac{f^{(n)}(z_m)}{f(z_m)} = \left(\frac{\nu_g(r_m)}{z_m}\right)^n (1+o(1)) \ \ (n \in \mathbb{N}) \tag{2.3}$$

and

$$\limsup_{r_m\to+\infty} \frac{\log\log \nu_g(r_m)}{\log r_m} = \rho_2(g). \tag{2.4}$$

By Lemma 2.4, for each given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ of a finite logarithmic measure such that

$$|F(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\}, \ \ |d(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\} \tag{2.5}$$

and

$$|A_j(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\} \ \ (j = 0, \dots, k-1) \tag{2.6}$$

hold for $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$. Since $M(r, g) \geqslant 1$ for r sufficiently large, it follows from (2.5) that

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r, g)} \leqslant \exp\left\{2r^{\rho+\varepsilon}\right\}. \tag{2.7}$$

Substituting (2.3), (2.6) and (2.7) into (2.2), we obtain

$$\left(\frac{\nu_g(r_m)}{r_m}\right)^k |1+o(1)| \leqslant \sum_{j=1}^{k-1} e^{r_m^{\rho+\varepsilon}} \left(\frac{\nu_g(r_m)}{r_m}\right)^j |1+o(1)| + e^{r_m^{\rho+\varepsilon}} + e^{2r_m^{\rho+\varepsilon}}.$$

It follows that

$$(\nu_g(r_m))^k |1+o(1)| \leqslant (k+1)\, e^{2r_m^{\rho+\varepsilon}} r_m^k (\nu_g(r_m))^{k-1} |1+o(1)|.$$

Hence,

$$\nu_g(r_m) \leqslant (k+1)\, A r_m^k e^{2r_m^{\rho+\varepsilon}}, \tag{2.8}$$

where the sequence $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfies

$$r_m \notin [0, 1] \cup E_3, \quad r_m \to +\infty, \quad \theta_m \in [0, 2\pi), \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0, 2\pi), \quad |g(z_m)| = M(r_m, g)$$

and $A > 0$ is some constant. Then by (2.8), Lemma 2.6 and $\varepsilon > 0$ being arbitrary, we obtain that $\rho_2(g) = \rho_2(f) \leqslant \rho$. □

Remark 2.1. For $F \equiv 0$, Lemma 2.7 was proved by Chen and Xu in [7].

Lemma 2.8. *([16]) Let $g(z)$ be a transcendental entire function and $\nu_g(r)$ be the central index of g. For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. Then there exist a constant δ_r (> 0) and a set E_5 of a finite logarithmic measure such that for all z satisfying $|z| = r \notin E_5$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\frac{g^{(n)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)) \ \ (n \geqslant 1 \textit{ is an integer}).$$

Lemma 2.9. *([8]) Let $f(z)$ be a transcendental meromorphic function of a finite order ρ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, 2, \dots, m)$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_6 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z obeying $|z| = r \notin [0, 1] \cup E_6$ and $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.10. *Let $f(z) = g(z)/d(z)$ be a meromorphic function with $\rho(f) = \rho \leqslant +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions:*
(i) g is transcendental and d is polynomial,
(ii) g, d are transcendental and $\lambda(d) = \rho(d) = \beta < \rho(g) = \rho$.
For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$ and let $\nu_g(r)$ be the central index of g. Then there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_7 of finite logarithmic measure such that the estimation

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z}\right)^n (1 + o(1)) \quad (n \geqslant 1 \text{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_7$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. By mathematical induction, we obtain

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.9}$$

where $C_{jj_1\cdots j_n}$ are constants and $j + j_1 + 2j_2 + \cdots + nj_n = n$. Hence,

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.10}$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. By Lemma 2.8, there exist a constant δ_r (> 0) and a set E_5 of a finite logarithmic measure such that for all z obeying $|z| = r \notin E_5$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, 2, \ldots, n), \tag{2.11}$$

where $\nu_g(r)$ is the central index of g. Substituting (2.11) into (2.10) yields

$$\begin{aligned} \frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n \Bigg[&(1 + o(1)) \\ &+ \sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z}\right)^{j-n} (1 + o(1)) \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \Bigg]. \end{aligned} \tag{2.12}$$

We can choose a constant σ such that $\beta < \sigma < \rho$. By Lemma 2.9, for any given ε $(0 < 2\varepsilon < \sigma - \beta)$, we have

$$\left|\frac{d^{(s)}(z)}{d(z)}\right| \leqslant r^{s(\beta - 1 + \varepsilon)} \quad (s = 1, 2, \ldots, n), \tag{2.13}$$

where $|z| = r \notin [0,1] \cup E_6$, $E_6 \subset (1, +\infty)$ with $lm(E_6) < +\infty$. From this and $j_1 + 2j_2 + \cdots + nj_n = n - j$, we have

$$|z|^{n-j} \left| \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leqslant |z|^{(n-j)(\beta+\varepsilon)} \tag{2.14}$$

for $|z| = r \notin [0,1] \cup E_6$. By $\rho(g) = \rho$, there exists a sequence $\{r'_m\}$ $(r'_m \to +\infty)$ satisfying

$$\lim_{r'_m \to +\infty} \frac{\log \nu_g(r'_m)}{\log r'_m} = \rho. \tag{2.15}$$

Setting the logarithmic measure of $E_7 = [0,1] \cup E_5 \cup E_6$, $lm(E_7) = \delta < +\infty$, there exists a point $r_m \in [r'_m, (\delta+1) r'_m] \setminus E_7$. Since

$$\frac{\log \nu_g(r_m)}{\log r_m} \geqslant \frac{\log \nu_g(r'_m)}{\log [(\delta+1) r'_m]} = \frac{\log \nu_g(r'_m)}{(\log r'_m)\left[1 + \frac{\log(\delta+1)}{\log r'_m}\right]}, \tag{2.16}$$

we get

$$\lim_{r_m \to +\infty} \frac{\log \nu_g(r_m)}{\log r_m} = \rho. \tag{2.17}$$

Hence, for sufficiently large m, we obtain

$$\nu_g(r_m) \geqslant r_m^{\rho-\varepsilon} \geqslant r_m^{\sigma-\varepsilon}, \tag{2.18}$$

where $\rho - \varepsilon$ can be replaced by a large enough number M if $\rho = +\infty$. This and (2.14) imply

$$\left| \left(\frac{\nu_g(r)}{z}\right)^{j-n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leqslant r_m^{(n-j)(\beta-\sigma+2\varepsilon)} \to 0,\ r_m \to +\infty, \tag{2.19}$$

where $|z| = r_m \notin E_7$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. From (2.12) and (2.19), we obtain our result. □

Lemma 2.11. *Let $f(z) = g(z)/d(z)$ be a meromorphic function with $\rho(f) = \rho \leqslant +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions*
(i) g is transcendental and d is polynomial,
(ii) g, d are transcendental and $\lambda(d) = \rho(d) = \beta < \rho(g) = \rho$.
For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. Then there exist a constant δ_r (>0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \leqslant r_m^{2n} \quad (n \geqslant 1 \textit{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_8, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. Let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. By Lemma 2.10, there exist a constant δ_r (>0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z}\right)^n (1 + o(1)) \quad (n \geqslant 1 \text{ is an integer}) \tag{2.20}$$

holds for all z satisfying $|z| = r_m \notin E_8, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. On the other hand, for any given $\varepsilon > 0$ and sufficiently large m we obtain

$$\nu_g(r_m) \geqslant r_m^{\rho-\varepsilon}, \tag{2.21}$$

where $\rho - \varepsilon$ can be replaced by a large enough number M if $\rho = +\infty$. Hence, we have

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \leqslant r_m^{2n}. \tag{2.22}$$

This completes the proof. □

Lemma 2.12. *([12]) Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O(\log T(r,f) + \log r)$, possibly outside a set $E_9 \subset (0,+\infty)$ with a finite linear measure. If f is of a finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2.13. *([6]) Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ are meromorphic functions of a finite order. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

3. Proof of Theorem 1.5

First, we prove that each meromorphic solution $f\,(\not\equiv 0)$ of the equation (1.4) is transcendental of order $\rho(f) \geqslant n$. We assume that $f\,(\not\equiv 0)$ is a meromorphic solution of equation (1.4) with $\rho(f) < n$. We can rewrite equation (1.4) as

$$\begin{aligned}(A_{k-1,1}(z)e^{P_{k-1}(z)} &+ A_{k-1,2}(z)e^{Q_{k-1}(z)})f^{(k-1)} \\ &+ \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = -f^{(k)}.\end{aligned} \tag{3.1}$$

Since

$$\max\{\rho(A_{ji}),\, j = 0,1,\ldots,k-1; i = 1,2\} < n$$

and

$$\rho(f) < n,$$

then $A_{ji}f^{(j)}$, $j = 0,1,\ldots,k-1; i = 1,2$ and $f^{(k)}$ are meromorphic functions of a finite order with

$$\rho\left(A_{ji}f^{(j)}\right) < n \quad \text{and} \quad \rho\left(f^{(k)}\right) < n.$$

We have also $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_j a_{0,n}, b_{j,n} = c_j b_{0,n}, c_j > 1,\ j = 1,\ldots,k-1$. Hence, $a_{j,n} \neq b_{j,n}$ and therefore $\deg(P_j - P_0) = \deg(Q_j - Q_0) = n$. Since $A_{0,1}(z)f \neq 0, A_{0,2}(z)f \neq 0$, by Lemma 2.1, we find that the order of growth of the left side of equation (3.1) is n, this contradicts the inequality $\rho\left(f^{(k)}\right) < n$. Thus, each meromorphic solution $f\,(\not\equiv 0)$ of equation (1.4) is transcendental with order $\rho(f) \geqslant n$.

Let $z = re^{i\theta}$, $a_{0,n} = |a_{0,n}|\, e^{i\theta_1}$, $b_{0,n} = |b_{0,n}|\, e^{i\theta_2}$, $\theta_1, \theta_2 \in [0, 2\pi)$. Then

$$\delta(P_0,\theta) = |a_{0,n}|\cos(n\theta + \theta_1),\ \delta(Q_0,\theta) = |b_{0,n}|\cos(n\theta + \theta_2). \tag{3.2}$$

Since $a_{j,n} = c_j a_{0,n},\ b_{j,n} = c_j b_{0,n}, c_j > 1,\ j = 1,\ldots,k-1$, and c_j are distinct numbers, we have

$$\delta(P_j,\theta) = c_j\delta(P_0,\theta),\ \ \delta(Q_j,\theta) = c_j\delta(Q_0,\theta), \tag{3.3}$$

and there exists exactly one c_s such that $c_s = \max\{c_j, j = 0,1,\ldots,k-1\}$. Let $c_0 = 1$.

We split our proof into two cases: $\theta_1 = \theta_2$ and $\theta_1 \neq \theta_2$

Case 1. As $\theta_1 = \theta_2$, because of $a_{0,n} \neq b_{0,n}$, we suppose $|a_{0,n}| < |b_{0,n}|$ without loss of generality. Assume that f is a meromorphic solution to equation (1.4) with poles of uniformly bounded multiplicity. From (1.4), we have

$$\begin{aligned}&|A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| \\ &\quad\leqslant \left|\frac{f}{f^{(s)}}\right| \left(\left|\frac{f^{(k)}}{f}\right| + \sum_{j=0,j\neq s}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right).\end{aligned} \tag{3.4}$$

Since f is transcendental, then by Lemma 2.2, for $\alpha = 2$, there exist a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T\left(2r, f\right)\right]^{k+1}, \ j = 1, 2, \ldots, k, \ j \neq s. \tag{3.5}$$

By (1.4), it follows that the poles of f can be located only at the poles of $A_{ji}(z)$, $j = 0, 1, \ldots, k-1$; $i = 1, 2$. We observe that the poles of f are of uniformly bounded multiplicity. Hence,

$$\lambda\left(1/f\right) \leqslant \max\{\rho\left(A_{ji}\right), j = 0, 1, \ldots, k-1; i = 1, 2\} < n.$$

By Hadamard factorization theorem, we know that f can be expressed as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda\left(d\right) = \rho\left(d\right) = \lambda\left(1/f\right) < n \leqslant \rho\left(f\right) = \rho\left(g\right).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g\left(z_r\right)| = M\left(r, g\right)$. By Lemma 2.11, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant r_m^{2s} \tag{3.6}$$

holds for all z satisfying $|z| = r_m \notin E_8$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

(i) If $\delta\left(P_0, \theta\right) > 0$, then by (3.3) we have

$$\delta\left(Q_j, \theta\right) > \delta\left(Q_0, \theta\right) > 0, \quad \delta\left(Q_j, \theta\right) > \delta\left(P_j, \theta\right) > \delta\left(P_0, \theta\right) > 0.$$

By Lemma 2.3, for any given ε obeying

$$0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right), j \neq s\right\},$$

there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, where

$$H = \{\theta \in [0; 2\pi) : \delta\left(P_0, \theta\right) = 0, \delta\left(Q_0, \theta\right) = 0\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,2}(z)e^{Q_s(z)}\right| - \left|A_{s,1}(z)e^{P_s(z)}\right| \\ &\geqslant \exp\left\{\left(1-\varepsilon\right)c_s\delta\left(Q_0, \theta\right)r^n\right\} - \exp\left\{\left(1+\varepsilon\right)c_s\delta\left(P_0, \theta\right)r^n\right\} \\ &\geqslant \frac{1}{2}\exp\left\{\left(1-\varepsilon\right)c_s\delta\left(Q_0, \theta\right)r^n\right\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} |A_{j,1}(z)e^{P_j(z)} &+ A_{j,2}(z)e^{Q_j(z)}| \leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{\left(1+\varepsilon\right)c_j\delta\left(P_0, \theta\right)r^n\right\} + \exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0, \theta\right)r^n\right\} \\ &\leqslant 2\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0, \theta\right)r^n\right\}, \ j = 0, 1, 2, \ldots, k-1, \ j \neq s. \end{aligned} \tag{3.8}$$

Substituting (3.5), (3.6), (3.7), (3.8) into (3.4), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_2 \cup E_8$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$ we obtain

$$\begin{aligned} \frac{1}{2}\exp\left\{\left(1-\varepsilon\right)c_s\delta\left(Q_0, \theta\right)r_m^n\right\} \leqslant r_m^{2s}\Bigg(& B\left[T\left(2r_m, f\right)\right]^{k+1} \\ & + B\left[T\left(2r_m, f\right)\right]^{k+1}\sum_{j=0, j\neq s}^{k-1} 2\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0, \theta\right)r_m^n\right\}\Bigg) \end{aligned}$$

$$\leqslant 4r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0,\theta\right)r_m^n\right\}$$

which gives

$$\exp\left\{\left(1-\varepsilon\right)c_s\delta\left(Q_0,\theta\right)r_m^n\right\}\leqslant 8r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0,\theta\right)r_m^n\right\}. \quad (3.9)$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.9) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

(ii) If $\delta\left(P_0,\theta\right)<0$, then by (3.2) and (3.3) we have

$$\delta\left(Q_j,\theta\right)<\delta\left(Q_0,\theta\right)<\delta\left(P_0,\theta\right)<0,\quad \delta\left(P_j,\theta\right)<\delta\left(P_0,\theta\right)<0.$$

By Lemma 2.3, for any given $0<\varepsilon<1$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H$, where $H=\{\theta\in[0;2\pi):\delta\left(P_0,\theta\right)=0,\delta\left(Q_0,\theta\right)=0\}$ is a finite set, we get

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|&\leqslant\left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\&\leqslant\exp\left\{\left(1-\varepsilon\right)\delta\left(P_j,\theta\right)r^n\right\}+\exp\left\{\left(1-\varepsilon\right)\delta\left(Q_j,\theta\right)r^n\right\}\\&\leqslant 2\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r^n\right\},\quad j=0,1,2,\ldots,k-1.\end{aligned} \quad (3.10)$$

By (1.4) we have

$$1\leqslant\left|\frac{f}{f^{(k)}}\right|\sum_{j=0}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}. \quad (3.11)$$

Substituting (3.5), (3.6) and (3.10) into (3.11), for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H$ we obtain

$$\begin{aligned}1&\leqslant r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left(\sum_{j=0}^{k-1}2\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r_m^n\right\}\right)\\&\leqslant 2kr_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r_m^n\right\}\end{aligned} \quad (3.12)$$

which gives

$$\exp\left\{\left(\varepsilon-1\right)\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 2kr_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}. \quad (3.13)$$

By Lemma 2.6 and (3.13) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log^+T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r\to+\infty}\frac{\log_2^+T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

Case 2 Assume that $\theta_1\neq\theta_2$.

(i) If $\delta(P_0,\theta)>0$, $\delta(Q_0,\theta)<0$, then by (3.3), we get

$$\delta(P_j,\theta)>\delta(P_0,\theta)>0, \quad \delta(Q_j,\theta)<\delta(Q_0,\theta)<0,$$

by Lemma 2.3, for any given $0<\varepsilon<\min\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right), j\neq s\}$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\{\theta\in[0,2\pi):\delta(P_0,\theta)=0,\delta(Q_0,\theta)=0,\delta(P_0,\theta)=\delta(Q_0,\theta)\}$$

is a finite set, we have

$$\begin{aligned}|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,1}(z)e^{P_s(z)}\right|-\left|A_{s,2}(z)e^{Q_s(z)}\right|\\ &\geqslant \exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\}-\exp\{(1-\varepsilon)c_s\delta(Q_0,\theta)r^n\}\\ &\geqslant \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\},\end{aligned} \tag{3.14}$$

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\ &\leqslant \exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\}+\exp\{(1-\varepsilon)c_j\delta(Q_0,\theta)r^n\}\\ &\leqslant 2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\},\ j=0,1,2,\ldots,k-1,\quad j\neq s.\end{aligned} \tag{3.15}$$

By (3.4), (3.5), (3.6), (3.14) and (3.15), for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$ we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\} \leqslant & r_m^{2s}\Bigg(B[T(2r_m,f)]^{k+1}\\ &+B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}\Bigg)\\ \leqslant & 4r_m^{2s}B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}\end{aligned}$$

which gives

$$\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\}\leqslant 8r_m^{2s}B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}. \tag{3.16}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.16) we obtain

$$\rho(f)=\limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m}=+\infty,$$

and

$$\rho_2(f)=\limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2(f)\leqslant n$, so $\rho_2(f)=n$.

(ii) If $\delta(P_0,\theta)<0$, $\delta(Q_0,\theta)>0$, by (3.3), we have

$$\delta(P_j,\theta)<\delta(P_0,\theta)<0, \quad \delta(Q_j,\theta)>\delta(Q_0,\theta)>0.$$

By Lemma 2.3, for any given $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right), j\neq s\right\}$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\{\theta\in[0,2\pi):\delta(P_0,\theta)=0,\delta(Q_0,\theta)=0,\delta(P_0,\theta)=\delta(Q_0,\theta)\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,2}(z)e^{Q_s(z)}\right| - \left|A_{s,1}(z)e^{P_s(z)}\right| \\ &\geqslant \exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\} - \exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\} \\ &\geqslant \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} |A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{(1+\varepsilon)\, c_j\delta\left({}_0,\theta\right) r^n\right\} + \exp\left\{(1-\varepsilon)\, c_j\delta\left(P_0,\theta\right) r^n\right\} \\ &\leqslant 2\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r^n\right\},\ j=0,1,2,\ldots,k-1,\ j\neq s. \end{aligned} \tag{3.18}$$

Proceeding as in the proof of (i) , for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_2 \cup E_8$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$ we obtain

$$\begin{aligned} \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r_m^n\right\} \leqslant r_m^{2s}\Bigg(& B\left[T\left(2r_m, f\right)\right]^{k+1} \\ & + B\left[T\left(2r_m, f\right)\right]^{k+1} \sum_{j=0, j\neq s}^{k-1} 2\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}\Bigg) \\ \leqslant & 4r_m^{2s} B\left[T\left(2r_m, f\right)\right]^{k+1} \sum_{j=0, j\neq s}^{k-1} \exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}, \end{aligned}$$

which gives

$$\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r_m^n\right\} \leqslant 8r_m^{2s} B\left[T\left(2r_m, f\right)\right]^{k+1} \sum_{j=0, j\neq s}^{k-1} \exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}. \tag{3.19}$$

Since $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.19) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T\left(r_m, f\right)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T\left(r_m, f\right)}{\log r_m} \geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2(f) \leqslant n$, so $\rho_2(f) = n$.

(iii) If $\delta(P_0,\theta) > 0$, $\delta(Q_0,\theta) > 0$, then by (3.3), we have

$$\delta\left(P_j,\theta\right) > \delta\left(P_0,\theta\right) > 0, \delta\left(Q_j,\theta\right) > \delta\left(Q_0,\theta\right) > 0.$$

We suppose $\delta(P_0,\theta) > \delta(Q_0,\theta)$ without loss of generality. By Lemma 2.3, for any given $0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right),\ j\neq s\right\}$, there exists a set $E_2 \subset [1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, where

$$H_1 = \left\{\theta \in [0,2\pi) : \delta\left(P_0,\theta\right) = 0, \delta\left(Q_0,\theta\right) = 0, \delta\left(P_0,\theta\right) = \delta\left(Q_0,\theta\right)\right\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,1}(z)e^{P_s(z)}\right| - \left|A_{s,2}(z)e^{Q_s(z)}\right| \\ &\geqslant \exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\} - \exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\} \\ &\geqslant \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\}, \end{aligned} \tag{3.20}$$

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{(1+\varepsilon)\,c_j\delta\left(P_0,\theta\right)r^n\right\}+\exp\left\{(1+\varepsilon)\,c_j\delta\left(Q_0,\theta\right)r^n\right\} \\ &\leqslant 2\exp\left\{(1+\varepsilon)\,c_j\delta\left(P_0,\theta\right)r^n\right\},\ j=0,1,2,\dots,k-1,\ j\neq s.\end{aligned}\tag{3.21}$$

From (3.4), (3.5), (3.6), (3.20) and (3.21), we have for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$

$$\frac{1}{2}\exp\left\{(1-\varepsilon)\,c_s\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 4r_m^{2s}\ B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\,c_j\delta\left(P_0,\theta\right)r_m^n\right\},$$

which gives

$$\exp\left\{(1-\varepsilon)\,c_s\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 8r_m^{2s}\ B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\,c_j\delta\left(P_0,\theta\right)r_m^n\right\}.\tag{3.22}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.22) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

(iv) If $\delta\left(P_0,\theta\right)<0$, $\delta\left(Q_0,\theta\right)<0$, then by (3.3), we have

$$\delta\left(P_j,\theta\right)<\delta\left(P_0,\theta\right)<0,\delta\left(Q_j,\theta\right)<\delta\left(Q_0,\theta\right)<0.$$

Let $\delta=\max\left\{\delta\left(P_0,\theta\right),\delta\left(Q_0,\theta\right)\right\}$. Then, by Lemma 2.3, for any given $0<\varepsilon<1$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\left\{\theta\in[0,2\pi):\delta\left(P_0,\theta\right)=0,\delta\left(Q_0,\theta\right)=0,\delta\left(P_0,\theta\right)=\delta\left(Q_0,\theta\right)\right\}$$

is a finite set, we get

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right| &\leqslant\left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant\exp\left\{(1-\varepsilon)\,c_j\delta\left(P_0,\theta\right)r^n\right\}+\exp\left\{(1-\varepsilon)\,c_j\delta\left(Q_0,\theta\right)r^n\right\} \\ &\leqslant 2\exp\left\{(1-\varepsilon)\,c_j\delta r^n\right\},\quad j=0,1,\dots,k-1.\end{aligned}\tag{3.23}$$

By (3.5), (3.6), (3.11) and (3.23) for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$ we have

$$\begin{aligned}1 &\leqslant r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left\{\sum_{j=0}^{k-1}2\exp\left\{(1-\varepsilon)\,c_j\delta r_m^n\right\}\right\} \\ &\leqslant 2r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left\{\sum_{j=0}^{k-1}\exp\left\{(1-\varepsilon)\,c_j\delta r_m^n\right\}\right\}.\end{aligned}\tag{3.24}$$

Since $c_j>1$, $j=1,\dots,k-1$ and $\delta<0$, we obtain

$$\exp\left\{(1-\varepsilon)\,c_j\delta r_m^n\right\}\leqslant\exp\left\{(1-\varepsilon)\,\delta r_m^n\right\},\quad j=1,\dots,k-1$$

so (3.24) becomes

$$1\leqslant 2r_m^{2k}kB\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{(1-\varepsilon)\,\delta r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon-1)\,\delta r_m^n\right\} \leqslant 2r_m^{2k}Bk\left[T\left(2r_m,f\right)\right]^{k+1}. \tag{3.25}$$

By Lemma 2.6 and (3.25) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$. This completes the proof of Theorem 1.5.

4. Proof of Theorem 1.6

(i) Suppose f_0 is a meromorphic solution of a finite order to equation (1.5) with poles of uniformly bounded multiplicities. If $f_1(\not\equiv f_0)$ is an another meromorphic solution of a finite order to equation (1.5) with poles of uniformly bounded multiplicities, the function f_1-f_0 is a nonzero meromorphic solution to equation (1.4) with $\rho(f_1-f_0)<+\infty$. This contradicts Theorem 1.5. Hence, equation (1.5) has at most one meromorphic solution of a finite order. We assume that $f(z)$ is a meromorphic solution of infinite order to (1.5) with poles of uniformly bounded multiplicity. By (1.5), it is easy to see that if f has a zero of order $\alpha\,(\alpha>k)$at z_0, and B_0, $B_1,\ldots,$ B_{k-1} are analytic at z_0, then F must have a zero at z_0 of order at least $\alpha-k$. Hence,

$$n\left(r,\frac{1}{f}\right)\leqslant k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}n\left(r,B_j\right)$$

and

$$N\left(r,\frac{1}{f}\right)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}N\left(r,B_j\right), \tag{4.1}$$

where $B_j(z)=A_{j1}(z)e^{P_j(z)}+A_{j2}(z)e^{Q_j(z)}$, $j=0,1,2,\ldots,k-1$. Now (1.5) can be rewritten as

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+B_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+B_1(z)\frac{f'}{f}+B_0(z)\right). \tag{4.2}$$

By Lemma 2.12 and (4.2), we get for $|z|=r$ outside a set E_9 of finite linear measure, we have

$$\begin{aligned} m\left(r,\frac{1}{f}\right)&\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k-1}m\left(r,B_j\right)+O\left(1\right)\\ &\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m\left(r,B_j\right)+O\left(\log rT\left(r,f\right)\right). \end{aligned} \tag{4.3}$$

Therefore, by (4.1), (4.3) and the first main theorem, there holds

$$T(r,f)=T(r,\frac{1}{f})+O\left(1\right)\leqslant T\left(r,F\right)+\sum_{j=0}^{k-1}T\left(r,B_j\right)+k\overline{N}\left(r,\frac{1}{f}\right)+O\left(\log rT\left(r,f\right)\right) \tag{4.4}$$

for all sufficiently large $r\notin E_9$. For sufficiently large r, we have

$$O\left(\log rT\left(r,f\right)\right)\leqslant\frac{1}{2}T(r,f). \tag{4.5}$$

Let $\rho_1 = \max\{n, \rho(F)\}$. By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ of a finite logarithmic measure such that

$$T(r,F) \leqslant r^{\rho_1+\varepsilon},\ T(r,B_j) \leqslant r^{\rho_1+\varepsilon},\ j = 0,1,\ldots,k-1, \tag{4.6}$$

when $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$. By (4.4), (4.5) and (4.6), for $r \notin [0,1] \cup E_3 \cup E_9$ sufficiently large, we obtain

$$T(r,f) \leqslant r^{\rho_1+\varepsilon} + kr^{\rho_1+\varepsilon} + k\overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{2}T(r,f)$$

which gives

$$T(r,f) \leqslant 2(k+1)r^{\rho_1+\varepsilon} + 2k\overline{N}\left(r, \frac{1}{f}\right). \tag{4.7}$$

Hence,

$$\rho_2(f) \leqslant \overline{\lambda}_2(f)$$

and therefore,

$$\rho_2(f) \leqslant \overline{\lambda}_2(f) \leqslant \lambda_2(f).$$

Since by the definition we have $\overline{\lambda}_2(f) \leqslant \lambda_2(f) \leqslant \rho_2(f)$, we get

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f).$$

On the other hand, $\max\{\rho(A_{ji}),\ j = 0,1,\ldots,k-1;\ i = 1,2\} < n$ and $\rho(B_j) < +\infty$ for all $j = 0,1,\ldots,k-1$, and $f(z)$ is a solution to (1.5) of infinite order. Hence, by Lemma 2.13 we obtain $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$. Since $\rho(B_j) \leqslant n$, by Lemma 2.7, we have $\rho_2(f) \leqslant \max\{n, \rho(F)\}$.

(ii) Suppose f_0 is a meromorphic solution of the equation (1.5) with finite order, by Lemma 2.12, we have $m\left(r, \frac{f_0^{(j)}}{f_0}\right) = O(\log r)$, $j = 1,\ldots,k-1$. Using (4.2), we can get for $|z| = r$ outside a set E_9 of finite linear measure, we have

$$\begin{aligned} m\left(r, \frac{1}{f_0}\right) \leqslant& m\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} m\left(r, \frac{f_0^{(j)}}{f_0}\right) + \sum_{j=0}^{k-1} m(r,B_j) + O(1) \\ \leqslant& m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r,B_j) + O(\log r) \end{aligned} \tag{4.8}$$

and

$$N\left(r, \frac{1}{f_0}\right) \leqslant k\overline{N}\left(r, \frac{1}{f_0}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r,B_j). \tag{4.9}$$

By (4.8) and (4.9), we get

$$T(r,f_0) = T(r, \frac{1}{f_0}) + O(1) \leqslant T\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} T(r,B_j) + k\overline{N}\left(r, \frac{1}{f_0}\right) + O(\log r). \tag{4.10}$$

By (4.6) and (4.10), we get

$$T(r,f_0) \leqslant (k+1)r^{\rho_1+\varepsilon} + k\overline{N}\left(r, \frac{1}{f_0}\right) + O(\log r).$$

Hence, we obtain

$$\rho(f_0) \leqslant \max\left\{\overline{\lambda}(f_0), \rho_1\right\} = \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}.$$

(iii) First we prove that each meromorphic solution f to equation (1.5) is transcendental of order $\rho(f) \geqslant n$. We assume that f is a meromorphic solution to equation (1.5) with $\rho(f) < n$. We can rewrite equation (1.5) as

$$\left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = B(z), \tag{4.11}$$

where

$$B(z) = F(z) - f^{(k)}.$$

Since $\max\{\rho(A_{ji}),\ j = 0, 1, \ldots, k-1;\ i = 1, 2, \rho(F)\} < n$ and $\rho(f) < n$, then $A_{ji}f^{(j)}$, $j = 0, 1, \ldots, k-1$, $i = 1, 2$, and $B(z)$ are meromorphic functions of a finite order with $\rho\left(A_{ji}f^{(j)}\right) < n$ and $\rho(B) < n$. We also have $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}$, $c_j > 1$, $j = 1, \ldots, k-1$,. Hence, $a_{j,n} \neq b_{j,n}$ and $\deg(P_j - P_0) = \deg(Q_j - Q_0) = n$. Since $A_{0,1}(z)f \not\equiv 0, A_{0,2}(z)f \not\equiv 0$, by Lemma 2.1 we find that the order of growth of the left hand side of equation (4.11) is n. This contradicts the inequality $\rho(B) < n$. Therefore, each meromorphic solution f to equation (1.5) is transcendental and is of order $\rho(f) \geqslant n$.

Let $z = re^{i\theta}$, $a_{0,n} = |a_{0,n}| e^{i\theta_1}$, $b_{0,n} = |b_{0,n}| e^{i\theta_2}$, $\theta_1, \theta_2 \in [0, 2\pi)$. Then

$$\delta(P_0, \theta) = |a_{0,n}| \cos(n\theta + \theta_1), \delta(Q_0, \theta) = |b_{0,n}| \cos(n\theta + \theta_2). \tag{4.12}$$

Since $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$, and c_j are distinct numbers, we have

$$\delta(P_j, \theta) = c_j \delta(P_0, \theta),\ \delta(Q_j, \theta) = c_j \delta(Q_0, \theta), \tag{4.13}$$

and there exists exactly one c_s such that $c_s = \max\{c_j, j = 0, 1, \ldots, k-1\}$. Let $c_0 = 1$, $\delta_1 = \max\{\delta(P_0, \theta), \delta(Q_0, \theta)\}$. We split our proof into two cases:

Case 1. Assume that $\delta_1 > 0$. By Lemma 2.3, for any given

$$0 < \varepsilon < \min\left\{n - \rho_1, \frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right),\ j \neq s\right\},$$

there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, where

$$H_3 = \{\theta \in [0, 2\pi) : \delta(P_0, \theta) = \delta(Q_0, \theta)\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,1}(z)e^{P_s(z)}\right| - \left|A_{s,2}(z)e^{Q_s(z)}\right| \\ &\geqslant \exp\{(1-\varepsilon) c_s \delta(P_0, \theta) r^n\} - \exp\{(1-\varepsilon) c_s \delta(Q_0, \theta) r^n\} \\ &\geqslant \frac{1}{2} \exp\{(1-\varepsilon) c_s \delta_1 r^n\}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\{(1+\varepsilon) c_j \delta(P_0, \theta) r^n\} + \exp\{(1+\varepsilon) c_j \delta(Q_0, \theta) r^n\} \\ &\leqslant 2\exp\{(1+\varepsilon) c_j \delta_1 r^n\},\ j = 0, 1, \ldots, k-1,\ j \neq s. \end{aligned} \tag{4.15}$$

By (1.5) we have

$$\begin{aligned} &|A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| \\ &\leqslant \left|\frac{f}{f^{(s)}}\right| \left\{\left|\frac{F(z)}{f}\right| + \left|\frac{f^{(k)}}{f}\right| + \sum_{j=0, j\neq s}^{k-1} \left\{\left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right| \left|\frac{f^{(j)}}{f}\right|\right\}\right\}. \end{aligned} \tag{4.16}$$

Since f is transcendental, from Lemma 2.2, there exists a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ and constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1$, we have (3.5) holds and by Lemma 2.11, there exists a set E_8 of finite logarithmic measure such that $|z| = r \notin E_8$,

$|g(z)| = M(r, g)$ and for r sufficiently large inequality (3.6) holds. We know that f is transcendental with $\rho(f) \geqslant n$, and by the assumptions, the poles of f are of uniformly bounded multiplicities. By Hadamard factorization theorem, we can express f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) < n, \quad \rho(g) = \rho(f) \geqslant n.$$

Let $\rho_1 = \max\{\rho(F), \rho(d)\} < n$. Since $|g(z)| = M(r, g) \geqslant 1$, then, by Lemma 2.4 we obtain

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)F(z)}{g(z)}\right| = \frac{|d(z)F(z)|}{M(r, g)} \leqslant \exp\left(r^{\rho_1+\varepsilon}\right)\exp\left(r^{\rho_1+\varepsilon}\right) = \exp\left(2r^{\rho_1+\varepsilon}\right) \tag{4.17}$$

as $|z| = r \notin [0, 1] \cup E_3$, $r \to +\infty$.

By (3.5), (3.6), (4.14),(4.15), (4.16) and (4.17), for all z satisfying $|z| = r_m \notin$ $\notin [0, 1] \cup E_1 \cup E_3 \cup E_8$, $r_m \to +\infty$, $|g(z)| = M(r_m, g)$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta_1 r_m^n\} \leqslant & r_m^{2s}\Bigg\{\exp\left(2r_m^{\rho_1+\varepsilon}\right) + B\left[T(2r_m, f)\right]^{k+1} \\ & + B\left[T(2r_m, f)\right]^{k+1}\sum_{j=0, j\neq s}^{k-1} 2\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}\Bigg\} \\ \leqslant & 4r_m^{2s}\exp\left(2r_m^{\rho_1+\varepsilon}\right)B\left[T(2r_m, f)\right]^{k+1}\sum_{j=0, j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}\end{aligned}$$

which gives

$$\exp\{(1-\varepsilon)c_s\delta_1 r_m^n\} \leqslant 8r_m^{2s}\exp\left(2r_m^{\rho_1+\varepsilon}\right)B\left[T(2r_m, f)\right]^{k+1}\sum_{j=0, j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}. \tag{4.18}$$

Since $\varepsilon < \min\left\{n - \rho_1, \frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right), \ j \neq s\right\}$ is arbitrary, so by Lemma 2.6 and (4.18) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty}\frac{\log^+ T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty}\frac{\log\log T(r_m, f)}{\log r_m} \geqslant n.$$

In addition, by Lemma 2.7 and equation (1.5), we have $\rho_2(f) \leqslant n$, so $\rho_2(f) = n$. Then, each meromorphic solution to (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

Case 2. Assume that $\delta_1 < 0$. By Lemma 2.3, for any given $\varepsilon > 0$ we obtain

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\{(1-\varepsilon)c_j\delta(P_0, \theta)r^n\} + \exp\{(1-\varepsilon)c_j\delta(Q_0, \theta)r^n\} \\ &\leqslant 2\exp\{(1-\varepsilon)c_j\delta_1 r^n\}, \ j = 0, 1, 2, \ldots, k-1.\end{aligned} \tag{4.19}$$

By (1.5) we get

$$1 \leqslant \left|\frac{f}{f^{(k)}}\right|\left(\left|\frac{F(z)}{f(z)}\right| + \sum_{j=0}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right). \tag{4.20}$$

As in Case 1, by (3.5), (3.6), (4.17),(4.19) and (4.20), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_3 \cup E_8$, $r_m \to +\infty$, at which $|g(z)| = M(r_m, g)$, and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, we have

$$1 \leqslant r_m^{2k} \left(\exp\left(2r_m^{\rho_1+\varepsilon}\right) + \sum_{j=0}^{k-1} B\left[T\left(2r_m, f\right)\right]^{k+1} 2\exp\left\{(1-\varepsilon)\, c_j \delta_1 r_m^n\right\} \right). \tag{4.21}$$

Since $c_j \geqslant 1$, $j = 0, \dots, k-1$, $r_m > R_1 > 1$ and $\delta_1 < 0$, we obtain

$$\exp\left\{(1-\varepsilon)\, c_j \delta_1 r_m^n\right\} \leqslant \exp\left\{(1-\varepsilon)\, \delta_1 r_m^n\right\},\ j = 0, \dots, k-1$$

so (4.21) becomes

$$1 \leqslant 2r_m^{2k}\,(k+1) \exp\left(r_m^{\rho_1+\varepsilon}\right) B\left[T\left(2r_m, f\right)\right]^{k+1} \exp\left\{(1-\varepsilon)\, \delta_1 r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon - 1)\, \delta_1 r_m^n - r_m^{\rho_1+\varepsilon}\right\} \leqslant 2r_m^{2k}\,(k+1)\, B\left[T\left(2r_m, f\right)\right]^{k+1}. \tag{4.22}$$

By Lemma 2.6 and (4.22) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T\left(r_m, f\right)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T\left(r_m, f\right)}{\log r_m} \geqslant n.$$

In addition, by Lemma 2.7 and equation (1.5) we get $\rho_2(f) \leqslant n$ and hence, $\rho_2(f) = n$. Then, each meromorphic solution to (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

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BIBLIOGRAPHY

1. M. Andasmas and B. Belaïdi. *On the order and hyper-order of meromorphic solutions of higher order linear differential equations* // Hokkaido Math. J. **42**:3, 357-383 (2013).
2. B. Belaïdi and H. Habib. *Relations between meromorphic solutions and their derivatives of differential equations and small functions* // Ann. Univ. Buchar. Math. Ser. (LXIV) **6**:1, 35-57 (2015).
3. Z. X. Chen. *On the hyper-order of solutions of some second order linear differential equations* // Acta Math. Sin. (Engl. Ser.) **18**:1, 79-88 (2002).
4. Z. X. Chen. *On the rate of growth of meromorphic solutions of higher order linear differential equations* // Acta Math. Sinica (Chin. Ser.) **42**:3, 551-558 (1999).
5. Z. X. Chen. *The zero, pole and order of meromorphic solutions of differential equations with meromorphic coefficients* // Kodai Math. J. **19**:3, 341-354 (1996).
6. Z. X. Chen. *Zeros of meromorphic solutions of higher order linear differential equations* // Analysis **14**:4, 425-438 (1994).
7. J. Chen and J. F. Xu. *Growth of meromorphic solutions of higher order linear differential equations* // Electron. J. Qual. Theory Differ. Equ. **2009**:1, 1-13 (2009).
8. G. G. Gundersen. *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates* // J. London Math. Soc. **37**: 1, 88-104 (1988).
9. G. G. Gundersen. *Finite order solutions of second order linear differential equations* // Trans. Amer. Math. Soc. **305**:1, 415-429 (1988).
10. H. Habib and B. Belaïdi. *On the growth of solutions of some higher order linear differential equations with entire coefficients* // Electron. J. Qual. Theory Differ. Equ. **2011**:93, 1-13 (2011).

11. K. Hamani and B. Belaïdi. *On the hyper-order of solutions of a class of higher order linear differential equations* // Bull. Belg. Math. Soc. Simon Stevin **20**:1, 27-39 (2013).
12. W. K. Hayman. *Meromorphic functions* // Oxford Mathematical Monographs Clarendon Press, Oxford (1964).
13. J. Jank and L. Volkmann. *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen* // Birkhäuser Verlag, Basel, (1985).
14. K. H. Kwon. *On the growth of entire functions satisfying second order linear differential equations* // Bull. Korean Math. Soc **33**:3, 487-496 (1996).
15. I. Laine. *Nevanlinna theory and complex differential equations* // de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, (1993).
16. J. Tu, H. Y. Xu, H. M. Liu and Y. Liu. *Complex oscillation of higher order Linear differential equations with coefficients being lacunary series of finite iterated order* // Abstr. Appl. Anal. **2013**, id 634739 (2013).
17. J. Wang and H. X. Yi. *Fixed points and hyper-order of differential polynomials generated by solutions of differential equations* // Complex Var. Theory Appl. **48**:1, 83-94 (2003).
18. C. C. Yang and H. X. Yi. *Uniqueness theory of meromorphic functions* // Mathematics and its Applications. **557**. Kluwer Academic Publ. Group, Dordrecht (2003).
19. Y. Zhan and L. Xiao. *The growth of solutions of higher order differential equations with coefficients having the same order* // J. Math. Res. Appl. **35**:4, 387-399 (2015).

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