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SOME FUNCTIONAL EQUATIONS IN SCHWARTZ SPACE AND THEIR APPLICATIONS

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Abstract. In the paper we consider functional equations of form

$$(B + r^2E)u(z) = 0,$$

where B is a constant complex $n \times n$ matrix, E is the unit $n \times n$ matrix, z is a complex variable, $r = |z|$, $u(z)$ is the sought generalized vector function. For this equation, we study the existence of non-trivial solutions and the manifold of all solutions in the functional space $D' = D'(\mathbb{C}, \mathbb{C}^n)$ of generalized vector function and in the space $S' = S'(\mathbb{C}, \mathbb{C}^n)$ of tempered distributions. We also study the existence of solutions growing at most polynomially at infinity.

Such study is motivated by the problem on finding the solutions in S' for elliptic systems of first order elliptic equations. Here an important role is played by the statement on the structure of distributions supported in a circumference. This statement provides an explicit representation of distributions supported in a circumference and this representation consists of a linear combinations of Cartesian product of periodic distributions and δ-function and its derivatives. The process of finding all solutions to this equation in the space D' consists of three stages. At the first stage, by reducing the matrix to the normal Jordan form, we split this equation into one-dimensional equations. At the second stage we prove that if the matrix B has non negative and zero eigenvalues, that is, $\sigma(B) \cap (-\infty, 0] = \varnothing$, where $\sigma(B)$ is the spectrum of the matrix B, then in the space D', this equation has only the trivial solution. At the third stage, in the case $\sigma(B) \cap (-\infty, 0] \neq \varnothing$, we find all solutions to this equation in the space D'. Subject to the eigenvalues of the matrix B, the set of all solutions to this equation in the space D' is either zero or depends on finitely many arbitrary 2π-periodic distributions of one variable and finitely many arbitrary constants. The number of these functions and constants depend on the order of the solution; the order is prescribed. As an application, we find solutions in the space S', in particular, polynomially growing solutions to elliptic systems of partial differential equations and to overdetermined systems. The results obtained in the work can be employed in studying the problems on solutions defined on the entire complex plane or a half-plane and in studying more general linear multi-dimensional elliptic systems and overdetermined systems of partial differential equations.

Keywords: functional equations, Schwarz spaces, distributions supported in a circumference.

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1. Introduction

In the work we consider functional equations of form

$$(B + r^2E)u(z) = 0 \tag{1.1}$$

in the Schwarz space $D'(\mathbb{C}, \mathbb{C}^n)$ [1], [2]. Here $\mathbb{C}$ is the complex plane, $\mathbb{C}^n$ is the n-dimensional complex space, B is a constant $n\times$ matrix, E is the unit $n \times n$ matrix, $z = x + iy$, $r = |z|$, $u(z)$ is the unknown generalized vector function. In the work we obtain the manifold of all solutions in the space $D'(\mathbb{C}, \mathbb{C}^n)$ and the demonstrate the applications of the results to finding the polynomially growing solutions to elliptic systems of form

$$\omega_{\overline{z}} + A\overline{\omega} = 0, \quad \omega \in \mathbb{C}^n, \tag{1.2}$$

and to the overdetermined systems of form

$$u_{\overline{z}_j} = A_j\overline{u}, \quad u \in \mathbb{C}^n, \quad j = 1, 2.$$

While studying equation (1.1) in the space $S'(\mathbb{C}, \mathbb{C}^n)$ of vector tempered distributions, one can employ the generalized spherical representation for distributions obtained in works [3], [4]. Many publications are devoted to solving systems of form (1.2) as well as to more general elliptic, hyperbolic and overdetermined systems of partial differential equations, see, for instance, [5]–[9].

2. Structure of generalized functions supported on circumference

We shall make use of the following spaces (see [2]): $C^m(\overline{G})$, $C^\infty(G)$, $D(G)$, $D'(G)$, where G is a domain in $\mathbb{C}$ or an interval in $\mathbb{R} = (-\infty, +\infty)$. As $G = \mathbb{C}$, we denote $C^\infty = C^\infty(G)$, $D = D(\mathbb{C})$, $D' = D'(\mathbb{C})$. By $D_{2\pi}$ we denote the space of infinitely differentiable 2π-periodic functions of one real variable, while $D'_{2\pi}$ stands for the space of 2π-periodic distributions. The value of a distribution f on a test function φ is denoted by $\langle f, \varphi\rangle$.

Similar to the distributions with a point support, the distributions supported in a circumference can be explicitly described [6]. We note that in [6], there was provided a theorem on the structure of distributions supported in a circumference, but this theorem was not proved. In that work, the problem on solutions to equation (1.1) in the space $S'(\mathbb{C}, \mathbb{C}^n)$ was also considered.

Given $r > 0$ and $\varphi \in D$, we let $\psi(r, \theta) = \varphi(re^{i\theta})$. For each value θ, the function ψ belongs to the space $C^\infty(0, +\infty)$, while for each $r > 0$, it belongs to the space $D_{2\pi}$. This is why for each distribution $c(\theta)$ in $D'_{2\pi}$ and $r_0 > 0$ we can define the direct product $c(\theta) \times \delta^{(j)}(r - r_0)$:

$$\langle c(\theta) \times \delta^{(j)}(r - r_0), \psi(r, \theta)\rangle = \langle c(\theta), \frac{\partial^j \psi(r_0, \theta)}{\partial r^j}\rangle. \tag{2.1}$$

The following theorem holds.

Theorem 2.1. *If a distribution $f \in D'$ is supported in a circumference $\Gamma = \{z : |z| = r_0\}$, it is uniquely represented as*

$$\langle f, \varphi\rangle = \sum_{j=0}^{N} \langle c_j(\theta) \times \delta^{(j)}(r - r_0), \psi(r, \theta)\rangle, \varphi \in D, \tag{2.2}$$

where N is the order of f and $c_j(\theta)$ are some 2π-periodic distributions.

Proof. Let $0 < \varepsilon < \min\{1, r_0\}$ and $\eta_\varepsilon(t) \in C^\infty(-1, 1)$ be a function such that

$$\eta_\varepsilon(t) = 1 \quad \text{as} \quad |t| \leqslant \varepsilon/3; \quad \eta_\varepsilon(t) = 0 \quad \text{as} \quad |t| \geqslant \varepsilon; \quad |\eta_\varepsilon^{(k)}(t)| \leqslant M_k\varepsilon^{-k}, \quad |t| \leqslant \varepsilon. \tag{2.3}$$

If $r = |z|$, the function $\eta_\varepsilon(r - r_0)$ belongs to C^∞, is equal to 1 in the vicinity of the circumference Γ and

$$\operatorname{supp} \eta_\varepsilon(r - r_0) \subset G_\varepsilon = \{z : r_0 - \varepsilon < |z| < r_0 + \varepsilon\}.$$

This is why $f = \eta_\varepsilon(r-r_0)f$ and $\eta_\varepsilon(r-r_0)\varphi = \eta_\varepsilon(r-r_0)\psi(r,\theta)$ for $\varphi \in D$. Therefore, for each $\varphi \in D$ we have

$$\begin{aligned}\langle f,\varphi\rangle =&\langle \eta_\varepsilon(r-r_0)f,\varphi\rangle = \langle f,\eta_\varepsilon(r-r_0)\varphi\rangle = \langle f,\eta_\varepsilon(r-r_0)\psi\rangle\\ =&\langle f,\eta_\varepsilon(r-r_0)(\psi-\psi_N)\rangle + \langle f,\eta_\varepsilon(r-r_0)\psi_N\rangle,\end{aligned} \tag{2.4}$$

where

$$\psi_N(r,\theta) = \sum_{j=0}^{N} \frac{\partial^j\psi(r_0,\theta)}{\partial r^j}\cdot\frac{(r-r_0)^j}{j!}.$$

Let us show that the first term in the right hand side of identity (2.4) tends to zero as $\varepsilon \to 0$. Since $\eta_\varepsilon(r-r_0)(\psi-\psi_N) \in D(G_\varepsilon)$, we obtain (see [2])

$$\begin{aligned}|\langle f,\eta_\varepsilon(r-r_0)(\psi-\psi_N)\rangle| \leqslant& c\|\eta_\varepsilon(r-r_0)(\psi-\psi_N)\|_{C^N(\overline{G}_\varepsilon)}\\ =&c\max_{(r,\theta)\in\overline{G}_\varepsilon}\left|\frac{\partial^{k+l}}{\partial r^k\partial\theta^l}[\eta_\varepsilon(r-r_0)(\psi-\psi_N)]\right|,\end{aligned} \tag{2.5}$$

where c is a constant. Let us estimate the expression in the right hand side in (2.5). We have

$$\frac{\partial^{k+l}}{\partial r^k\partial\theta^l}[\eta_\varepsilon(r-r_0)(\psi-\psi_N)] = \sum_{|\nu|\leqslant k+l}\frac{d^{\nu_1}\eta_\varepsilon(r-r_0)}{dr^{\nu_1}}\cdot\frac{\partial^{\nu_2+\nu_3}(\psi-\psi_N)}{\partial r^{\nu_2}\partial\theta^{\nu_3}},$$

where $|\nu| = \nu_1+\nu_2+\nu_3$. By (2.3), the inequality

$$|\eta_\varepsilon^{(\nu_1)}(r-r_0)| \leqslant M_{\nu_1}\varepsilon^{-\nu_1} \quad \text{as} \quad |r-r_0| \leqslant \varepsilon$$

holds. Hence, by the identity

$$\frac{\partial^{\nu_2+\nu_3}(\psi-\psi_N)}{\partial r^{\nu_2}\partial\theta^{\nu_3}} = \sum_{j>N}\frac{\partial^{\nu_3}}{\partial\theta^{\nu_3}}\left[\frac{\partial^j\psi(r_0,\theta)}{\partial r^j}\right]\cdot\frac{j(j-1)\dots(j-\nu_2+1)}{j!}(r-r_0)^{j-\nu_2},$$

for $|r-r_0| \leqslant \varepsilon$ we have

$$\left|\frac{\partial^{k+l}}{\partial r^k\partial\theta^l}[\eta_\varepsilon(r-r_0)(\psi-\psi_N)]\right| \leqslant c_{kl}\sum_{\nu_1+\nu_2\leqslant k+l}\varepsilon^{-\nu_1}\varepsilon^{N-\nu_2+1}. \tag{2.6}$$

For $k+l \leqslant N$ and sufficiently small ε, the right hand side of inequality (2.6) can be estimated as follows:

$$\begin{aligned}c_{kl}\sum_{\nu_1+\nu_2\leqslant k+l}\varepsilon^{-\nu_1}\varepsilon^{N-\nu_2+1} \leqslant& \max_{k+l\leqslant N} c_{kl}\varepsilon^{N+1}\sum_{\nu_1+\nu_2\leqslant N}\varepsilon^{-(\nu_1+\nu_2)}\\ =&c_1\varepsilon^{N+1}\frac{1-\varepsilon^{-N}}{1-\varepsilon^{-1}} = c_1\frac{\varepsilon^{N+1}-\varepsilon}{\varepsilon-1} \leqslant c_2\varepsilon.\end{aligned}$$

Therefore,

$$|\langle f,\eta_\varepsilon(r-r_0)(\psi-\psi_N)\rangle| \leqslant c_3\varepsilon$$

and

$$\langle f,\eta_\varepsilon(r-r_0)(\psi-\psi_N)\rangle \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The second term in the right hand side of identity (2.4) is independent of ε and is equal to $< F,\psi_N >$, where F is the continuation of f on C^∞.

Passing in (2.4) to the limit as $\varepsilon \to 0$, we obtain

$$\langle f,\varphi\rangle = \langle f,\psi_N\rangle = \sum_{j=0}^{N}\langle F,\frac{\partial^j\psi(r_0,\theta)}{\partial r^j}\cdot\frac{(r-r_0)^j}{j!}\rangle. \tag{2.7}$$

We define 2π-periodic distributions $c_j(\theta)$ by letting

$$\langle c_j, \mu\rangle = (-1)^j \langle F, \frac{(r-r_0)^j}{j!}\mu(\theta)\rangle.$$

for $\mu(\theta) \in D_{2\pi}$. Then identity (2.7) can be rewritten as

$$\langle f, \varphi\rangle = \sum_{j=0}^{N}(-1)^j \left\langle c_j, \frac{\partial^j \psi(r_0,\theta)}{\partial r^j}\right\rangle.$$

But

$$\frac{\partial^j \psi(r_0,\theta)}{\partial r^j} = (-1)^j \langle \delta^{(j)}(r-r_0), \psi(r,\theta)\rangle,$$

and hence, employing (2.1), we get

$$\langle f, \varphi\rangle = \sum_{j=0}^{N}\langle c_j(\theta), \langle \delta^{(j)}(r-r_0), \psi(r,\theta)\rangle\rangle = \sum_{j=0}^{N}\langle c_j(\theta)\times \delta^{(j)}(r-r_0), \psi(r,\theta)\rangle, \ \varphi \in D.$$

This implies representation (2.2).

Let us prove the uniqueness of representation (2.2). Assume that apart of (2.2), we are given an another representation

$$f = \sum_{j=0}^{N} c_j'(\theta)\times \delta^{(j)}(r-r_0),$$

where $c_j'(\theta)$ are 2π-periodic distributions. Then we have

$$\sum_{j=0}^{N}[c_j(\theta) - c_j'(\theta)]\times \delta^{(j)}(r-r_0) = 0.$$

Hence, for $\varphi = \eta_\varepsilon(r-r_0)(r-r_0)^k\psi(\theta)$, where ψ is an arbitrary function in $D_{2\pi}$, we get

$$\sum_{j=0}^{N}\langle [c_j(\theta) - c_j'(\theta)], \langle \delta^{(j)}(r-r_0), \varphi\rangle\rangle = 0$$

or

$$\sum_{j=0}^{N}\langle [c_j(\theta) - c_j'(\theta)], (-1)^j \frac{d^j}{dr^j}(r-r_0)^k|_{r=r_0}\psi(\theta)\rangle = 0.$$

Then

$$\langle (c_k - c_k'), (-1)^k k!\psi(\theta)\rangle = 0.$$

This is why $c_k = c_k'$, $k = 0, \ldots, N$. This proves the uniqueness of representation (2.2). The proof is complete. □

Remark. While proving Theorem 2.1, we have employed the definition of a periodic distribution given in [10]. We note this definition is equivalent to the definition in [2].

3. Solutions of functional equations

In this section we consider the problem on finding all solutions to the equation

$$(B + r^2 E)u(z) = 0 \tag{3.1}$$

in the space $D'(\mathbb{C}, \mathbb{C}^n)$. Here B is a constant complex $n\times n$ matrix, E is the unit $n\times n$ matrix, $r = |z|$, $u(z)$ is an unknown vector function.

If V is a non-degenerate $n\times n$ matrix, equation (3.1) is equivalent to

$$(VBV^{-1} + r^2 E)v(z) = 0 \tag{3.2}$$

in the following sense: if $u \in D'(C, C^n)$ solves equation (3.1), then the function $\upsilon = Vu$ solves equation (3.2). And vice versa, if $\upsilon \in D'(\mathbb{C}, \mathbb{C}^n)$ solves equation (3.2), then $u = V^{-1}\upsilon$ solves equation (3.1).

We choose the matrix V so that to transform the matrix B to the normal Jordan form:

$$VBV^{-1} = \operatorname{diag}[J_1(\lambda_1), J_2(\lambda_2), \ldots, J_s(\lambda_s)], (s \leqslant n),$$

where

$$J_k(\lambda_k) = \begin{pmatrix} \lambda_k & 0 & 0 & \ldots & 0 \\ 1 & \lambda_k & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & \lambda_k \end{pmatrix}$$

are Jordan blocks of order m_k, λ_k are the eigenvalues of the matrix B, $k = 1, \ldots, s$, $m_k \geqslant 1$, $m_1 + m_2 + \ldots + m_s = n$. Let $\upsilon = (\upsilon_1, \ldots, \upsilon_n)^T$. Then equation (3.2) splits into the following equations:

$$(\lambda_{k+1} + r^2)\upsilon_{1+l(k)} = 0, \qquad k = 0, \ldots, s-1, \tag{3.3}$$

$$\upsilon_j + (\lambda_{k+1} + r^2)\upsilon_{j+1} = 0, \qquad l(k) < j < l(k+1), \tag{3.4}$$

where $l(k) = m_1 + m_2 + \ldots + m_k$, $l(0) = 0$.

Let us show that if the matrix B has no negative and zero eigenvalues, then equation (3.1) has only the trivial solution in $D'(\mathbb{C}, \mathbb{C}^n)$. Indeed, let $u \in D'(\mathbb{C}, \mathbb{C}^n)$ be a solution to equation (3.1). Then for an arbitrary $\varphi \in D$, by equations (3.3) and (3.4) we obtain

$$\langle \upsilon_{1+l(k)}, (\lambda_{k+1} + r^2)\varphi \rangle = 0, \quad k = 0, \ldots, s-1, \tag{3.5}$$

$$\langle \upsilon_j, \varphi \rangle + \langle \upsilon_{j+1}, (\lambda_{k+1} + r^2)\varphi \rangle = 0, \quad l(k) < j < l(k+1). \tag{3.6}$$

Since $\inf\limits_{k,r} |\lambda_{k+1} + r^2| > 0$, letting $\varphi = (\lambda_{k+1} + r^2)^{-1}\psi, (\varphi \in D)$ in (3.5) for an arbitrary $\psi \in D$, we get

$$\langle \upsilon_{1+l(k)}, \psi \rangle = 0, \quad \psi \in D,$$

that is, $\upsilon_{1+l(k)} = 0, \quad k = 0, \ldots, s-1$. This is why it follows from (3.6) that

$$\langle \upsilon_{2+l(k)}, (\lambda_{k+1} + r^2)\varphi \rangle = 0, \quad \varphi \in D.$$

In same way as above, $\upsilon_{2+l(k)} = 0$. Continuing this procedure, one can show that $\upsilon_j = 0$ for each j. Thus, $\upsilon = 0$ and therefore, $u = V^{-1}\upsilon = 0$.

In view of this, we assume that the matrix B can have eigenvalues in the semi-axis $(-\infty, 0]$. To be definite, we assume that $\lambda_1, \ldots, \lambda_p \in (-\infty, 0]$ and $\lambda_{p+1}, \ldots, \lambda_n \notin (-\infty, 0]$, $\quad (1 \leqslant p \leqslant n)$. Let us show that the support of the solution $u \in D'(\mathbb{C}, \mathbb{C}^n)$ to equation (3.1) is contained in the set

$$\Gamma = \bigcup_{j=1}^{p} \{z : |z| = |\lambda_j|^{1/2}\}.$$

Indeed, let $\varphi \in D$ and $\operatorname{supp}\varphi \bigcap \Gamma = \emptyset$. We let

$$\psi(z) = \begin{cases} (\lambda_{k+1} + r^2)^{-1}\varphi(z) & \text{as} \quad |z| \neq |\lambda_j|^{1/2}, \\ 0 & \text{as} \quad |z| = |\lambda_j|^{1/2}. \end{cases}$$

Then $\psi \in D$ and by (3.5)

$$\langle \upsilon_{1+l(k)}, \varphi \rangle = \langle \upsilon_{1+l(k)}, (\lambda_{k+1} + r^2)\psi \rangle = 0, \quad k = 0, \ldots, p-1,$$

that is, $\operatorname{supp}\upsilon_{1+l(k)} \subset \Gamma$. In the same way, employing relations (3.6), we can establish that $\operatorname{supp}\upsilon_j \subset \Gamma, j \leqslant l(p)$. As it was shown above, other υ_j vanish, that is, $\upsilon_j = 0$ as $j > l(p)$. Therefore, $\operatorname{supp} u = \operatorname{supp} \upsilon \subset \Gamma$. Thus, the support of the solution to equation (3.1) is contained in the set Γ, which is the union of concentric circumferences and the point $z = 0$ (of course, if some $\lambda_j = 0$). This is why, in order to find solutions to equation (3.1), we shall make use the representation for distributions supported in a circumference and at a point.

We proceed to solving equation (3.1) in the space $D'(\mathbb{C}, \mathbb{C}^n)$. In order to do this, we solve equivalent system of equations (3.3), (3.4). Reproducing the proofs of the inclusions $\operatorname{supp} \upsilon_j \subset \Gamma$, $j \leqslant l(p)$, we obtain more precise inclusions:

$$\operatorname{supp} \upsilon_j \subset \Gamma_k, \quad j \leqslant l(k+1), \quad \Gamma_k = \{z : |z| = |\lambda_{k+1}|^{1/2}\}, \quad k = \overline{0, p-1}.$$

In order to employ the representation of the functions υ_j, $j \leqslant l(p)$, we need to know which Γ_k are circumferences and which are points. Assume that $\lambda_1, \ldots, \lambda_q$ are negative and $\lambda_{q+1}, \ldots, \lambda_p$ are zero ($1 \leqslant q \leqslant p$). Then $\Gamma_1, \ldots, \Gamma_q$ are circumferences and $\Gamma_{q+1}, \ldots, \Gamma_p$ are the point $z = 0$. Therefore, the functions υ_j can be represented as

$$\upsilon_j = \begin{cases} \sum\limits_{\nu=0}^{N} c_{\nu j}(\theta) \times \delta^{(\nu)}(r - r_k) & \text{as} \quad l(k) < j \leqslant l(k+1), \quad k = \overline{0, q-1}, \\ \sum\limits_{\alpha+\beta \leqslant N} a_{\alpha\beta}^{(j)} \frac{\partial^{\alpha+\beta}\delta(z)}{\partial z^\alpha \partial \bar{z}^\beta} & \text{as} \quad l(q) < j \leqslant l(p), \end{cases} \tag{3.7}$$

where $c_{\nu j} \in D'_{2\pi}$, $r_k = |\lambda_{k+1}|^{1/2}$, $a_{\alpha\beta}^{(j)}$ are constants; for the sake of simplicity, we assume that the orders of the distributions υ_j are N.

Let $\eta(t)$ be a function in C^∞ such that $\eta(t) = 1$ in the vicinity of the point $t = r_k$ and $\eta(t) = 0$ as $0 \leqslant t \leqslant \rho_1$ and as $t \geqslant \rho_2$, where $0 < \rho_1 < \rho_2$. Then the function $\varphi_s = (r - r_k)^s \eta(r) h(\theta)$, where $r = |z|$, $\theta = \arg z$, $h \in D_{2\pi}$, s is a non-negative integer, belongs to D. This is why

$$\langle \delta^{(\nu)}(r - r_k), (\lambda_{k+1} + r^2)\varphi_s \rangle = 2\nu r_k \frac{\partial^{\nu-1}\varphi_s}{\partial r^{\nu-1}} + \nu(\nu-1)\frac{\partial^{\nu-2}\varphi_s}{\partial r^{\nu-2}}, (0 \leqslant k \leqslant q-1). \tag{3.8}$$

This and (3.5), (3.7) for $j = 1 + l(k)$, $\varphi = \varphi_s$ imply

$$\sum_{\nu=0}^{N} \langle c_{\nu j}(\theta), 2\nu r_k \frac{\partial^{\nu-1}\varphi_s}{\partial r^{\nu-1}} + \nu(\nu-1)\frac{\partial^{\nu-2}\varphi_s}{\partial r^{\nu-2}} \rangle = 0. \tag{3.9}$$

Since

$$\left.\frac{\partial^\nu \varphi_s}{\partial r^\nu}\right|_{r=r_k} = \begin{cases} h(\theta) & \text{as} \quad \nu = s, \\ 0 & \text{as} \quad \nu \neq s, \end{cases}$$

then by (3.9) as $s = N - 1$ we get

$$\langle c_{Nj}(\theta), 2N r_k h(\theta) \rangle = 0, h \in D_{2\pi},$$

that is, $c_{Nj}(\theta) = 0$. Letting $s = N - 2$ in (3.9), we obtain

$$\langle c_{N-1,j}(\theta), 2(N-1) r_k h(\theta) \rangle = 0, \quad h \in D_{2\pi},$$

that is, $c_{N-1,j}(\theta) = 0$. Continuing this process, we find:

$$c_{N-2,j} = c_{N-3,j} = \ldots = c_{1,j} = 0.$$

The left hand side in (3.9) does not involve $c_{0j}(\theta)$ and it remains arbitrary. Thus, υ_j for $j = 1 + l(k)$, $0 \leqslant k \leqslant q-1$, are determined by the formula

$$\upsilon_j = c_{0j}(\theta) \times \delta(r - r_k), \tag{3.10}$$

where $c_{0j}(\theta)$ are arbitrary 2π-periodic distributions.

Letting $j = 1 + l(k)$, $0 \leqslant k \leqslant q-1$, $\varphi = \varphi_s$ in (3.6), in view of relations (3.7), (3.8) and (3.10) we obtain

$$\langle c_{0j}(\theta) \times \delta(r - r_k), \varphi_s(r, \theta) \rangle + \sum_{\nu=0}^{N} \left\langle c_{\nu,j+1}(\theta), 2\nu r_k \frac{\partial^{\nu-1}\varphi_s(r_k, \theta)}{\partial r^{\nu-1}} + \nu(\nu-1)\frac{\partial^{\nu-1}\varphi_s(r_k, \theta)}{\partial r^{\nu-1}} \right\rangle = 0.$$

Substituting $s = N-1, \dots, 2, 1$ into this identity, we get

$$c_{N,j+1} = c_{N-1,j+1} = \dots = c_{2,j+1} = 0, \quad \langle c_{0j}, h(\theta)\rangle + \langle c_{1,j+1}, 2r_k h(\theta)\rangle = 0,$$

that is, $c_{1,j+1}(\theta) = -c_{0j}(\theta)/2r_k$. We observe that $c_{0,j+1}(\theta)$ remain arbitrary. This is why as $j = 2l(k)$,

$$\upsilon_j = c_{0j}(\theta) \times \delta(r - r_k) - \frac{1}{2r_k} c_{0,j-1}(q) \times \delta'(r - r_k).$$

Continuing this procedure, we find

$$\upsilon_j = c_{0j}(\theta) \times \delta(r - r_k) + \sum_{\alpha=1}^{j-1-l(k)} A_{\alpha j}(\theta) \times \delta^{(\alpha)}(r - r_k), \tag{3.11}$$

where $l(k) \leqslant j \leqslant l(k+1)$, $0 \leqslant k \leqslant q-1$, $c_{0j}(\theta)$ are arbitrary distributions in $D'_{2\pi}$, $A_{\alpha j}(\theta)$ are expressed linearly via $c_{0t}(\theta)$, $l(k) < t < j$. Thus, we have determined υ_j involved in the left hand sides of (3.5), (3.6) with the subscript j: $l(k) < j \leqslant l(k+1)$, $0 \leqslant k \leqslant q-1$. These υ_j are given by formula (3.11).

Now let us define υ_j with the subscript j: $l(k) < j \leqslant l(k+1)$, $q \leqslant k \leqslant p-1$. By (3.5) we have

$$\langle \upsilon_j, r^2\varphi\rangle = 0, \quad j = 1 + l(k), \quad q \leqslant k \leqslant p-1, \quad \varphi \in D. \tag{3.12}$$

This implies that either $\upsilon_j = 0$ or υ_j supported at the point $z = 0$. This is why the theorem on the structure of distributions with a point support we have

$$\upsilon_j = \sum_{\alpha+\beta\leqslant N} c_{\alpha\beta} \frac{\partial^{\alpha+\beta}\delta}{\partial z^\alpha \partial \overline{z}^\beta}, \tag{3.13}$$

where $c_{\alpha\beta}$ are constants, N is the order of the distribution υ_j. This is why identity (3.5) becomes

$$\sum_{\alpha+\beta\leqslant N} c_{\alpha\beta} \langle \frac{\partial^{\alpha+\beta}\delta}{\partial z^\alpha \partial \overline{z}^\beta}, z\overline{z}\varphi\rangle = 0, \quad \varphi \in D. \tag{3.14}$$

Letting here

$$\varphi = \eta(z) z^\nu \overline{z}^t, \tag{3.15}$$

where $\nu \geqslant 0$, $t \geqslant 0$, $\eta \in D$, and $\eta(z) = 1$ in the vicinity of zero, we obtain

$$(\nu+1)!(t+1)! c_{\nu+1,t+1} = 0,$$

that is, $c_{\alpha\beta} = 0$ as $\alpha\beta \neq 0$. We open the brackets in the left hand side in identity (3.14) and we see that the terms with the coefficients $c_{\alpha 0}$, $c_{0\beta}$ vanish. This is why the coefficients $c_{\alpha\beta}$ at $\alpha\beta = 0$ remain arbitrary. Therefore,

$$\upsilon_j = \sum_{\nu=0}^{N} \Big(c_{\nu j} \frac{\partial^\nu \delta}{\partial z^\nu} + c'_{\nu j} \frac{\partial^\nu \delta}{\partial \overline{z}^\nu} \Big), \ j = 1 + l(k), \ q \leqslant k \leqslant p-1, \tag{3.16}$$

where $c_{\nu j}$, $c'_{\nu j}$ are arbitrary constants.

As $q \leqslant k \leqslant p-1$, $j = 1 + l(k)$, by (3.6) we get

$$\langle \upsilon_j, \varphi\rangle + \langle \upsilon_{j+1}, r^2\varphi\rangle = 0, \quad \varphi \in D.$$

By this we conclude that the support of υ_{j+1} consists of the point $z = 0$. This is why, representing the distribution υ_{j+1} as (3.13) with coefficients $d_{\alpha\beta}$ ($\alpha + \beta \leqslant N$) and choosing function (3.15) as φ, by (3.16) we obtain that $d_{\alpha 0}$, $d_{0\beta}$ are arbitrary and

$$\alpha! c_{\alpha j} + \beta! c'_{\beta j} + (\alpha+1)!(\beta+1)! d_{\alpha+1,\beta+1} = 0.$$

Then

$$\upsilon_j = \sum_{\nu=0}^{N}\Big(c_{\nu j}\frac{\partial^\nu \delta}{\partial z^\nu} + c'_{\nu j}\frac{\partial^\nu \delta}{\partial \overline{z}^\nu}\Big) - \sum_{\substack{\alpha,\beta\geqslant 1\\ \alpha+\beta\leqslant N}} \frac{1}{\alpha!\beta!}\Big[(\alpha-1)!c_{\alpha-1,j} + (\beta-1)!c'_{\beta-1,j}\Big]\frac{\partial^{\alpha+\beta}\delta}{\partial z^\alpha \partial \overline{z}^\beta}, \quad j = 1 + l(k),$$

where $c_{\nu,j+1}$, $c'_{\nu,j+1}$ are arbitrary constants. In the same way we find other υ_j with the subscript j: $l(k) < j \leqslant l(k+1)$, $q \leqslant k \leqslant p-1$:

$$\upsilon_j = \sum_{\nu=0}^{N}\Big(c_{\nu j}\frac{\partial^\nu \delta}{\partial z^\nu} + c'_{\nu j}\frac{\partial^\nu \delta}{\partial \overline{z}^\nu}\Big) + \sum_{\substack{\alpha,\beta\geqslant 1\\ \alpha+\beta\leqslant N}} B^{(j)}_{\alpha\beta}\frac{\partial^{\alpha+\beta}\delta}{\partial z^\alpha \partial \overline{z}^\beta},$$

where $c_{\nu j}$, $c'_{\nu j}$ are arbitrary constants, $B^{(j)}_{\alpha\beta}$ are expressed linearly in $c_{\nu t}$, $c'_{\nu t}$, $t < j$, and $B^{(j)}_{\alpha\beta} = 0$ as $j = 1 + l(k)$.

It remains to determine υ_j for $j > l(p)$. Let us show that all such υ_j vanish. By (3.5) with $k = p$ we obtain

$$\langle \upsilon_{1+l(p)}, (\lambda_{p+1} + r^2)\varphi\rangle = 0, \ \varphi \in D. \tag{3.17}$$

Since $\inf|\lambda_{p+1} + r^2| > 0$ (we recall that $\lambda_{p+1} \notin (-\infty, 0]$), given an arbitrary $\psi \in D$, we let $\varphi = (\lambda_{p+1} + r^2)^{-1}\psi$, $\varphi \in D$, in (3.17) and we get

$$\langle \upsilon_{1+l(p)}, \psi\rangle = 0 \qquad \forall \psi \in D,$$

that is, $\upsilon_{1+l(p)} = 0$. It follows from (3.6) that

$$\langle \upsilon_{2+l(p)}, (\lambda_{p+2} + r^2)\varphi\rangle = 0 \qquad \forall \varphi \in D,$$

and as above, we obtain $\upsilon_{2+l(p)} = 0$. Continuing this procedure, we can show that $\upsilon_j = 0$ as $j > l(p)$.

Thus, we have determined all components υ_j of the distribution υ, that is, we have described the set of all solutions to equation (3.2) in the space $D'(\mathbb{C}; \mathbb{C}^n)$. Then the set of all solutions to equation (3.1) in $D'(\mathbb{C}; \mathbb{C}^n)$ is given by the formula $u = V^{-1}\upsilon$. We observe that subject to the eigenvalues of the matrix B, the set of all solutions to equation (3.1) in the space $D'(\mathbb{C}; \mathbb{C}^n)$ is either zero or consists of finitely many arbitrary 2π-periodic distributions and finitely many arbitrary constants, and the number of these functions and constants depend on the order of the solution. We can prescribe the order of the solution.

4. Applications to systems of partial differential equations

Problems on solving multi-dimensional elliptic systems

$$\omega_{\overline{z}} + A\overline{\omega} = 0, \quad \omega \in \mathbb{C}^n, \tag{4.1}$$

and overdetermined systems of form

$$u_{\overline{z}_j} = A_j\overline{u}, \quad u \in \mathbb{C}^n, \quad j = 1, 2, \tag{4.2}$$

in the space $S'(\mathbb{C}, \mathbb{C}^n)$ are reduced to a particular case of equation (3.1).

4.1. Elliptic systems. We consider elliptic system (4.1) in the space $S'(\mathbb{C}, \mathbb{C}^n)$ and make the Fourier transform:

$$i\zeta\widetilde{\omega}(\zeta) + 2A\widetilde{\overline{\omega}}(\zeta) = 0. \tag{4.3}$$

Replacing ζ by $-\zeta$ and passing to the complex-conjugate quantities, in view of the identity $\overline{\omega(-\zeta)} = \widetilde{\overline{\omega}}(\zeta)$ we obtain one more equation

$$2\overline{A}\widetilde{\omega}(\zeta) + i\overline{\zeta}\ \widetilde{\overline{\omega}}(\zeta) = 0. \tag{4.4}$$

We exclude $\widetilde{\overline{\omega}}(\zeta)$ from equations (4.3) and (4.4) and we get the following equation:

$$(4A\overline{A} + |\zeta|^2 E)\widetilde{\omega}(\zeta) = 0, \tag{4.5}$$

which is a particular case of equation (3.1). As it was shown in Section 2, if the spectrum $\sigma(A\overline{A})$ of the matrix $A\overline{A}$ does not intersect the semi-axis $(-\infty, 0]$, then equation (4.5) has only the trivial solution in the space $S'(\mathbb{C}, \mathbb{C}^n)$ and hence, system (4.1) has also only the trivial solution in $S'(\mathbb{C}, \mathbb{C}^n)$. If $\sigma(A\overline{A}) \cap (-\infty, 0] \neq \varnothing$, then similar to equation (3.1), we can find solution to equation (4.5) in $S'(\mathbb{C}, \mathbb{C}^n)$ and to determine the solutions to systems (4.1).

We note (see [6]), that if for system (4.1) we consider the problem on solutions $\omega(z)$ growing at most as the power function z^N, $N \in \{0, 1, \dots\}$, then in the case $\sigma(A\overline{A}) \cap (-\infty, 0) \neq \varnothing$, the space P_N of solutions to such problem is infinite-dimensional as a linear space over the field of real numbers; in the case $\sigma(A\overline{A}) \cap (-\infty, 0] = \{0\}$ it is finite-dimensional and

$$\dim P_N = 2n(N+1) - 2\sum_{j=0}^{N} rank B_j,$$

where $B_{2k} = \overline{A}(A\overline{A})^k$, $B_{2k+1} = (A\overline{A})^{k+1}$, $k = 0, \dots, [\frac{N}{2}]$.

4.2. Overdetermined systems. We consider overdetermined system (4.2), in which $u = u(z_1, z_2)$ is the unknown vector function of complex variables z_1 and z_2, $z_j = x_j + iy_j$, $u_{\bar{z}_j} = \frac{1}{2}(u_{x_j} + iu_{y_j})$, $j = 1, 2$, A_1 and A_2 are constant complex $n \times n$ matrices.

For system (4.2) we seek solutions of power growth, that is, solutions $u(z_1, z_2)$ defined in C^2 and satisfying the condition

$$\|u(z_1, z_2)\|_{C^n} \leqslant K(1 + |z_1| + |z_2|)^N, \tag{4.6}$$

where K is a constant depending in general on $u(z_1, z_2)$, N is a non-negative integer.

In the space $S'(\mathbb{C}^2, \mathbb{C}^n)$, system (4.2) is equivalent to the system of functional equations

$$\begin{cases} i\zeta_1 v(\zeta_1, \zeta_2) - 2A_1 w(\zeta_1, \zeta_2) = 0, \\ i\zeta_2 v(\zeta_1, \zeta_2) - 2A_2 w(\zeta_1, \zeta_2) = 0, \end{cases} \tag{4.7}$$

where $v(\zeta_1, \zeta_2)$ and $w(\zeta_1, \zeta_2)$ are the Fourier images $u(z_1, z_2)$ and $\overline{u(z_1, z_2)}$, respectively, in the following sense: if $u \in S'(\mathbb{C}^2, \mathbb{C}^n)$ is a solution to system (4.2), then the pair (v, w) satisfies system (4.7). If the pair (v, w), $v, w \in S'(\mathbb{C}^2, \mathbb{C}^n)$ is a solution to system (4.7) and the identity

$$\overline{w(-\zeta_1, -\zeta_2)} = v(\zeta_1, \zeta_2) \tag{4.8}$$

holds, then the function $u = F^{-1}v$, where F^{-1} is the inverse Fourier transform, is a solution to system (4.2) in the space $S'(\mathbb{C}^2, \mathbb{C}^n)$. Employing relation (4.8) in system (4.7), we obtain the following equations:

$$\begin{cases} 2\overline{A}_1 v(\zeta_1, \zeta_2) - i\bar{\zeta}_1 w(\zeta_1, \zeta_2) = 0, \\ 2\overline{A}_2 v(\zeta_1, \zeta_2) - i\bar{\zeta}_2 w(\zeta_1, \zeta_2) = 0. \end{cases} \tag{4.9}$$

We exclude $w(\zeta_1, \zeta_2)$ from equations (4.7), (4.9) and for $v(\zeta_1, \zeta_2)$ we obtain the system

$$(4A_1\overline{A}_1 + |\zeta_1|^2 E)v(\zeta_1, \zeta_2) = 0, \tag{4.10}$$

$$(4A_2\overline{A}_2 + |\zeta_2|^2 E)v(\zeta_1, \zeta_2) = 0, \tag{4.11}$$

where E is the unit $n \times n$ matrix. We can apply the results of Section 2 to equations (4.10), (4.11).

By σ_1 and σ_2 we denote the spectra of the matrices $A_1\overline{A}_1$ and $A_2\overline{A}_2$, respectively. If the condition

$$(\sigma_1 \cup \sigma_2) \cap (-\infty, 0] = \varnothing \tag{4.12}$$

holds, then it follows from system (4.10), (4.11) that $v = 0$ and hence, $u = 0$.

If condition (4.12) fails, the support of distribution $v(\zeta_1, \zeta_2)$ can be a single point, the Cartesian product of two circumferences and even a non-compact set. In the first two cases, thanks to the theorem on the structure of distributions with a point support and Theorem 2.1, we can determine solutions to system (4.10), (4.11) and then to overdetermined system (4.2).

Let P_N be the manifold of solutions to problem (4.2), (4.6). We obviously have $P_N \subset S'(\mathbb{C}^2, \mathbb{C}^n)$ and P_N is a linear space over the field of real numbers. The space P_N can infinite-dimensional or finite-dimensional. We consider some examples.

Example 1. Let $A_1 = 0$ in system (4.2). Then the solutions of this system are holomorphic in z_1 and by the Liouville theorem, the solutions in P_N are to be of the form

$$u(z_1, z_2) = \sum_{j=0}^{N} c_j(z_2) z_1^j,$$

where the functions $c_j(z_2)$ satisfy the condition

$$\|c_j(z_2)\|_{\mathbb{C}^n} \leqslant K(1 + |z_2|^N). \tag{4.13}$$

Then by the second equation in system (4.2) we have

$$\sum_{j=0}^{N} \frac{\partial c_j}{\partial \overline{z}_2} z_1^j = \sum_{j=0}^{N} A_2 \overline{c_j(z_2)} \cdot \overline{z}_1^j.$$

This implies the following equations for $c_j(z_2)$:

$$\frac{\partial c_0}{\partial \overline{z}_2} = A_2 \overline{c}_0, \tag{4.14}$$

$$\begin{cases} \dfrac{\partial c_j}{\partial \overline{z}_2} = 0, \\ A_2 \overline{c}_j = 0, \ j = 1, \dots, N. \end{cases} \tag{4.15}$$

If $\sigma_2 \cap (-\infty, 0) \neq \varnothing$, then the manifold of solutions to equation (4.14) satisfying condition (4.13) is infinite-dimensional. If $\sigma_2 \cap (-\infty, 0) = \varnothing$, then as $\det A_2 \neq 0$, we have $c_0(z_2) \equiv 0$, while as $\det A_2 = 0$, the solutions to problem (4.13), (4.14), as this was mentioned in Section 4.1, form a finite-dimensional space. By (4.15) we see that as $\det A_2 \neq 0$, all $c_j(z_2)$, , $j = 1, \dots, N$, are identically zero, while if $\det A_2 = 0$, then $c_j(z_2)$ are holomorphic polynomials of form $\sum\limits_{k=0}^{N} d_{jk} z_2^k$, where d_{jk} are the eigenvectors of the matrix $\overline{A}_2$ associated with the zero eigenvalue.

Example 2. Let $A_1 = A_2 = A$ in system (4.2). Then we get

$$u_{\overline{z}_1} = u_{\overline{z}_2}. \tag{4.16}$$

The solutions of this equation are the functions of form $u = \varphi(z_1, z_2)$ and $u = \psi(z_1 + z_2)$, where φ is a holomorphic in z_1 and z_2 vector function, and $\psi(z_1)$ is a vector function possessing the partial derivatives ψ_{x_1} and ψ_{y_1}. The set of solutions to equation (4.16) is larger than the set of solutions to system (4.2). We substitute the aforementioned solutions of equation (4.16) into system (4.2):

$$\varphi_{z_j}(z_1, z_2) = A\overline{\varphi(z_1, z_2)}, \tag{4.17}$$

$$\psi_{z_j}(z_1 + z_2) = A\overline{\psi(z_1 + z_2)}, \ j = 1, 2. \tag{4.18}$$

In (4.17), the left hand sides are zero and this is why

$$A\overline{\varphi(z_1, z_2)} = 0. \tag{4.19}$$

If $\det A \neq 0$, then $\varphi \equiv 0$. If $\det A = 0$, then $\varphi(z_1, z_2) = f(z_1, z_2)\upsilon$, where f is a holomorphic in z_1 and z_2 scalar function, υ is an eigenvector of the matrix A associated with the zero eigenvalue. Then the vector function

$$u(z_1, z_2) = P_N(z_1, z_2)\upsilon,$$

where P_N is a holomorphic polynomial in z_1 and z_2 of degree at most N, solve problem (4.2), (4.6).

If in the first (respectively, in the second) equation in (4.18) we fix z_2 (respectively, z_1), then for ψ we obtain system (4.1) considered in Subsection 4.1.

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