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ASYMPTOTICS IN PARAMETER OF SOLUTION TO ELLIPTIC BOUNDARY VALUE PROBLEM IN VICINITY OF OUTER TOUCHING OF CHARACTERISTICS TO LIMIT EQUATION

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Abstract. In a bounded domain $Q \subset \mathbb{R}^3$ with a smooth boundary Γ we consider the boundary value problem

$$\varepsilon A u - \frac{\partial u}{\partial x_3} = f(x), \quad u|_\Gamma = 0.$$

Here A is a second order elliptic operator, ε is a small parameter. The limiting equation, as $\varepsilon = 0$, is the first order equation. Its characteristics are the straight lines parallel to the axis Ox_3. For the domain $\overline{Q}$ we assume that the characteristic either intersects Γ at two points or touches Γ from outside. The set of touching point forms a closed smooth curve. In the paper we construct the asymptotics as $\varepsilon \to 0$ for the solutions to the studied problem in the vicinity of this curve. For constructing the asymptotics we employ the method of matching asymptotic expansions.

Keywords: small parameter, asymptotic, elliptic equation.

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Formulation of problem

In a bounded simply-connected domain $Q \subset \mathbb{R}^3$ with piecewise smooth boundary Γ we consider the boundary valu problem

$$\varepsilon A(x, D) u(x, \varepsilon) - D_3 u(x, \varepsilon) = f(x), \quad x \in Q, \tag{0.1}$$

$$u = 0, \quad x \in \Gamma. \tag{0.2}$$

Here $\varepsilon > 0$ is a small parameter, $x = (x_1, x_2, x_3)$, $D = (D_1, D_2, D_3)$, $D_j = \frac{\partial}{\partial x_j}$,

$$A(x, D) = \sum_{|\alpha| \leqslant 2} a_\alpha(x) D^\alpha$$

is an elliptic differentiation operator with a positive definite quadratic form

$$a_2(x, \xi) = \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \geqslant a_0 |\xi|^2, \quad a_0 > 0,$$

a_0 is a constant, α is a multi-index.

Assume that the data of problem (0.1)—(0.2) are smooth (belong to C^∞), then for each $\varepsilon > 0$ there exists the unique solution $u(x, \varepsilon) \in C^\infty(\overline{Q})$.

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The limiting equation for (0.1) is that as $\varepsilon = 0$, i.e., the first order equation

$$-D_3 u_0(x) = f(x). \tag{0.3}$$

Its characteristics are straight lines parallel the axis Ox_3. Regarding the domain $\overline{Q} = Q \cup \Gamma$ we assume that the characteristics of equation (0.3) either intersect Γ at two points or they have first order touching with Γ from outside and the set of touching points is a smooth closed curve S_0. In what follows we assume that the curve S_0 lies in the plane $x_3 = 0$. This can be achieved by a smooth change of variables, which keeps the form of equation (0.1).

The curve S_0 partitions Γ into two parts $\Gamma^{\pm}$ as $x_3 \gtrless 0$, respectively. The limiting problem for (0.1)–(0.2) is the problem

$$-D_3 u_0(x) = f(x), \qquad u_0|_{\Gamma^-} = 0. \tag{0.4}$$

Everywhere in the domain Q except the vicinity of the curve S_0, an asymptotic solution of problem (0.1)–(0.2) as $\varepsilon \to 0$ is found by the Vishik-Lyusternik method [1]. In the present work we construct an asymptotic solution to problem (0.1)–(0.2) in the vicinity of S_0. In order to construct asymptotic solution, we employ the method of matching asymptotic solutions by A.M. Il'in [2]. The two-dimensional case for equations with constant coefficients was considered in [3] (see also [2]).

1. Estimate of solution in a subdomain

Let $d(x_1, x_2)$ be the distance along the interior normal to S_0. By S_{d_0} we denote the curve in the plane $x_3 = 0$ separated from S_0 by the distance $d(x, y) = d_0$, where d_0 is chosen so that the normals do not intersect. The characteristics of equation (0.3) passing S_{d_0} separate the domain Q_0 bordered by these characteristics X_{d_0} by Γ_{d_0}, which is a part Γ containing S_0. Let Q_δ be the subdomain

$$Q_0: \; Q_\delta = \{x \in Q_0: \; 0 < d(x, y) < d_0 - \delta\},$$

where $0 < \delta < d_0$. Given a domain G in $\mathbb{R}^3$ and an integer $p \geqslant 0$, by $H^p(G)$ we denote the Sobolev space with the norm

$$\|u\|^2_{p,G} = \sum_{|\alpha| \leqslant p} \int |D^\alpha u|^2 \, dx.$$

Theorem 1. *Let Q_0 and Q_δ be the domain defined above. Then for sufficiently small $\varepsilon > 0$ and $\delta \geqslant C\varepsilon^\gamma$, where $C > 0$ is a constant independent of ε, $0 < \gamma < \frac{1}{2}$, the solution of problem (0.1)—(0.2) satisfies the estimate*

$$\varepsilon \|u\|^2_{1,Q_\delta} + \|u\|^2_{0,Q_\delta} \leqslant C_1 \left[\|f\|^2_{0,Q_0} + \varepsilon^{\frac{1}{2}-\gamma} (\varepsilon \|u\|^2_{1,Q_0} + \|u\|^2_{0,Q_0}) \right] \tag{1.1}$$

with a constant C_1 independent of ε.

Proof. Let $\psi_\delta(x_1, x_2)$ be a smooth cut-off function

$$\psi_\delta(x_1, x_2) = \begin{cases} 1, & 0 \leqslant d(x_1, x_2) \leqslant \delta_0 - \delta, \\ 0, & d(x_1, x_2) \geqslant \delta_0, \end{cases}$$

for which the estimates

$$\|D_1^k D_2^m \psi_\delta\| \leqslant C_{k,m} \delta^{-(k+m)}, \quad k, m = 0, 1, 2,$$

hold with constants $C_{k,m}$ independent of δ.

We consider the expression $u_\delta(x) = e^{-\lambda x_3} v_\delta(x)$, where

$$u_\delta(x) = u(x)\psi_\delta(x_1, x_2), \quad v_\delta(x) = v(x)\psi_\delta(x_1, x_2).$$

By equation (0.1)

$$\varepsilon A v_\delta - D_3 v_\delta - \lambda v_\delta = e^{\lambda x_3} f \psi_\delta - \varepsilon A' v, \tag{1.2}$$

where $A'v = e^{\lambda x_3}[A, e^{-\lambda x_3}\psi_\delta]v(x)$, $[\,\cdot\,,\cdot\,]$ is the commutator.

Multiplying (1.2) by $-(v_\delta(x))$ and integrating over the domain Q_0, we obtain

$$-\varepsilon\langle A v_\delta, v_\delta\rangle + \langle D_3 v_\delta, v_\delta\rangle + \lambda\|v_\delta\|^2_{0,Q_0} \leqslant |\langle e^{\lambda x_3} f\psi_\delta, v_\delta\rangle| + \varepsilon|\langle A'v, v_\delta\rangle|, \quad \langle u, v\rangle = \int\limits_{Q_0} uv\,dx. \tag{1.3}$$

Integrating by parts in the left hand side of inequality (1.3) and taking into consideration that $v_\delta = 0$ on $\partial Q_0 = X_{d_0} \cup \Gamma_{d_0}$ as well as the ellipticity of the operator A, we get

$$-\varepsilon\langle A v_\delta, v_\delta\rangle + \langle D_3 v_\delta, v_\delta\rangle + \lambda\|v_\delta\|^2 \geqslant \varepsilon\alpha_0\|v_\delta\|^2_{1,Q_0} + \left(\lambda - \frac{1}{2} - C_2\varepsilon\right)\|v_\delta\|^2_{0,Q_0}.$$

Hereinafter, C_j, $j = 1, 2, 3, \ldots$ are positive constants independent of ε.

Estimating the right hand side in (1.3), we get

$$\begin{aligned}|\langle e^{\lambda x_3} f\psi_\delta, v_\delta\rangle| + \varepsilon|\langle A'v, v_\delta\rangle| \leqslant& C_3\left(\frac{1}{2}\|f\|^2_{0,Q_0} + \frac{1}{2}\|v_\delta\|^2_{0,Q_0}\right) + C_4\varepsilon\left[\frac{\varepsilon^{\frac{1}{2}}}{\delta}\|v\|^2_{1,Q_0} + \frac{1}{\varepsilon^{\frac{1}{2}}\delta}\|v\|^2_{0,Q_0}\right] \\ \leqslant& \frac{C_3}{2}\|f\|^2_{0,Q_0} + \frac{C_3}{2}\|v_\delta\|^2_{0,Q_0} + C_5\varepsilon^{\frac{1}{2}-\gamma}\left[\varepsilon\|v\|^2_{1,Q_0} + \|v\|^2_{0,Q_0}\right].\end{aligned}$$

The obtained estimates for the right and left hand sides in inequality (1.3) imply:

$$\varepsilon\alpha_0\|v_\delta\|^2_{1,Q_0} + \left(\lambda - \frac{1}{2} - C_2\varepsilon - \frac{C_3}{2}\right)\|v_\delta\|^2_{0,Q_0} \leqslant \frac{C_3}{2}\|f\|^2_{0,Q_0} + C_5\varepsilon^{\frac{1}{2}-\gamma}\left(\varepsilon\|v\|^2_{1,Q_0} + \|v\|^2_{0,Q_0}\right).$$

Choosing

$$\lambda > \alpha_0 + \frac{1}{2} + C_1\varepsilon + \frac{C_3}{2}$$

and taking into consideration that

$$\|v_\delta\|^2_{0,Q_0} \geqslant \|v\|^2_{0,Q_\delta}, \quad \|v_\delta\|^2_{1,Q_0} \geqslant \|v\|^2_{0,Q_\delta}$$

and that the norm $\|v\|^2_{1,Q_\delta}$ is equivalent to $\|u\|^2_{0,Q_\delta}$, we arrive at inequality (1.1). The proof is complete. □

Corollary. *If*

$$\|f\|^2_{0,Q_0} = O(\varepsilon^k) \quad \textit{and} \quad \varepsilon\|u\|^2_{1,Q_0} + \|u\|^2_{0,Q_0} = O(\varepsilon^m),$$

where $m < k$, then under the assumptions of Theorem 1 we have

$$\varepsilon\|u\|^2_{1,Q_\delta} + \|u\|^2_{0,Q_\delta} = O(\varepsilon^k).$$

Proof. Indeed, applying inequality (1.1) to the domains $Q_{\frac{\delta}{2^n}}$, $n = 1, 2, \ldots$, in finitely many steps we obtain the required estimate. The proof is complete. □

Theorem 1 shows that the construction of asymptotic solution can be localized.

2. External expansion

It follows from the assumption on the order of touching of characteristics and the curve S_0 that the equation of Γ_{d_0} can be transformed to the form

$$d(x_1, x_2) = x_3^2.$$

Assuming this in the domain Q_0, we introduced the variables straightening Γ_{d_0}:

$$z_1 = d(x_1, x_2) - x_3^2, \quad z_2 = x_3, \quad z_3 = s(x_1, x_2), \tag{2.1}$$

where $s(x_1, x_2)$ is a coordinate on S_0, $0 \leqslant s \leqslant s_1$.

The mapping $\varkappa: x \to z$ is a diffeomorphism and at that,

$$\begin{aligned}
&Q_0 \to \omega(0, d_0) = \{z: \quad 0 < z_1 + z_2^2 < d_0, \quad |z_2| < \sqrt{d_0}, \quad 0 \leqslant z_3 \leqslant s_1\},\\
&\Gamma_{d_0} \to \gamma_0 = \{z: \quad z_1 = 0, \quad |z_2| \leqslant \sqrt{d_0}, \quad 0 \leqslant z_3 \leqslant s_1\},\\
&\gamma_0^{\pm} = \{z \in \gamma_0, \quad z_2 \gtrless 0\}.
\end{aligned}$$

If we let $u \circ \varkappa^{-1} = v(z, \varepsilon)$, $(A_\varepsilon u) \circ \varkappa^{-1} = B_\varepsilon v$, then problem (0.1)–(0.2) is rewritten as

$$B_\varepsilon v = \varepsilon B(z, D) v(z, \varepsilon) + B_0(z, D) v(z, \varepsilon) = g(z), \quad z \in \omega(0, d_0), \tag{2.2}$$

$$v|_{\gamma_0} = v(0, z_2, z_3) = 0, \tag{2.3}$$

where $z = (z_1, z_2, z_3)$, $D = (D_1, D_2, D_3)$, $D_j = \frac{\partial}{\partial z_j}$,

$$B(z, D) = \sum_{|\alpha| \leqslant 2} b_\alpha(z) D^\alpha$$

is an elliptic differential operator, $B_0(z, D) = 2z_2 D_1 - D_2$.

A formal asymptotic solutions (FAS) to problem (2.2)–(2.3) is sought as

$$V = \sum_{k=0}^{\infty} \varepsilon^k v_k(z). \tag{2.4}$$

For $v_k(z)$ we get the recurrent system of equations

$$\begin{cases} B_0 v_0 = (2z_2 D_1 - D_2) v_0(z) = g(z), \quad v_0|_{\gamma_0} = 0,\\ B_0 v_k = -B v_{k-1}, \quad v_k|_{\gamma_0^-} = 0. \end{cases} \tag{2.5}$$

The solutions of this system are written explicitly

$$\begin{cases} v_0(z) = \displaystyle\int\limits_{-\sqrt{z_1+z_2^2}}^{z_2} g_0(z_1 + z_2^2 - t^2, t, z_3)\, dt,\\ v_k(z) = -\displaystyle\int\limits_{\sqrt{z_1+z_2^2}}^{z_2} B v_{k-1}\, dt, \quad k = 1, 2, \dots \end{cases} \tag{2.6}$$

By (2.6) we see that $v_0(z)$ is continuous as $z \in \overline{\omega}(0, d_0)$, by its derivatives has in z_1, z_2 have singularities as $r = \sqrt{z_1 + z_2^2} \to 0$. Let us study the asymptotics of $v_k(z)$ as $r = \sqrt{z_1 + z_2^2} \to 0$.

Lemma 2.1. *The functions $v_k(z)$, $k = 0, 1, 2, \dots$, can be represented as*

$$v_k(z) = r^{1-3k} \varphi_k(r, \theta, z_3), \tag{2.7}$$

where

$$r = \sqrt{z_1 + z_2^2}, \quad \theta = \frac{z_2}{r}, \quad \Pi_{d_0} = [0, \sqrt{d_0}] \times [-1, 0) \times (0, 1) \times [0, s_1], \quad \varphi_k(r, \theta, z_3) \in C^\infty(\Pi_{d_0}),$$

and as $r \to 0$, they have the asymptotics

$$v_k(z) \sim r^{1-3k} \sum_{m=0}^{\infty} \varphi_{k,m}(\theta, z_3) r^m, \tag{2.8}$$

where $\varphi_{k,m}(\theta, z_3) \in C^\infty(I_0 \times [0, s_1])$, $I_0 = [-1, 1] \setminus \{0\}$.

Proof. By induction, as $k = 0$,

$$v_0(z) = -\int_{-r}^{z_2} g(r^2 - t^2, t, z_3)\, dt = -r \int_{-1}^{\theta} g(r^2(1 - \xi^2), r\xi, z_3)\, d\xi = r\varphi_0(r, \theta, z_3),$$

where

$$\varphi_0 = -\int_{-1}^{\theta} g(r^2(1 - \xi^2), r\xi, z_3)\, d\xi \in C^\infty(\Pi_{d_0}).$$

For integer $p < 0$, by V_p we denote the class of functions $\tilde{v}_p(z)$, which can be represented as $\tilde{v}_p(z) = r^p \varphi_p(r, \theta, z_3)$, where $\varphi_p(r, \theta, z_3) \in C^\infty(\Pi_{d_0})$. The functions in V_p possess the following properties:

1° $\tilde{v}_p(z) \in V_p \to D_1\tilde{v}_p \in V_{p-2}$, $\quad D_2\tilde{v}_p \in V_{p-2}$, $\quad D_3\tilde{v}_p \in V_p$;

2° $V_{p'} \subset V_p$ as $p' > p$.

Let $v_m(z) \in V_{1-3m}$ as $1 \leqslant m \leqslant k - 1$. We are going to prove that $v_k(z) \in V_{1-3k}$:

$$v_k(z) = \int_{r}^{z_2} B v_{k-1}\, dt = \sum_{|\alpha| \leqslant 2} \int_{-r}^{z_2} b_\alpha(r^2 - t^2, t, z_3) D^\alpha v_{k-1}\, dt$$

$$= \sum_{|\alpha| \leqslant 2} \int_{-r}^{z_3} b_\alpha(r^2 - t^2, t, z_3) r^{-3k} \tilde{\varphi}_{k-1}\left(r, \frac{t}{r}, z_3\right) dt = r^{1-3k}\varphi_k(r, \theta, z_3),$$

$$\varphi_k = \sum_{|\alpha| \leqslant 2} \int_{-1}^{\theta} b_\alpha(r^2(1 - \xi^2), r\xi, z_3) \tilde{\varphi}_{k-1}(r, t, z_3)\, d\xi \in C^\infty(\Pi_{d_0}).$$

Asymptotics (2.8) follows (2.7) by expanding $\varphi_k(r, \theta, z_1)$ into the Taylor series as $r = 0$. The proof is complete. □

Corollary. *On γ_0^+, the functions $v_k(z)$ take the values*

$$v_k(z)|_{\gamma_0^+} = v_k(0, z_2, z_3) = z_2^{1-3k} \varphi_k^+(z_2, z_3), \quad z_2 > 0, \tag{2.9}$$

where $\varphi_k^+(z_2, z_3)$ are smooth functions and as $z_2 \to +0$, for $v_k(z)|_{\gamma_0^+}$ we have the asymptotic expansions

$$v_k(z)|_{\gamma_0^+} \sim z_2^{1-3k} \sum_{m=0}^{\infty} \varphi_{k,m}^+(z_3) z_2^m. \tag{2.10}$$

The errors on γ^+ can be removed by a regular boundary layer:

$$\hat{Y}(t, z_2, z_3, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t, z_2, z_3), \tag{2.11}$$

where $t = \varepsilon^{-1} z_1$, $y_k(t, z_2, z_3) \to 0$ as $t \to \infty$.

In order to write out equations for determining $y_k(t, z_2, z_3)$, we need to split the operator B_ε in powers of ε. We represent B_ε as

$$B_\varepsilon = \varepsilon^{-1}[b_{2,0,0}(\varepsilon t, z') D_t^2 + 2 z_3 D_2] + (q_1(\varepsilon t, z', D') - D_2) + \varepsilon q_2(\varepsilon t, z', D'),$$

where $z' = (z_2, z_3)$, $D' = (D_2, D_3)$, $q_1(\varepsilon t, z', D')$ is a first order differential operator, $q_2(\varepsilon t, z', D')$ is a second order differential operator. Then we expand the coefficients of B_ε into the Taylor series at $\varepsilon = 0$ and we obtain

$$B_\varepsilon = \varepsilon^{-1} M_0 + \sum_{k=0}^{\infty} \varepsilon^k M_{k+1}, \tag{2.12}$$

where

$$M_0 = b_{2,0,0}(0, z') D_t^2 + 2z_2 D_t, \qquad M_1 = q_1(0, z', D') D_t - D_2,$$
$$M_k = t^k b_{2,0,0}(0, z') D_t^2 + t^{k-1} q_1^{(k-1)}(0, z', D') D_t + t^{k-2} q_2(0, z', D').$$

Employing (2.11), (2.12) for $y_k(t, z')$, we obtain the system of ordinary differential equations in the variable t:

$$\begin{cases} M_0 y_0 = \left(\dfrac{1}{\lambda} D_t^2 + 2z_2 D_t\right) y_0 = 0, \\ y_0(0, z') = -v_0(0, z'), \quad z_3 > 0, \quad y_0 \to 0, \quad t \to +\infty, \\ M_0 y_k = \displaystyle\sum_{j=1}^{k} M_j y_{k-j}, \\ y_k(0, z') = -v_0(0, z'), \quad z_2 > 0, \quad y_k \to 0, \quad t \to +\infty, \end{cases} \tag{2.13}$$

where $\frac{1}{\lambda} = b_{2,0,0}(0, z') > 0$.

The solutions to this system are written explicitly

$$\begin{cases} y_0(t, z') = -v_0(0, z') e^{-2\lambda z_2 t}, \\ y_k(t, z') = e^{-2\lambda z_2 t} P_{2k}(t, z'), \end{cases} \tag{2.14}$$

where $P_{2k}(t, z')$ are polynomials in t of degree $2k$.

Let us find out the behavior of $y_k(t, z_2, z_3)$ as $z_2 \to 0$.

Lemma 2.2. *The functions $y_k(t, z_2, z_3)$ are represented as*

$$y_k(t, z_2, z_3) = z_2^{1-3k} e^{-\lambda\sigma} P_{2k}(\sigma, z_2, z_3), \tag{2.15}$$

where $\sigma = 2z_2 t$, $P_{2k}(\sigma, z_2, z_3)$ are polynomials in σ of order $2k$, whose coefficients are smooth functions of (z_2, z_3). As $z_2 \to +0$, the functions $y_k(t, z_2, z_3)$ are expanded into the asymptotic series

$$y_k(t, z_2, z_3) \sim z_2^{1-3k} e^{-\lambda_0 \sigma} \sum_{m=0}^{\infty} P_{2k+m}(\sigma, z_3) z_2^m, \tag{2.16}$$

where $\lambda_0 = \frac{1}{b_{2,0,0}(0,0,z_3)}$, $P_{2k+m}(\sigma, z_3)$ are polynomials in σ of order $2k+m$ with smooth coefficients in $z_3 \in [0, s_1]$.

Proof. By induction, as $k = 0$,

$$y_0(t, z') = -e^{-2\lambda z_3 t} v_0(0, z_2, z_3) = z_3 e^{-\lambda\sigma} Q_0(z_2, z_3), \quad z_3 > 0,$$

where $Q_0(z_2, z_3) = -\varphi_0^+(z_2, z_3)$ by Corollary of Lemma 2.1. By $Y_{p,m}$ we denote the class of functions of form

$$y_{p,m}(t, z_2, z_3) = z_3^{1-p} e^{-\lambda\sigma} P_m(\sigma, z_2, z_3),$$

where $p > 1$ is integer, $P_m(\sigma, z_2, z_3)$ are polynomials of order m, whose coefficients are smooth functions of (z_2, z_3). The following properties of $Y_{p,m}$ hold:

1° $Y_{p',m'} \subset Y_{p,m}$ as $p' > p$, $m' \leqslant m$,

2° if $y_{p,m}(t, z_2, z_3) \in Y_{p,m}$, then $D_t y_{p,m} \in Y_{p+1,m}$, $D_2 y_{p,m} \in Y_{p,m+1}$, $D_3 y_{p,m} \in Y_{p-1,m+1}$, $t^j y_{p,m} \in Y_{p-j,m+j}$.

Assuming that $y_j(t,z') \in Y_{1-3j,2j}$ as $1 \leqslant j \leqslant k-1$, let us show that $y_k(t,z') \in Y_{1-3k,2k}$. We let $y_j(t,z') = \tilde{y}_j(\sigma,z')$, $0 \leqslant j \leqslant k$, then the equation for $\tilde{y}_k(\sigma,z')$ becomes

$$\left(\frac{1}{\lambda}D_\sigma^2 + D_\sigma\right)\tilde{y}_k = -z_2^{-2}\sum_{j=1}^{k} M_j\tilde{y}_{k-j}, \quad \tilde{y}_k(0,z') = -z_2^{1-3k}\tilde{\varphi}_k(z'),$$
$$\tilde{y}_k(\sigma,z') \to 0, \quad \sigma \to +\infty.$$

Employing the assumption of the induction, the properties of $Y_{1-3j,2j}$, $0 \leqslant j \leqslant k-1$, and the form of the operator M_j, see (2.12), one can show easily that $z_2^{-2}\sum\limits_{j=1}^{k} M_j\tilde{y}_{k-j} \in Y_{1-3k,2k-1}$, which implies that the problem for $\tilde{y}_k$ has a solution of the form:

$$\tilde{y}_k = z_2^{1-3k}P_{2k}(\sigma,z')e^{-\lambda\sigma}$$

and thus, we have proved (2.15).

Representing y_k as

$$y_k = z_2^{1-3k}e^{-\lambda_0\sigma}[e^{(\lambda_0-\lambda)\sigma}P_k(\sigma,z_2,z_3)]$$

and expanding the expression in the square brackets into the Taylor series as $z_2 = 0$, we arrive at (2.16). The proof is complete. □

We consider n-th partial sums of series (2.4) and (2.11):

$$V_n(z,\varepsilon) = \sum_{k=0}^{n} v_k(z)\varepsilon^k, \quad Y_n(t,z_2,z_3,\varepsilon) = \sum_{k=0}^{n} y_k(t,z_2,z_3)\varepsilon^k$$

and we let

$$U_n(z,\varepsilon) = V_n(z,\varepsilon) + Y_n(t,z_2,z_3,\varepsilon)\chi\left(\frac{z_2}{\varepsilon^{\frac{1}{3}}}\right), \tag{2.17}$$

where

$$\chi(t) = \begin{cases} 1, & t \geqslant 2 \\ 0, & t \leqslant 1 \end{cases}$$

is a smooth cut-off function.

Lemma 2.3. *The function $U_n(z,\varepsilon)$ is formal asymptotic solution to problem (2.2), (2.3) in the domain*

$$\overline{\omega}(\varepsilon^\beta, d_0) = \{z : \quad \varepsilon^\beta \leqslant r \leqslant d_0, \quad 0 \leqslant z_3 \leqslant s_1\}$$

up to $O(\varepsilon^{(1-3\beta)n})$, where $0 < \beta < \frac{1}{3}$.

Proof. By (2.5) and (2.13), in the domain $\overline{\omega}(\varepsilon^\beta, d_0)$ we have

$$B_\varepsilon U_n = g(z) + R_n^\beta(z,\varepsilon), \quad U_n(0,z_2,z_3,\varepsilon) = 0, \quad |z_2| \geqslant \varepsilon^\beta,$$

where

$$R_n(z,\varepsilon) = \varepsilon^{n+1}Bv_n + \varepsilon^n\sum_{k=1}^{n}\varepsilon^k\left(\sum_{j=k}^{n} M_j y_{n+k-j}\right) + \left(B_\varepsilon - \sum_{j=0}^{n}\varepsilon^{j-1}M_j\right)Y_n.$$

It follows from Lemmata 2.1 and 2.2 that as $r \geqslant \varepsilon^\beta$, $z_2 \geqslant \varepsilon^\beta$, $0 < \beta < \frac{1}{3}$, we have

$$|R_n^\beta(z,\varepsilon)| \leqslant C_n\left[\left(\frac{\varepsilon}{r^3}\right)^n + \left(\frac{\varepsilon}{z_2^3}\right)^n\right] \leqslant 2C_n\varepsilon^{(1-3\beta)n},$$

where C_n is a constant independent of ε. The proof is complete. □

3. External expansion

In order to construct the formal asymptotic solution in the vicinity of the curve S_0, we introduce the rescaled variables:

$$z_1 = \varepsilon^{\frac{2}{3}}\xi, \quad z_2 = \varepsilon^{\frac{1}{3}}\tau, \quad z_3 = z_3. \tag{3.1}$$

Let $\varkappa_\varepsilon : z \to (\xi, \tau, z_3)$ and

$$v \circ \varkappa_\varepsilon = w(\xi, \tau, z_3, \varepsilon), \quad (B_\varepsilon v) \circ \varkappa_\varepsilon = \mathcal{L}_\varepsilon w, \quad g \circ \varkappa_\varepsilon = h(\xi, \tau, z_3, \varepsilon). \tag{3.2}$$

We rewrite problem (2.3),(2.4) in the variables (ξ, τ, z_3):

$$\mathcal{L}_\varepsilon w = h, \quad w(0, \tau, z_3, \varepsilon) = 0. \tag{3.3}$$

Here the splitting of the operator $\mathcal{L}_\varepsilon$ into the powers of ε reads as

$$\mathcal{L}_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^{\frac{k-1}{3}} L_k, \tag{3.4}$$

where

$$\begin{aligned} L_0 =& \lambda_0^{-1} D_\xi^2 + 2\tau D_\xi - D_\tau, \\ L_k =& \frac{1}{k!} D_\mu^k b_{2,0,0}(\mu^2\xi, \mu\tau, z_3)|_{\mu=0} D_\xi^2 + \frac{1}{(k+1)!} D_\mu^{k-1} b_{1,1,0}(\mu^2\xi, \mu\tau, z_3)|_{\mu=0} D_\xi D_\tau + \ldots \end{aligned}$$

are second order differential operators, whose coefficients are quasi-homogeneous polynomials in ξ, τ, and the coefficients as the powers of ξ, τ are smooth functions of z_2.

We seek the formal asymptotic solution to problem (3.3) as

$$W = \sum_{k=0}^{\infty} \varepsilon^{\frac{k+1}{3}} w_k(\xi, \tau, z_3). \tag{3.5}$$

Expanding $h(\xi, \tau, z_3, \varepsilon)$ into the powers of ε, we find that

$$h = \sum_{k=0}^{\infty} h_k(\xi, \tau, z_3) \varepsilon^{\frac{k}{3}},$$

where

$$h_k(\xi, \tau, z_3) = \frac{1}{k!} D_\mu^k g(\mu^2\xi, \mu\tau, z_3)|_{\mu=0}.$$

Then in the standard way we obtain a system of parabolic equations for finding $w_k(\xi, \tau, z_3)$ in the domain

$$\mathbb{R}_+^2 \times [0, s_1] = \{0 < \xi < \infty, \, |\tau| < \infty, \, 0 \leqslant z_3 \leqslant s_1\}.$$

This system is

$$\begin{cases} L_0 w_0 = (\lambda_0^{-1} D_\xi^2 + 2\tau D_\xi - D_\tau) w_0 = h_0, \\ L_0 w_k + \displaystyle\sum_{j=1}^{k} L_j w_{k-j} = h_k, \quad k = 1, 2, \ldots \end{cases} \tag{3.6}$$

subject to the boundary conditions

$$w_k(0, \tau, z_3) = 0, \quad k = 0, 1, \ldots \tag{3.7}$$

To find out additional conditions for solutions to (3.6)—(3.7), we employ the matching conditions [2].

We denote

$$V_n^{(3n)} = \sum_{k=0}^{n} v_k^{(3n)}(z) \varepsilon^k, \quad Y_n^{(3n)} = \sum_{k=0}^{n} y_k^{(3n)}(t, z_2, z_3), \tag{3.8}$$

where $v_k^{(3n)}$, $y_k^{(3n)}$ are $3n$-th partial sums of asymptotic series (2.4), (2.16) for the functions $v_k(z)$, $y_k(t, z_2, z_3)$, respectively. Let

$$U_n^{(3n)}(z, \varepsilon) = V_n^{(3n)}(z, \varepsilon) + Y_n^{(3n)}(t, z_2, z_3)\chi\left(\frac{z_2}{\varepsilon^{\frac{1}{3}}}\right), \tag{3.9}$$

where $\chi(\tau)$ is a smooth cut-off function:

$$\chi(\tau) = \begin{cases} 1, & \tau \geqslant 2 \\ 0, & \tau \leqslant 1 \end{cases}$$

and $G_n(z, \varepsilon) = U_n(z, \varepsilon) - U_n^{(3n)}(z, \varepsilon)$.

In view of expansions (2.4), (2.16), Lemmata 2.1 and 2.2 imply that the estimates

$$G_n(z, \varepsilon) = O(\varepsilon^{\mu(n+1)}), \quad B_\varepsilon G_n(z, \varepsilon) = O(\varepsilon^{\mu n})$$

in the domain

$$\omega(\varepsilon^\beta, \varepsilon^\mu) = \{z | \, \varepsilon^\beta \leqslant r \leqslant \varepsilon^\mu\},$$

where $0 < \mu < \beta < \frac{1}{3}$. These estimates and Lemma 2.3 imply the estimate

$$B_\varepsilon U_n^{(3n)} - g(z) = O(\varepsilon^{\mu_0 n}), \tag{3.10}$$

in the domain $\omega(\varepsilon^\beta, \varepsilon^\mu)$, where $\mu_0 = \min(\mu, 1 - 3\beta)$.

We rewrite (3.9) in the variables (ξ, τ, z_3):

$$U_n^{(3n)} \circ \varkappa_\varepsilon = W_n^{(3n)}(\xi, \tau, z_3, \varepsilon). \tag{3.11}$$

Here

$$W_n^{(3n)} = \sum_{k=0}^{\infty} w_k^{(n)}(\xi, \tau, z_3)\varepsilon^{\frac{k+1}{3}},$$

$$w_k^{(n)} = \sum_{m=0}^{n} \rho^{k+1-3m}\varphi_{k,m}(\theta_1, z_3) + e^{-\lambda_0\sigma_1}\left(\sum_{m=0}^{n} \tau^{k+1-3m}P_{2k+m}(\sigma_1, z_3)\right)\chi(\tau),$$

where $\rho = \sqrt{\xi + \tau^2}$, $\theta_1 = \frac{\tau}{\rho}$, $\sigma_1 = 2\xi\tau$. At that, $w_k^{(n)}|_{\xi=0} = 0$ as $\tau \neq 0$, which is implied by the explicit formulae and Lemma 2.3.

Formula (3.11) is exactly the matching condition of external and internal expansions. This means that the solutions to system of equations (3.6), (3.7) should be sought in the class of functions growing as $\rho \to +\infty$ not faster than a power of ρ and having the asymptotics $w_k \sim w_k^{(n)}$ as $\rho \to \infty$.

We rewrite (3.10) in the variables (ξ, τ, z_3):

$$\begin{aligned} B_\varepsilon U_n^{(3n)} \circ \varkappa_\varepsilon - g(z) \circ \varkappa_\varepsilon = &\mathcal{L}_\varepsilon W_{3n}^{(n)} - h(\xi, \tau, \varepsilon) \\ &= \left(L_0 w_0^{(n)} - h_0\right) + \varepsilon^{\frac{1}{3}}\left(L_0 w_1^{(n)} + L_1 w_0^{(n)} - h_1\right) + \cdots \\ &+ \varepsilon^{\frac{k}{3}}\left(L_0 w_k^{(n)} + \sum_{j=1}^{k} L_j w_{k-j}^{(n)} - h_k\right) + \cdots = O(\varepsilon^{\mu_0 n}). \end{aligned} \tag{3.12}$$

It follows from (3.12) that under the mapping

$$\varkappa_\varepsilon : \omega(\varepsilon^\beta, \varepsilon^\mu) \to \omega'(\varepsilon^{\beta-\frac{1}{3}}, \varepsilon^{\mu-\frac{1}{3}}) = \{(\varepsilon, \tau, z_3)| \quad \varepsilon^{\beta-\frac{1}{3}} < \rho < \varepsilon^{\mu-\frac{1}{3}}, z_3 \in [0, s_1]\}$$

we have

$$L_0 w_0^{(n)} - h_0 = O(\varepsilon^{\mu_0 n}), \quad L_0 w_k^{(n)} + \sum_{j=1}^{k} L_j w_{k-j}^{(n)} - h_k = O(\varepsilon^{\mu_0 n - \frac{k}{3}}), \quad k = 1, 2, \ldots, k_1. \tag{3.13}$$

in the domain ω'. As $\rho \to \infty$, these relations are equivalent to

$$L_0 w_0^{(n)} - h_0 = O(\rho^{-\mu_1 n}),\ L_0 w_k^{(n)} + \sum_{j=1}^{k} L_j w_{k-j}^{(n)} - h_k = O(\rho^{-\mu_1 n + \mu_2 k}), \quad k = 1, 2, \dots, k_1. \tag{3.14}$$

Letting $n \to \infty$ in (3.14), we obtain

$$L_0 \widehat{w}_0 - h_0 = O(\rho^{-\infty}), \quad L_0 \widehat{w}_k + \sum_{j=1}^{k} L_j \widehat{w}_{k-j} - h_k = O(\rho^{-\infty}), \quad k = 1, 2, \dots, \tag{3.15}$$

where

$$\widehat{w}_k = \sum_{j=0}^{\infty} \rho^{k+1-3j} \varphi_{k,j}(\theta_1, z_3) + \chi(\varepsilon) e^{-\lambda_0 \sigma_1} \sum_{j=0}^{\infty} \tau^{k+1-3j} P_{2k+j}(\sigma_1, z_3) \tag{3.16}$$

are formal asymptotic series.

Lemma 3.1. *There exist the unique solutions $w_k(\xi, \tau, z_3) \in C^\infty(\mathbb{R}^2_+ \times [0, s_1])$ to system of equations (3.6) subject to boundary conditions (3.7). As $\rho \to \infty$, these solutions are expanded into asymptotic series (3.16): $w_k(\xi, \tau, z_3) \sim \widehat{w}_k(\xi, \tau, z_3)$.*

Proof. We denote by $w_{a,k}(\xi, \tau, z_3)$ smooth functions, which are expanded into asymptotic series (3.16) as $\rho \to \infty$ and which vanish as $\xi = 0$: $w_{a,k} \sim \widehat{w}_k$, $w_{a,k}(0, \tau, z_3) = 0$. Such functions are known to exist. We let

$$w_k(\xi, \tau, z_3) = w_{a,k}(\xi, \tau, z_3) + r_k(\xi, \tau, z_3), \quad k = 0, 1, \dots$$

By (3.6) and (3.7) we get

$$L_0 r_0 = \psi_0, \quad L_0 r_k + \sum_{j=1}^{k} L_j r_{k-j} = \psi_k, \quad r_0(0, \tau, z_3) = r_k(0, \tau, z_3) = 0, \quad k = 1, 2, \dots,$$

where $\psi_0 = h_0 - L_0 w_{a,0}$, $\psi_k = \sum\limits_{j=0}^{k} L_j r_{k-j}$ are smooth functions decaying faster each power of ρ^{-1} with all their derivatives. We denote the class of such functions by $S(\overline{\mathbb{R}^2_+} \times [0, s_1])$.

We consider the problem

$$L_0 R_0 = \psi(\xi, \tau, z_3), \quad R_0(0, \tau, z_3) = 0,$$

where $\psi \in S(\overline{\mathbb{R}^2_+} \times [0, s_1])$. In [2] there was proved the unique solvability of this problem in the class S in the case, when L_0 and ψ are independent of z_3. It is obvious that this result is true also in the case of a smooth dependence of L_0 and ψ of z_3. This implies the statement of the lemma as $k = 0$. After that, the proof is completed by the induction in k. $\square$

We consider $3n$-th partial sum of the just determined formal asymptotic solutions (3.5):

$$W_{3n} = \sum_{k=0}^{3n} \varepsilon^{\frac{k+1}{3}} w_k(\xi, \tau, z_3). \tag{3.17}$$

Lemma 3.2. *Series (3.17) is a formal asymptotic solution to problem (2.2),(2.3) in the domain*

$$\overline{\omega}(0, \varepsilon^\mu) = \{z |\ 0 \leqslant r \leqslant \varepsilon^\mu,\ z_3 \in [0, s_1]\}$$

up to $O(\varepsilon^{\mu n})$, where $0 < \mu < \frac{1}{3}$.

Proof. By (3.6) we have

$$B_\varepsilon W_{3n} = \mathcal{L}_\varepsilon W_{3n} = h + \left[R_{1,n}(z,\varepsilon) + R_{2,n}(z,\varepsilon) + R_{3,n}(z,\varepsilon)\right],$$

where

$$R_{1,n}(z,\varepsilon) = \left(\sum_{k=0}^{3n} \varepsilon^{\frac{k}{3}} h_k - h\right),$$

$$R_{2,n}(z,\varepsilon) = \varepsilon^n \sum_{k=1}^{3n} \varepsilon^{\frac{k}{3}} \left(\sum_{j=k}^{3n} L_j w_{3n+k-j}\right),$$

$$R_{3,n}(z,\varepsilon) = \left(\mathcal{L}_\varepsilon - \sum_{k=0}^{3n} \varepsilon^{\frac{k-1}{3}} L_k\right) W_{3n}.$$

By employing the asymptotic expansions for $w_k(\xi,\tau,z_3)$ and the form of the operators L_k, it is easy to see that each term in the square brackets does not exceed $C_n\varepsilon^n\rho^{3n} = C_n r^n \leqslant C_n\varepsilon^{\mu n}$, where the constant C_n is independent of ε. This implies that

$$B_\varepsilon W_{3n} = g(z) + R_n^\mu(z,\varepsilon), \quad z \in \overline{\omega}(0,\varepsilon^\mu), \tag{3.18}$$

in the domain $\overline{\omega}(0,\varepsilon^\mu)$, where $R_n^\mu(z,\varepsilon) = O(\varepsilon^{\mu n})$. □

We introduce the composed expansion [2]:

$$V_{a,n} = U_n(z,\varepsilon) + W_{3n}(\xi,\tau,z_3,\varepsilon) - U_n^{(3n)}(z,\varepsilon). \tag{3.19}$$

Theorem 2. *Composed asymptotic expansion (3.19) is the uniform asymptotic expansion to problem (2.2)–(2.3) up to $O(\varepsilon^{\mu_0 n})$ in the domain $\overline{\omega}(0,d_0)$.*

Proof. By Lemmata 2.3, 3.2 and formula (3.10) we have

$$B_\varepsilon(v - V_{a,n}) = R_n(z,\varepsilon) = \begin{cases} R_n^\beta(z,\varepsilon), & z \in \overline{\omega}(\varepsilon^\beta, d_0) \\ R_n^\mu(z,\varepsilon), & z \in \overline{\omega}(\varepsilon^\mu, d_0) \\ -R_n^\beta(z,\varepsilon) + R_n^0(z,\varepsilon), & z \in \overline{\omega}(\varepsilon^\beta, \varepsilon^\mu) \end{cases}$$

where $R_n(z,\varepsilon) = O(\varepsilon^{\mu_0 n})$. It follows from Theorem 1 that

$$\varepsilon\|v - V_{a,n}\|^2_{1,\overline{\omega}(0,d_0)} + \|v - V_{a,n}\|^2_{0,\overline{\omega}(0,d_0)} \leqslant C\varepsilon^{\mu_0 n}.$$

□

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