ON A HILBERT SPACE OF ENTIRE FUNCTIONS

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Abstract. We consider the Hilbert space $F^2_\varphi$ of entire functions of $n$ variables constructed by means of a convex function $\varphi$ in $\mathbb{C}^n$ depending on the absolute value of the variable and growing at infinity faster than $a|z|$ for each $a > 0$. We study the problem on describing the dual space in terms of the Laplace transform of the functionals. Under certain conditions for the weight function $\varphi$, we obtain the description of the Laplace transform of linear continuous functionals on $F^2_\varphi$. The proof of the main result is based on using new properties of Young-Fenchel transform and one result on the asymptotics of the multi-dimensional Laplace integral established by R.A. Bashmakov, K.P. Isaev, R.S. Yulmukhametov.

Keywords: Hilbert space, Laplace transform, entire functions, convex functions, Young-Fenchel transform.

Mathematics Subject Classification: 32A15, 42B10, 46E10, 46F05, 42A38

1. Introduction

1.1. Problem. Let $H(C^n)$ be the space of entire functions in $C^n$, $d\mu_n$ be the Lebesgue measure in $C^n$ and for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ ($C^n$) we define $u := (|u_1|, \ldots, |u_n|)$.

We denote by $\mathcal{V}(\mathbb{R}^n)$ the set of all convex functions $g$ in $\mathbb{R}^n$ such that
1) $g(x_1, \ldots, x_n) = g(|x_1|, \ldots, |x_n|), \ (x_1, \ldots, x_n) \in \mathbb{R}^n$;
2) the restriction $g$ on $[0, \infty)^n$ is non-decaying in each variable;
3) $\lim_{x \to \infty} \frac{g(x)}{||x||} = +\infty$; $||x||$ is the Euclidean norm of a point $x \in \mathbb{R}^n$.

To each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ we associate the Hilbert space

$$F^2_\varphi = \left\{ f \in H(C^n) : \|f\|_\varphi = \left( \int_{C^n} |f(z)|^2 e^{-2\varphi(|z|)} \ d\mu_n(z) \right)^{\frac{1}{2}} < \infty \right\}$$

with the scalar product

$$(f, g)_\varphi = \int_{C^n} f(z)g(z)e^{-2\varphi(|z|)} \ d\mu_n(z), \ f, g \in F^2_\varphi.$$ 

If $\varphi(x) = \frac{||x||^2}{2}$, then $F^2_\varphi$ is the Fock space.

It is obvious that for each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and each $\lambda \in C^n$, the function $f_\lambda(z) = e^{(\lambda, z)}$ belongs to $F^2_\varphi$. This is why for each linear continuous functional $S$ on the space $F^2_\varphi$, the function

$$\hat{S}(\lambda) = S(e^{(\lambda, z)}), \ \lambda \in C^n,$$

is well defined in $C^n$; this function is the Laplace transform of the functional $S$. It is easy to see that $\hat{S}$ is an entire function.

By $(F^2_\varphi)^*$ we denote the dual space for $F^2_\varphi$. 

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The work is financially supported by the Russian Foundation for Basic Researches (grant no. 15-01-01661) and the Program of the Presidium of RAS (project “Complex Analysis and Functional Equations”).

The aim of the work is to find the conditions for $\varphi \in \mathcal{V}(\mathbb{R}^n)$, under which the space $(F^2_\varphi)^*$ of the Laplace transforms of the linear continuous functionals on $F^2_\varphi$ can be described as $F^2_{\varphi^*}$.

If $\varphi(x) = \frac{\|x\|^2}{2}$, then $(F^2_\varphi)^* = F^2_{\varphi}$. Indeed, in this case the problem on describing the space $(F^2_\varphi)^*$ in terms of the Laplace transform of the functionals is easily solved thanks to the classical representation: for each $f \in F^2_\varphi$,

$$f(\lambda) = \pi^{-n} \int_{\mathbb{C}^n} f(z) e^{(\lambda, z) - \|z\|^2} d\mu_n(z), \quad \lambda \in \mathbb{C}^n.$$  

If the function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ is radial, the mentioned problem was solved by V.V. Napalkov and S.V. Popenov [5], [6].

1.2. Notations and definitions. For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n(\mathbb{C}^n)$ we let $\langle u, v \rangle := u_1v_1 + \cdots + u_nv_n$, $\|u\|$ is the Euclidean norm of $u$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ we denote the length of the multi-index $\alpha$, $\bar{\alpha} := (\alpha_1 + 1, \ldots, \alpha_n + 1)$, and we denote $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$,

$$D^\alpha := \frac{\partial^{1|\cdots|n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$  

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, $\varphi \in \mathcal{V}(\mathbb{R}^n)$, we define

$$c_\alpha(\varphi) := \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\text{abs } z)} d\mu_n(z).$$

For a function $u$ with a domain containing the set $(0, \infty)^n$, we define a function $u[e]$ in $\mathbb{R}^n$ by the rule:

$$u[e](x) = u(e^{x_1}, \ldots, e^{x_n}), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$  

By $\mathcal{B}(\mathbb{R}^n)$ we denote the set of all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition

$$\lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty.$$  

The Young-Fenchel transform of the function $u : \mathbb{R}^n \to [0, +\infty]$ is the function $u^* : \mathbb{R}^n \to [0, +\infty]$ defined by the formula

$$u^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - u(y)), \quad x \in \mathbb{R}^n.$$  

If $E$ is a convex domain in $\mathbb{R}^n$, $h$ is a convex set in $E$, $\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}$, $p > 0$, then

$$D^h_y(p) := \{x \in E : h(x) + h^*(y) - \langle x, y \rangle \leq p\}, \quad y \in \tilde{E}.$$  

By $V(D)$ we denote the $n$-dimensional volume of a set $D \subset \mathbb{R}^n$.

1.3. Main result.

Theorem. Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and for some $K > 0$ and each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ the inequalities

$$\frac{1}{K} \leq V(D^\alpha[e] \left(\frac{1}{2}\right)) V(D^{\varphi^*[e]} \left(\frac{1}{2}\right)) \prod_{j=1}^n \alpha_j \leq K$$

hold. Then the mapping $\mathcal{L} : S \in (F^2_\varphi)^* \to \dot{S}$ makes an isomorphism between the spaces $(F^2_\varphi)^*$ and $F^2_{\varphi^*}$.

The proof of Theorem in Subsection 3.2 is based on new properties of Young-Fenchel transform, see Subsection 2.1, and one result on the asymptotics of the multi-dimensional Laplace integral in work [9], see Subsection 2.2.
2. Auxiliary data and results

2.1. On some properties of Young-Fenchel transform. It is easy to confirm that the following statement holds.

**Proposition 1.** Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then $(u[e])^*(x) > -\infty$ as $x \in \mathbb{R}^n$, $(u[e])^*(x) = +\infty$ as $x \notin [0, \infty)^n$ and $(u[e])^*(x) < +\infty$ as $x \in [0, \infty)^n$.

We note that the last statement of Proposition 1 is implied, for instance, by the fact that for each $M > 0$ there exists a constant $A > 0$ such that

$$(u[e])^*(x) \leq \sum_{1 \leq j \leq n; x_j \neq 0} (x_j \ln \frac{x_j}{M} - x_j) + A, \quad x \in [0, \infty)^n.$$  

**Proposition 2.** Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\lim_{x \to +\infty, \|x\| \to \infty} \frac{(u[e])^*(x)}\|x\| = +\infty.$$  

**Proof.** For each $x \in [0, \infty)^n$ and $t \in \mathbb{R}^n$ we have

$$(u[e])^*(x) \geq \langle x, t \rangle - (u[e])(t).$$

Employing this inequality, we obtain that for each $M > 0$

$$(u[e])^*(x) \geq M\|x\| - u[e]\left(\frac{Mx}{\|x\|}\right), \quad x \in [0, \infty)^n \setminus \{0\}.$$  

This completes the proof. \qed

The next three statements were proved in work \cite{1}, see there Lemma 6, Proposition 3, Proposition 4.

**Proposition 3.** Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then

$$(u[e])^*(x) + (u^*[e])^*(x) \leq \sum_{1 \leq j \leq n; x_j \neq 0} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) \leq 0.$$  

**Proposition 4.** Let $u \in \mathcal{B}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ be a convex function. Then

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in (0, \infty)^n.$$  

**Proposition 5.** Let $u \in \mathcal{V}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ be a convex function. Then

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{1 \leq j \leq n; x_j \neq 0} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$  

Propositions 4 and 5 can be strengthened by employing the results by D. Azagra \cite{2}, \cite{3}. He proved the following theorem.

**Theorem A.** Let $U \subseteq \mathbb{R}^n$ be an open convex set. For each convex function $f : U \to \mathbb{R}$ and each $\varepsilon > 0$ there exists a real analytic convex function $g : U \to \mathbb{R}$ such that

$$f(x) - \varepsilon \leq g(x) \leq f(x), \quad x \in U.$$  

Thus, the following corollary hold \cite{3}. 

Corollary A. Let $U \subseteq \mathbb{R}^n$ be an open convex set. For each convex function $f : U \to \mathbb{R}$ and each $\varepsilon > 0$ there exists an infinitely differentiable convex function $g : U \to \mathbb{R}$ such that
\[ f(x) - \varepsilon \leq g(x) \leq f(x), \quad x \in U. \]

Employing Proposition 6 and Corollary A, we easily confirm the following statement.

Proposition 6. Let $u \in \mathcal{B}(\mathbb{R}^n)$ be a convex function. Then
\[
(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in (0, \infty)^n.
\]

Moreover, the following proposition is true.

Proposition 7. Let $u \in \mathcal{V}(\mathbb{R}^n)$ be a convex function. Then
\[
(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n \setminus \{0\};
\]
\[
(u[e])^*(0) + (u^*[e])^*(0) = 0.
\]

Proof. According Proposition 6, our statement is true for the points $x \in (0, \infty)^n$. Assume that $x = (x_1, \ldots, x_n)$ belongs to the boundary of $[0, \infty)^n$ and $x \neq 0$. For the sake of simplicity we consider the case when the first $k$ ($1 \leq k \leq n - 1$) coordinates of $x$ are positive and all other are equal to zero. For each $\xi = (\xi_1, \ldots, \xi_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ we have
\[
(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^{k} x_j (\xi_j + \mu_j) - (u(e^{\xi_1}, \ldots, e^{\xi_n}) + u^*(e^{\mu_1}, \ldots, e^{\mu_n})).
\]

By this inequality we obtain that
\[
(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^{k} x_j (\xi_j + \mu_j)
\]
\[
- (u(e^{\xi_1}, \ldots, e^{\xi_k}, 0, \ldots, 0) + u^*(e^{\mu_1}, \ldots, e^{\mu_k}, 0, \ldots, 0)).
\]

We define a function $u_k$ on $\mathbb{R}^k$ by the rule: $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \to u(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$. We observe that for each $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$, $\hat{t} = (t_1, \ldots, t_k, 0, \ldots, 0) \in \mathbb{R}^n$ we have
\[
 u^*(\hat{t}) = \sup_{v \in \mathbb{R}^n} \langle (\hat{t}, v) - u(v) \rangle
\]
\[
 \leq \sup_{v_1, \ldots, v_k \in \mathbb{R}} \left( \sum_{j=1}^{k} t_j v_j - u(v_1, \ldots, v_k, 0, \ldots, 0) \right) = \sup_{v \in \mathbb{R}^k} \langle (t, v) - u_k(v) \rangle = u_k^*(t).
\]

Employing this and the above inequality, for $\hat{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and each $\check{\xi} = (\xi_1, \ldots, \xi_k), \check{\mu} = (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k$ we have
\[
 (u[e])^*(\hat{x}) + (u^*[e])^*(\hat{x}) \geq \langle \check{x}, \check{\xi} \rangle - u_k[e](\check{\xi}) + \langle \check{x}, \check{\mu} \rangle - u_k^*[e](\check{\mu}).
\]

Therefore,
\[
 (u[e])^*(\hat{x}) + (u^*[e])^*(\hat{x}) \geq (u_k[e])^*(\hat{x}) + (u_k^*[e])^*(\hat{x}).
\]

Since by the Proposition 6,
\[
 (u_k[e])^*(\hat{x}) + (u_k^*[e])^*(\hat{x}) = \sum_{j=1}^{k} (x_j \ln x_j - x_j),
\]
Then \((u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^{k} (x_j \log x_j - x_j)\). By Proposition 3 this implies the first statement of the proposition.

If \(x = 0\), then
\[
(u[e])^*(0) = -\inf_{\xi \in \mathbb{R}^n} u[e](\xi) = -u(0),
\]
\[
(u^*[e])^*(0) = -\inf_{\xi \in \mathbb{R}^n} u^*[e](\xi) = -u^*(0) = \inf_{\xi \in \mathbb{R}^n} u(\xi) = u(0).
\]

Therefore, \((u[e])^*(0) + (u^*[e])^*(0) = 0\). \(\square\)

### 2.2. Asymptotics of multi-dimensional Laplace integral.

In work \cite{9} there was established the following theorem.

**Theorem B.** Let \(E\) be a convex domain in \(\mathbb{R}^n\), \(h\) be a convex function in \(E\), \(\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}\) and the interior of \(\tilde{E}\) is non-empty. Let
\[
D^h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : h(x) + h^*(y) - \langle x, y \rangle \leq 1\},
\]
\[
D_y^h = \{x \in \mathbb{R}^n : (x, y) \in D\}, \quad y \in \mathbb{R}^n.
\]

Then
\[
e^{-1}V(D_y^h)e^{h^*(y)} \leq \int_{\mathbb{R}^n} e^{\langle x, y \rangle - h(x)} \, dx \leq (1 + n!)V(D_y^h)e^{h^*(y)}, \quad y \in \tilde{E}.
\]

Here we assume that \(h(x) = +\infty\) as \(x \notin E\).

### 3. Description of dual space

#### 3.1. Auxiliary lemmata.

In the proof of Theorem the following four lemmata will be useful.

**Lemma 1.** Let \(\varphi \in \mathcal{V}(\mathbb{R}^n)\). Then the system \(\{\exp(\lambda, z)\}_{\lambda \in \mathbb{C}^n}\) is complete in \(F^2_{\varphi}\).

**Proof.** Let \(S\) be a linear continuous functional on the space \(F^2_{\varphi}\) such that \(S(e^{(\lambda, z)}) = 0\) for each \(\lambda \in \mathbb{C}^n\). Since for each multi-index \(\alpha \in \mathbb{Z}^n_+\) we have \((D^\lambda_{\alpha}S)(\lambda) = S(z^\alpha e^{(\lambda, z)})\), this identity implies that \(S(z^\alpha) = 0\). Since the function \(\varphi([z_1, \ldots, z_n])\) is convex in \(\mathbb{C}^n\), it follows from the result by B.A. Taylor on the weight approximation of entire functions by polynomials \cite[Thm. 2]{4} that the polynomials are dense in \(F^2_{\varphi}\). Hence, \(S\) is the zero functional. By the known corollary of Khan-Banach theorem we obtain that the system \(\{\exp(\lambda, z)\}_{\lambda \in \mathbb{C}^n}\) is complete in \(F^2_{\varphi}\). \(\square\)

We note that the system \(\{z^\alpha\}_{|\alpha| \geq 0}\) is orthogonal in \(F^2_{\varphi}\). Moreover, it is dense in \(F^2_{\varphi}\). Therefore, the system \(\{z^\alpha\}_{|\alpha| \geq 0}\) is a basis in \(F^2_{\varphi}\).

**Lemma 2.** Let \(\varphi \in \mathcal{V}(\mathbb{R}^n)\). Then
\[
c_\alpha(\varphi) \geq \frac{\pi^n}{\alpha_1 \cdots \alpha_n} e^{2(\varphi[e])^*(\tilde{\alpha})}, \quad \alpha \in \mathbb{Z}^n_+.
\]

In particular, for each \(M > 0\) there exists a constant \(C_M > 0\) such that \(c_\alpha(\varphi) \geq C_M M^{|\alpha|}\) for each \(\alpha \in \mathbb{Z}^n_+\).
Employing now Proposition 2, we obtain easily the second statement of the lemma. This implies that for each \( t \in \mathbb{R}^n \)

\[
c_\alpha(\varphi) = (2\pi)^n \int_0^\infty \cdots \int_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1, \ldots, r_n)} \, dr_1 \cdots dr_n
\]

\[
\geq (2\pi)^n \int_0^\infty \cdots \int_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(R_1, \ldots, R_n)} \, dr_1 \cdots dr_n
\]

\[
= (2\pi)^n \frac{r_1^{2\alpha_1+2}}{2\alpha_1 + 2} \cdots \frac{r_n^{2\alpha_n+2}}{2\alpha_n + 2} e^{-2\varphi(R_1, \ldots, R_n)}.
\]

This implies that for each \( t \in \mathbb{R}^n \)

\[
c_\alpha(\varphi) \geq \frac{\pi^n}{\alpha_1 \cdots \alpha_n} e^{(2\alpha, t) - 2\varphi(\varepsilon)(t)}.
\]

Therefore,

\[
c_\alpha(\varphi) \geq \frac{\pi^n}{\alpha_1 \cdots \alpha_n} e^{2(\varphi(\varepsilon))^*(\tilde{\alpha})}.
\]

Employing now Proposition 2, we obtain easily the second statement of the lemma.

**Lemma 3.** Assume that an entire in \( \mathbb{C}^n \) function satisfies \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \in F^2_{\varphi} \). Then

\[
\sum_{|\alpha| \geq 0} |a_\alpha|^2 c_\alpha(\varphi) < \infty \quad \text{and} \quad \|f\|^2_\varphi = \sum_{|\alpha| \geq 0} |a_\alpha|^2 c_\alpha(\varphi).
\]

And vice versa, let the sequence \( (a_\alpha)_{|\alpha| \geq 0} \) of complex number \( a_\alpha \) is such that the series \( \sum_{|\alpha| \geq 0} |a_\alpha|^2 c_\alpha(\varphi) \) converges. Then \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \in H(\mathbb{C}^n) \). At that, \( f \in F^2_{\varphi} \).

**Proof.** Let

\[
f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha
\]

be an entire function in \( \mathbb{C}^n \) in the class \( F^2_{\varphi} \). Then

\[
\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(\abs{z})} \, d\lambda(z) = \int_{\mathbb{C}^n} \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \sum_{|\beta| \geq 0} \overline{a_\beta} \overline{z}^\beta e^{-2\varphi(\abs{z})} \, d\mu_n(z)
\]

\[
= \sum_{|\alpha| \geq 0} |a_\alpha|^2 \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\abs{z})} \, d\mu_n(z) = \sum_{|\alpha| \geq 0} |a_\alpha|^2 c_\alpha(\varphi).
\]

Vice versa, the convergence of the series \( \sum_{|\alpha| \geq 0} |a_\alpha|^2 c_\alpha(\varphi) \) and Lemma 2 implies that for each \( \varepsilon > 0 \) there exists a constant \( c_\varepsilon > 0 \) such that \( |a_\alpha| \leq c_\varepsilon \varepsilon^{|\alpha|} \) for each \( \alpha \in \mathbb{Z}_+^n \). This means that \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \) is an entire function in \( \mathbb{C}^n \). It is easy to see that \( f \in F^2_{\varphi} \).

**Lemma 4.** Let \( \varphi \in \mathcal{V}(\mathbb{R}^n) \). Then

\[
(2\pi)^n e^{-1} V(D^{\varphi(\varepsilon)}_{\alpha} \left( \frac{1}{2} \right)) e^{2(\varphi(\varepsilon))^*(\tilde{\alpha})} \leq c_\alpha(\varphi) \leq (2\pi)^n (1 + n!) V(D^{\varphi(\varepsilon)}_{\alpha} \left( \frac{1}{2} \right)) e^{2(\varphi(\varepsilon))^*(\tilde{\alpha})}
\]

for each \( \alpha \in \mathbb{Z}_+^n \).
Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. Then
\[
c_\alpha(\varphi) = (2\pi)^n \int_0^\infty \cdots \int_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1, \ldots, r_n)} \, dr_1 \cdots dr_n
\]
\[
= (2\pi)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{(2\alpha_1+2)t_1 + \cdots + (2\alpha_n+2)t_n - 2\varphi(t_1, \ldots, t_n)} \, dt_1 \cdots dt_n.
\]
That is,
\[
c_\alpha(\varphi) = (2\pi)^n \int_{\mathbb{R}^n} e^{(2\alpha_1 t_1 + \cdots + 2\alpha_n t_n - 2\varphi(t_1, \ldots, t_n))} \, dt.
\]
By Theorem B we have
\[
(2\pi)^n e^{-1} V(D_{2\alpha}^{\varphi[e]}) e^{2(\varphi[e])^*(\hat{\alpha})} \leq c_\alpha(\varphi) \leq (2\pi)^n (1 + n!) V(D_{2\alpha}^{\varphi[e]}) e^{2(\varphi[e])^*(\hat{\alpha})}
\]
Since $D_{2\alpha}^{\varphi[e]} = D_{\alpha}^{\varphi[e]}(\frac{1}{2})$, by the previous inequality this completes the proof.

3.2. Proof of Theorem. Let us prove that the mapping $\mathcal{L}$ acts from $(F_\varphi^2)^*$ into $F_{\varphi^*}^2$. Let $S \in (F_{\varphi^*}^2)^*$. Then there exists a function $g_S \in F_{\varphi^*}^2$ such that $S(f) = (f, g_S)_\varphi$, that is,
\[
S(f) = \int_{c^n} f(z) g_S(z) e^{-2\varphi(\text{abs } z)} \, d\mu_n(z), \quad f \in F_{\varphi^*}^2.
\]
At that, $\|S\| = \|g_S\|_{\varphi^*}$. If $g_S(z) = \sum_{|\alpha| \geq 0} b_\alpha z^\alpha$, then $\hat{S}(\lambda) = \sum_{|\alpha| \geq 0} \frac{c_\alpha(\varphi) b_\alpha}{\alpha!} \lambda^\alpha$, $\lambda \in \mathbb{C}^n$. Therefore,
\[
\|\hat{S}\|_{\varphi^*}^2 = \sum_{|\alpha| \geq 0} \left( \frac{c_\alpha(\varphi) |b_\alpha|}{\alpha!} \right)^2 c_\alpha(\varphi^*).
\]
By Lemma 3,
\[
c_\alpha(\varphi) \leq (2\pi)^n (1 + n!) V\left( D_{\alpha}^{\varphi[e]} \left( \frac{1}{2} \right) \right) e^{2(\varphi[e])^*(\hat{\alpha})},
\]
\[
c_\alpha(\varphi^*) \leq (2\pi)^n (1 + n!) V\left( D_{\alpha}^{\varphi[e]} \left( \frac{1}{2} \right) \right) e^{2(\varphi^*[e])^*(\hat{\alpha})}
\]
for each $\alpha \in \mathbb{Z}_+^n$.
Therefore,
\[
c_\alpha(\varphi) c_\alpha(\varphi^*) \leq (2\pi)^{2n} (1 + n!)^2 V\left( D_{\alpha}^{\varphi[e]} \left( \frac{1}{2} \right) \right) V\left( D_{\alpha}^{\varphi[e]} \left( \frac{1}{2} \right) \right) e^{2(\varphi[e])^*(\hat{\alpha}) + 2(\varphi^*[e])^*(\hat{\alpha})}
\]
for each $\alpha \in \mathbb{Z}_+^n$.
According Proposition 6, for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we have
\[
(\varphi[e])^*(\hat{\alpha}) + (\varphi^*[e])^*(\hat{\alpha}) = \sum_{j=1}^n ((\alpha_j + 1) \ln(\alpha_j + 1) - (\alpha_j + 1)).
\]
Since by the Stirling’s formula \cite{10}, for each $m \in \mathbb{Z}_+$ we have
\[
(m + 1) \ln(m + 1) - (m + 1) = \ln \Gamma(m + 1) - \ln \sqrt{2\pi} + \frac{1}{2} \ln(m + 1) - \frac{\theta}{12(m + 1)},
\]
where $\theta \in (0, 1)$ depends on $m$, then
\[
(\varphi[e])^*(\hat{\alpha}) + (\varphi^*[e])^*(\hat{\alpha}) = -n \ln \sqrt{2\pi} + \sum_{j=1}^n \left( \ln \Gamma(\alpha_j + 1) + \frac{1}{2} \ln(\alpha_j + 1) - \frac{\theta_j}{12(\alpha_j + 1)} \right),
\]
where $\theta_j \in (0, 1)$ depends on $\alpha_j$. Then
\[
\frac{e^{2((\varphi'[\epsilon])^*(\tilde{\alpha}) + (\varphi'^*[\epsilon])^*(\tilde{\alpha}))}}{\alpha!^2} = \frac{1}{(2\pi)^n} \prod_{j=1}^n (\alpha_j + 1) e^{-\frac{\theta_j}{6(\alpha_j + 1)}}. \tag{2}
\]
Thus,
\[
c_\alpha(\varphi)c_\alpha(\varphi^*) \leq (2\pi)^n (1 + n!)^2 V \begin{pmatrix} D_{\alpha}^\varphi[e] \left( \frac{1}{2} \right) \end{pmatrix} V \begin{pmatrix} D_{\alpha}^{\varphi^*}[e] \left( \frac{1}{2} \right) \end{pmatrix} \prod_{j=1}^n \hat{\alpha}_j.
\]
Employing the condition for $\varphi$, we obtain that
\[
c_\alpha(\varphi)c_\alpha(\varphi^*) \leq (2\pi)^n (1 + n!)^2 K
\]
for each $\alpha \in \mathbb{Z}_+^n$. Letting $M_1 = (2\pi)^n (1 + n!)^2 K$, by (1) we obtain
\[
\|\hat{S}\|_{\varphi^*}^2 \leq M_1 \sum_{|\alpha| \geq 0} c_\alpha(\varphi)|b_\alpha|^2 = M_1\|gS\|_{\varphi^*}^2 = M_1\|S\|^2.
\]
Hence, $\hat{S} \in F_\varphi^2$. Moreover, the latter estimate implies that the linear mapping $\mathcal{L}$ acts continuously from $(F_\varphi^2)^* \rightarrow F_\varphi^2$.

We observe that the mapping $\mathcal{L}$ is injective from $(F_\varphi^2)^* \rightarrow F_\varphi^2$, since by Lemma 1 the system \{exp($\lambda$, $z$)\}$_{\lambda \in \mathbb{C}^n}$ is complete in $F_\varphi^2$.

Let us show that the mapping $\mathcal{L}$ acts from $(F_\varphi^2)^*$ onto $F_\varphi^2$. Assume that $G \in F_\varphi^2$. Employing the representation of an entire function $G$ by the Taylor series
\[
G(\lambda) = \sum_{|\alpha| \geq 0} d_\alpha \lambda^\alpha, \quad \lambda \in \mathbb{C}^n,
\]
we get
\[
\|G\|_{\varphi^*}^2 = \sum_{|\alpha| \geq 0} |d_\alpha|^2 c_\alpha(\varphi^*).
\]
For each $\alpha \in \mathbb{Z}_+^n$ we define the numbers $g_\alpha = \frac{\alpha!}{c_\alpha(\varphi)}$ and consider the convergence of the series
\[
\sum_{|\alpha| \geq 0} |g_\alpha|^2 c_\alpha(\varphi). \quad \text{We have}
\]
\[
\sum_{|\alpha| \geq 0} |g_\alpha|^2 c_\alpha(\varphi) = \sum_{|\alpha| \geq 0} \left| \frac{\alpha!}{c_\alpha(\varphi)} \right|^2 c_\alpha(\varphi) = \sum_{|\alpha| \geq 0} \frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} |d_\alpha|^2 c_\alpha(\varphi^*).
\]
By Lemma 4,
\[
c_\alpha(\varphi) \geq e^{-1} V \begin{pmatrix} D_{\alpha}^\varphi[e] \left( \frac{1}{2} \right) \end{pmatrix} e^{2(\varphi'[\epsilon])^*(\tilde{\alpha})}, \quad c_\alpha(\varphi^*) \geq e^{-1} V \begin{pmatrix} D_{\alpha}^{\varphi^*}[e] \left( \frac{1}{2} \right) \end{pmatrix} e^{2(\varphi'^*[\epsilon])^*(\tilde{\alpha})}
\]
for each $\alpha \in \mathbb{Z}_+^n$. Therefore,
\[
c_\alpha(\varphi)c_\alpha(\varphi^*) \geq e^{-2} V \begin{pmatrix} D_{\alpha}^\varphi[e] \left( \frac{1}{2} \right) \end{pmatrix} V \begin{pmatrix} D_{\alpha}^{\varphi^*}[e] \left( \frac{1}{2} \right) \end{pmatrix} e^{2((\varphi'[\epsilon])^*(\tilde{\alpha}) + (\varphi'^*[\epsilon])^*(\tilde{\alpha}))}
\]
for each $\alpha \in \mathbb{Z}_+^n$. By identity (2) this implies
\[
\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} \leq V \begin{pmatrix} D_{\alpha}^\varphi[e] \left( \frac{1}{2} \right) \end{pmatrix} V \begin{pmatrix} D_{\alpha}^{\varphi^*}[e] \left( \frac{1}{2} \right) \end{pmatrix} \prod_{j=1}^n (\alpha_j + 1)
\]
for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. 

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Employing the condition for $\varphi$, we obtain that $\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} \leq Ke^2(2\pi)^n$, $\forall \alpha \in \mathbb{Z}_+^n$. Therefore, for the considered series we have

$$\sum_{|\alpha| \geq 0} |g_\alpha|^2 c_\alpha(\varphi) \leq Ke^2(2\pi)^n \sum_{|\alpha| \geq 0} |d_\alpha|^2 c_\alpha(\varphi^*) = Ke^2(2\pi)^n \|G\|^2_{\varphi^*}. \tag{3}$$

Thus, the series $\sum_{|\alpha| \geq 0} |g_\alpha|^2 c_\alpha(\varphi)$ converges. But by Lemma 3 the function

$$g(\lambda) = \sum_{|\alpha| \geq 0} g_\alpha \lambda^\alpha, \quad \lambda \in \mathbb{C}^n,$$

is entire and by (3), $g$ belongs to $F^2_{\varphi}$ and

$$\|g\|^2_{\varphi} \leq Ke^2(2\pi)^n \|G\|^2_{\varphi^*}. \tag{4}$$

We define a functional $S$ on $F^2_{\varphi}$ by the formula

$$S(f) = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-2\varphi(\text{abs}z)} d\mu_\alpha(z), \quad f \in F^2_{\varphi}.$$

It is clear that $S$ is a linear continuous functional on $F^2_{\varphi}$. At that, $\hat{S} = G$. Since $\|S\| = \|g\|_{\varphi}$, estimate (4) shows that the inverse mapping $L^{-1}$ is continuous. Thus, $L$ makes an isomorphism between the spaces $(F^2_{\varphi})^*$ and $F^2_{\varphi^*}$. The proof is complete.

**BIBLIOGRAPHY**


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