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ON A HILBERT SPACE OF ENTIRE FUNCTIONS

I.KH. MUSIN

Abstract. We consider the Hilbert space F^2_φ of entire functions of n variables constructed by means of a convex function φ in $\mathbb{C}^n$ depending on the absolute value of the variable and growing at infinity faster than $a|z|$ for each $a > 0$. We study the problem on describing the dual space in terms of the Laplace transform of the functionals. Under certain conditions for the weight function φ, we obtain the description of the Laplace transform of linear continuous functionals on F^2_φ. The proof of the main result is based on using new properties of Young-Fenchel transform and one result on the asymptotics of the multi-dimensional Laplace integral established by R.A. Bashmakov, K.P. Isaev, R.S. Yulmukhametov.

Keywords: Hilbert space, Laplace transform, entire functions, convex functions, Young-Fenchel transform.

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1. Introduction

1.1. Problem. Let $H(\mathbb{C}^n)$ be the space of entire functions in $\mathbb{C}^n$, $d\mu_n$ be the Lebesgue measure in $\mathbb{C}^n$ and for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ $(\mathbb{C}^n)$ we define $\operatorname{abs} u := (|u_1|, \dots, |u_n|)$.

We denote by $\mathcal{V}(\mathbb{R}^n)$ the set of all convex functions g in $\mathbb{R}^n$ such that

1) $g(x_1, \dots, x_n) = g(|x_1|, \dots, |x_n|),\ (x_1, \dots, x_n) \in \mathbb{R}^n$;

2) the restriction g on $[0, \infty)^n$ is non-decaying in each variable;

3) $\lim\limits_{x\to\infty} \dfrac{g(x)}{\|x\|} = +\infty$; $\|x\|$ is the Euclidean norm of a point $x \in \mathbb{R}^n$).

To each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ we associate the Hilbert space

$$F^2_\varphi = \left\{ f \in H(\mathbb{C}^n) : \|f\|_\varphi = \left(\int_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(\operatorname{abs} z)}\, d\mu_n(z) \right)^{\frac{1}{2}} < \infty \right\}$$

with the scalar product

$$(f, g)_\varphi = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(\operatorname{abs} z)}\, d\mu_n(z),\ f, g \in F^2_\varphi.$$

If $\varphi(x) = \frac{\|x\|^2}{2}$, then F^2_φ is the Fock space.

It is obvious that for each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and each $\lambda \in \mathbb{C}^n$, the function $f_\lambda(z) = e^{\langle \lambda, z\rangle}$ belongs to F^2_φ. This is why for each linear continuous functional S on the space F^2_φ, the function

$$\hat{S}(\lambda) = S(e^{\langle \lambda, z\rangle}), \quad \lambda \in \mathbb{C}^n,$$

is well defined in $\mathbb{C}^n$; this function is the Laplace transform of the functional S. It is easy to see that $\hat{S}$ is an entire function.

By $(F^2_\varphi)^*$ we denote the dual space for F^2_φ.

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The aim of the work is to find the conditions for $\varphi \in \mathcal{V}(\mathbb{R}^n)$, under which the space $\widehat{(F^2_\varphi)^*}$ of the Laplace transforms of the linear continuous functionals on F^2_φ can be described as $F^2_{\varphi^*}$.

If $\varphi(x) = \frac{\|x\|^2}{2}$, then $\widehat{(F^2_\varphi)^*} = F^2_\varphi$. Indeed, in this case the problem on describing the space $(F^2_\varphi)^*$ in terms of the Laplace transform of the functionals is easily solved thanks to the classical representation: for each $f \in F^2_\varphi$,

$$f(\lambda) = \pi^{-n} \int_{\mathbb{C}^n} f(z) e^{\langle \lambda, \overline{z} \rangle - \|z\|^2} \, d\mu_n(z), \quad \lambda \in \mathbb{C}^n.$$

If the function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ is radial, the mentioned problem was solved by V.V. Napalkov and S.V. Popenov [5], [6].

1.2. Notations and definitions. For $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n(\mathbb{C}^n)$ we let $\langle u, v \rangle := u_1 v_1 + \dots + u_n v_n$, $\|u\|$ is the Euclidean norm of u.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, by $|\alpha| := \alpha_1 + \ldots + \alpha_n$ we denote the length of the multi-index α, $\tilde{\alpha} := (\alpha_1 + 1, \dots, \alpha_n + 1)$, and we denote $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $D^\alpha_z := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, $\varphi \in \mathcal{V}(\mathbb{R}^n)$, we define

$$c_\alpha(\varphi) := \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\operatorname{abs} z)} \, d\mu_n(z).$$

For a function u with a domain containing the set $(0, \infty)^n$, we define a function $u[e]$ in $\mathbb{R}^n$ by the rule:

$$u[e](x) = u(e^{x_1}, \dots, e^{x_n}), \ x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

By $\mathcal{B}(\mathbb{R}^n)$ we denote the set of all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition

$$\lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty.$$

The Young-Fenchel transform of the function $u : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $u^* : \mathbb{R}^n \to [-\infty, +\infty]$ defined by the formula

$$u^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - u(y)), \quad x \in \mathbb{R}^n.$$

If E is a convex domain in $\mathbb{R}^n$, h is a convex set in E, $\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}$, $p > 0$, then

$$D^h_y(p) := \{x \in E : h(x) + h^*(y) - \langle x, y \rangle \leqslant p\}, \quad y \in \tilde{E}.$$

By $V(D)$ we denote the n-dimensional volume of a set $D \subset \mathbb{R}^n$.

1.3. Main result.

Theorem. *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and for some $K > 0$ and each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ the inequalities*

$$\frac{1}{K} \leqslant V\left(D^{\varphi[e]}_\alpha\left(\frac{1}{2}\right)\right) V\left(D^{\varphi^*[e]}_\alpha\left(\frac{1}{2}\right)\right) \prod_{j=1}^n \alpha_j \leqslant K$$

hold. Then the mapping $\mathcal{L} : S \in (F^2_\varphi)^* \to \hat{S}$ makes an isomorphism between the spaces $(F^2_\varphi)^*$ and $F^2_{\varphi^*}$.

The proof of Theorem in Subsection 3.2 is based on new properties of Young-Fenchel transform, see Subsection 2.1, and one result on the asymptotics of the multi-dimensional Laplace integral in work [9], see Subsection 2.2.

2. Auxiliary data and results

2.1. On some properties of Young-Fenchel transform. It is easy to confirm that the following statement holds.

Proposition 1. *Let* $u \in \mathcal{B}(\mathbb{R}^n)$*. Then* $(u[e])^*(x) > -\infty$ *as* $x \in \mathbb{R}^n$*,* $(u[e])^*(x) = +\infty$ *as* $x \notin [0,\infty)^n$ *and* $(u[e])^*(x) < +\infty$ *as* $x \in [0,\infty)^n$*.*

We note that the last statement of Proposition 1 is implied, for instance, by the fact that for each $M > 0$ there exists a constant $A > 0$ such that

$$(u[e])^*(x) \leqslant \sum_{1\leqslant j\leqslant n: x_j\neq 0} (x_j \ln \frac{x_j}{M} - x_j) + A, \quad x \in [0,\infty)^n.$$

Proposition 2. *Let* $u \in \mathcal{B}(\mathbb{R}^n)$*. Then*

$$\lim_{\substack{x\to\infty,\\ x\in[0,\infty)^n}} \frac{(u[e])^*(x)}{\|x\|} = +\infty.$$

Proof. For each $x \in [0,\infty)^n$ and $t \in \mathbb{R}^n$ we have

$$(u[e])^*(x) \geqslant \langle x, t\rangle - (u[e])(t).$$

Employing this inequality, we obtain that for each $M > 0$

$$(u[e])^*(x) \geqslant M\|x\| - u[e]\left(\frac{Mx}{\|x\|}\right), \quad x \in [0,\infty)^n \setminus \{0\}.$$

This completes the proof. □

The next three statements were proved in work [1], see there Lemma 6, Proposition 3, Proposition 4.

Proposition 3. *Let* $u \in \mathcal{B}(\mathbb{R}^n)$*. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) \leqslant \sum_{\substack{1\leqslant j\leqslant n:\\ x_j\neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1,\ldots,x_n) \in [0,\infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) \leqslant 0.$$

Proposition 4. *Let* $u \in \mathcal{B}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1,\ldots,x_n) \in (0,\infty)^n.$$

Proposition 5. *Let* $u \in \mathcal{V}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1\leqslant j\leqslant n:\\ x_j\neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1,\ldots,x_n) \in [0,\infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

Propositions 4 and 5 can be strengthen by employing the results by D. Azagra [2], [3]. He proved the following theorem.

Theorem A. *Let* $U \subseteq \mathbb{R}^n$ *be an open convex set. For each convex function* $f : U \to \mathbb{R}$ *and each* $\varepsilon > 0$ *there exists a real analytic convex function* $g : U \to \mathbb{R}$ *such that*

$$f(x) - \varepsilon \leqslant g(x) \leqslant f(x), \quad x \in U.$$

Thus, the following corollary hold [3].

Corollary A. *Let $U \subseteq \mathbb{R}^n$ be an open convex set. For each convex function $f : U \to \mathbb{R}$ and each $\varepsilon > 0$ there exists an infinitely differentiable convex function $g : U \to \mathbb{R}$ such that*

$$f(x) - \varepsilon \leqslant g(x) \leqslant f(x), \quad x \in U.$$

Employing Proposition 4 and Corollary A, we easily confirm the following statement.

Proposition 6. *Let $u \in \mathcal{B}(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n}(x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in (0, \infty)^n.$$

Moreover, the following proposition is true.

Proposition 7. *Let $u \in \mathcal{V}(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1 \leqslant j \leqslant n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

Proof. According Proposition 6, our statement is true for the points $x \in (0, \infty)^n$. Assume that $x = (x_1, \dots, x_n)$ belongs to the boundary of $[0, \infty)^n$ and $x \neq 0$. For the sake of simplicity we consider the case when the first k $(1 \leqslant k \leqslant n-1)$ coordinates of x are positive and all other are equal to zero. For each $\xi = (\xi_1, \dots, \xi_n)$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ we have

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant \sum_{j=1}^{k} x_j(\xi_j + \mu_j) - (u(e^{\xi_1}, \dots, e^{\xi_n}) + u^*(e^{\mu_1}, \dots, e^{\mu_n})).$$

By this inequality we obtain that

$$\begin{aligned}(u[e])^*(x) + (u^*[e])^*(x) \geqslant & \sum_{j=1}^{k} x_j(\xi_j + \mu_j) \\ & -(u(e^{\xi_1}, \dots, e^{\xi_k}, 0, \dots, 0) + u^*(e^{\mu_1}, \dots, e^{\mu_k}, 0, \dots, 0)).\end{aligned}$$

We define a function u_k on $\mathbb{R}^k$ by the rule: $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \to u(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$. We observe that for each $t = (t_1, \dots, t_k) \in \mathbb{R}^k$, $\breve{t} = (t_1, \dots, t_k, 0, \dots, 0) \in \mathbb{R}^n$ we have

$$\begin{aligned}u^*(\breve{t}) &= \sup_{v \in \mathbb{R}^n} (\langle \breve{t}, v \rangle - u(v)) \\ &\leqslant \sup_{v_1, \dots, v_k \in \mathbb{R}} (\sum_{j=1}^{k} t_j v_j - u(v_1, \dots, v_k, 0, \dots, 0)) = \sup_{v \in \mathbb{R}^k} (\langle t, v \rangle - u_k(v)) = u_k^*(t).\end{aligned}$$

Employing this and the above inequality, for $\tilde{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ and each $\tilde{\xi} = (\xi_1, \dots, \xi_k), \tilde{\mu} = (\mu_1, \dots, \mu_k) \in \mathbb{R}^k$ we have

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant \langle \tilde{x}, \tilde{\xi} \rangle - u_k[e](\tilde{\xi}) + \langle \tilde{x}, \tilde{\mu} \rangle - u_k^*[e](\tilde{\mu}).$$

Therefore,

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant (u_k[e])^*(\tilde{x}) + (u_k^*[e])^*(\tilde{x}).$$

Since by the Proposition 6,

$$(u_k[e])^*(\tilde{x}) + (u_k^*[e])^*(\tilde{x}) = \sum_{j=1}^{k}(x_j \ln x_j - x_j),$$

then $(u[e])^*(x)+(u^*[e])^*(x) \geqslant \sum\limits_{j=1}^{k}(x_j \ln x_j - x_j)$. By Proposition 3 this implies the first statement of the proposition.

If $x = 0$, then

$$\begin{aligned}(u[e])^*(0) &= -\inf_{\xi\in\mathbb{R}^n} u[e](\xi) = -u(0),\\ (u^*[e])^*(0) &= -\inf_{\xi\in\mathbb{R}^n} u^*[e](\xi) = -u^*(0) = \inf_{\xi\in\mathbb{R}^n} u(\xi) = u(0).\end{aligned}$$

Therefore, $(u[e])^*(0) + (u^*[e])^*(0) = 0$. □

2.2. Asymptotics of multi-dimensional Laplace integral. In work [9] there was established the following theorem.

Theorem B. *Let E be a convex domain in $\mathbb{R}^n$, h be a convex function in E, $\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}$ and the interior of $\tilde{E}$ is non-empty. Let*

$$\begin{aligned}D^h &= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : h(x) + h^*(y) - \langle x, y\rangle \leqslant 1\},\\ D^h_y &= \{x \in \mathbb{R}^n : (x, y) \in D\},\ \ y \in \mathbb{R}^n.\end{aligned}$$

Then

$$e^{-1}V(D^h_y)e^{h^*(y)} \leqslant \int_{\mathbb{R}^n} e^{\langle x,y\rangle - h(x)}\,dx \leqslant (1 + n!)V(D^h_y)e^{h^*(y)}, \quad y \in \tilde{E}.$$

Here we assume that $h(x) = +\infty$ as $x \notin E$.

3. Description of dual space

3.1. Auxiliary lemmata. In the proof of Theorem the following four lemmata will be useful.

Lemma 1. *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then the system $\{\exp\langle\lambda, z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in F^2_φ.*

Proof. Let S be a linear continuous functional on the space F^2_φ such that $S(e^{\langle\lambda,z\rangle}) = 0$ for each $\lambda \in \mathbb{C}^n$. Since for each multi-index $\alpha \in \mathbb{Z}^n_+$ we have $(D^\alpha_\lambda \hat{S})(\lambda) = S(z^\alpha e^{\langle z,\lambda\rangle})$, this identity implies that $S(z^\alpha) = 0$. Since the function $\varphi(|z_1|, \dots, |z_n|)$ is convex in $\mathbb{C}^n$, it follows from the result by B.A. Taylor on the weight approximation of entire functions by polynomials [4, Thm. 2] that the polynomials are dense in F^2_φ. Hence, S is the zero functional. By the known corollary of Khan-Banach theorem we obtain that the system $\{\exp\langle\lambda, z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in F^2_φ. □

We note that the system $\{z^\alpha\}_{|\alpha|\geqslant 0}$ is orthogonal in F^2_φ. Moreover, it is dense in F^2_φ. Therefore, the system $\{z^\alpha\}_{|\alpha|\geqslant 0}$ is a basis in F^2_φ.

Lemma 2. *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then*

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} e^{2(\varphi[e])^*(\tilde{\alpha})}, \quad \alpha \in \mathbb{Z}^n_+.$$

In particular, for each $M > 0$ there exists a constant $C_M > 0$ such that $c_\alpha(\varphi) \geqslant C_M M^{|\alpha|}$ for each $\alpha \in \mathbb{Z}^n_+$

Proof. For each $\alpha \in \mathbb{Z}_+^n$ and each positive numbers $R_1, \ldots, R_n$ we have

$$\begin{aligned} c_\alpha(\varphi) =&(2\pi)^n \int_0^\infty \cdots \int_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1,\cdots,r_n)}\, dr_1 \cdots dr_n \\ \geqslant&(2\pi)^n \int_0^{R_1} \cdots \int_0^{R_n} r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(R_1,\cdots,R_n)}\, dr_1 \cdots dr_n \\ =&(2\pi)^n \frac{R_1^{2\alpha_1+2}}{2\alpha_1+2} \cdots \frac{R_n^{2\alpha_n+2}}{2\alpha_n+2} e^{-2\varphi(R_1,\cdots,R_n)}. \end{aligned}$$

This implies that for each $t \in \mathbb{R}^n$

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} e^{\langle 2\tilde{\alpha}, t\rangle - 2\varphi[e](t)}.$$

Therefore,

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} e^{2(\varphi[e])^*(\tilde{\alpha})}.$$

Employing now Proposition 2, we obtain easily the second statement of the lemma. □

Lemma 3. *Assume that an entire in $\mathbb{C}^n$ function satisfies $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha \in F_\varphi^2$. Then*

$$\sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi) < \infty \quad \textit{and} \quad \|f\|_\varphi^2 = \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi).$$

And vice versa, let the sequence $(a_\alpha)_{|\alpha|\geqslant 0}$ of complex number a_α is such that the series $\sum\limits_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi)$ converges. Then $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha \in H(\mathbb{C}^n)$. At that, $f \in F_\varphi^2$.

Proof. Let

$$f(z) = \sum_{|\alpha|\geqslant 0} a_\alpha z^\alpha$$

be an entire function in $\mathbb{C}^n$ in the class F_φ^2. Then

$$\begin{aligned} \|f\|_\varphi^2 =& \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(\text{abs } z)}\, d\lambda(z) = \int_{\mathbb{C}^n} \sum_{|\alpha|\geqslant 0} a_\alpha z^\alpha \sum_{|\beta|\geqslant 0} \overline{a}_\beta \overline{z}^\beta e^{-2\varphi(\text{abs } z)}\, d\mu_n(z) \\ =& \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\text{abs } z)}\, d\mu_n(z) = \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi). \end{aligned}$$

Vice versa, the convergence of the series $\sum\limits_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi)$ and Lemma 2 implies that for each $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $|a_\alpha| \leqslant c_\varepsilon \varepsilon^{|\alpha|}$ for each $\alpha \in \mathbb{Z}_+^n$. This means that $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha$ is an entire function in $\mathbb{C}^n$. It is easy to see that $f \in F_\varphi^2$. □

Lemma 4. *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then*

$$(2\pi)^n e^{-1} V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha})} \leqslant c_\alpha(\varphi) \leqslant (2\pi)^n (1+n!) V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha})}$$

for each $\alpha \in \mathbb{Z}_+^n$.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. Then

$$\begin{aligned} c_\alpha(\varphi) =& (2\pi)^n \int_0^\infty \cdots \int_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1,\cdots,r_n)}\, dr_1 \cdots dr_n \\ =& (2\pi)^n \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{(2\alpha_1+2)t_1+\cdots+(2\alpha_n+2)t_n-2\varphi[e](t_1,\dots,t_n)}\, dt_1 \cdots dt_n. \end{aligned}$$

That is,

$$c_\alpha(\varphi) = (2\pi)^n \int_{\mathbb{R}^n} e^{\langle 2\tilde{\alpha}, t\rangle - 2\varphi[e](t)}\, dt.$$

By Theorem B we have

$$(2\pi)^n e^{-1} V\big(D_{2\tilde{\alpha}}^{2\varphi[e]}\big) e^{2(\varphi[e])^*(\tilde{\alpha})} \leqslant c_\alpha(\varphi) \leqslant (2\pi)^n (1+n!) V\big(D_{2\tilde{\alpha}}^{2\varphi[e]}\big) e^{2(\varphi[e])^*(\tilde{\alpha})}$$

Since $D_{2\tilde{\alpha}}^{2\varphi[e]} = D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)$, by the previous inequality this completes the proof. □

3.2. Proof of Theorem. Let us prove that the mapping $\mathcal{L}$ acts from $(F_\varphi^2)^*$ into $F_{\varphi^*}^2$. Let $S \in (F_\varphi^2)^*$. Then there exists a function $g_S \in F_\varphi^2$ such that $S(f) = (f, g_S)_\varphi$, that is,

$$S(f) = \int_{\mathbb{C}^n} f(z)\overline{g_S(z)} e^{-2\varphi(\text{abs } z)}\, d\mu_n(z), \quad f \in F_\varphi^2.$$

At that, $\|S\| = \|g_S\|_\varphi$. If $g_S(z) = \sum\limits_{|\alpha|\geqslant 0} b_\alpha z^\alpha$, then $\hat{S}(\lambda) = \sum_{|\alpha|\geqslant 0} \frac{c_\alpha(\varphi)\overline{b_\alpha}}{\alpha!}\lambda^\alpha,\ \lambda \in \mathbb{C}^n$. Therefore,

$$\|\hat{S}\|_{\varphi^*}^2 = \sum_{|\alpha|\geqslant 0} \left(\frac{c_\alpha(\varphi)|b_\alpha|}{\alpha!}\right)^2 c_\alpha(\varphi^*). \tag{1}$$

By Lemma 3,

$$\begin{aligned} c_\alpha(\varphi) &\leqslant (2\pi)^n (1+n!) V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha})}, \\ c_\alpha(\varphi^*) &\leqslant (2\pi)^n (1+n!) V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi^*[e])^*(\tilde{\alpha})} \end{aligned}$$

for each $\alpha \in \mathbb{Z}_+^n$.

Therefore,

$$c_\alpha(\varphi)c_\alpha(\varphi^*) \leqslant (2\pi)^{2n}(1+n!)^2 V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha})+2(\varphi^*[e])^*(\tilde{\alpha})}$$

for each $\alpha \in \mathbb{Z}_+^n$.

According Proposition 6, for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we have

$$(\varphi[e])^*(\tilde{\alpha}) + (\varphi^*[e])^*(\tilde{\alpha}) = \sum_{j=1}^n ((\alpha_j+1)\ln(\alpha_j+1) - (\alpha_j+1)).$$

Since by the Stirling's formula [10], for each $m \in \mathbb{Z}_+$ we have

$$(m+1)\ln(m+1) - (m+1) = \ln\Gamma(m+1) - \ln\sqrt{2\pi} + \frac{1}{2}\ln(m+1) - \frac{\theta}{12(m+1)},$$

where $\theta \in (0,1)$ depends on m, then

$$(\varphi[e])^*(\tilde{\alpha}) + (\varphi^*[e])^*(\tilde{\alpha}) = -n\ln\sqrt{2\pi} + \sum_{j=1}^n \left(\ln\Gamma(\alpha_j+1) + \frac{1}{2}\ln(\alpha_j+1) - \frac{\theta_j}{12(\alpha_j+1)}\right),$$

where $\theta_j \in (0,1)$ depends on α_j. Then

$$\frac{e^{2((\varphi[e])^*(\tilde\alpha)+(\varphi^*[e])^*(\tilde\alpha))}}{\alpha!^2} = \frac{1}{(2\pi)^n}\prod_{j=1}^n (\alpha_j+1)e^{-\frac{\theta_j}{6(\alpha_j+1)}}. \tag{2}$$

Thus,

$$\frac{c_\alpha(\varphi)c_\alpha(\varphi^*)}{\alpha!^2} \leqslant (2\pi)^n(1+n!)^2 V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac12\right)\right)V\left(D_{\tilde\alpha}^{\varphi^*[e]}\left(\frac12\right)\right)\prod_{j=1}^n \tilde\alpha_j.$$

Employing the condition for φ, we obtain that

$$\frac{c_\alpha(\varphi)c_\alpha(\varphi^*)}{\alpha!^2} \leqslant (2\pi)^n(1+n!)^2 K$$

for each $\alpha\in\mathbb{Z}_+^n$. Letting $M_1=(2\pi)^n(1+n!)^2K$, by (1) we obtain

$$\|\hat S\|^2_{\varphi^*} \leqslant M_1\sum_{|\alpha|\geqslant 0} c_\alpha(\varphi)|b_\alpha|^2 = M_1\|g_S\|^2_\varphi = M_1\|S\|^2.$$

Hence, $\hat S\in F^2_{\varphi^*}$. Moreover, the latter estimate implies that the linear mapping $\mathcal{L}$ acts continuously from $(F^2_\varphi)^*$ into $F^2_{\varphi^*}$.

We observe that the mapping $\mathcal{L}$ is injective from $(F^2_\varphi)^*$ into $F^2_{\varphi^*}$ since by Lemma 1 the system $\{\exp\langle\lambda,z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in F^2_φ.

Let us show that the mapping $\mathcal{L}$ acts from $(F^2_\varphi)^*$ onto $F^2_{\varphi^*}$. Assume that $G\in F^2_{\varphi^*}$. Employing the representation of an entire function G by the Taylor series

$$G(\lambda)=\sum_{|\alpha|\geqslant 0} d_\alpha\lambda^\alpha, \quad \lambda\in\mathbb{C}^n,$$

we get

$$\|G\|^2_{\varphi^*}=\sum_{|\alpha|\geqslant 0}|d_\alpha|^2c_\alpha(\varphi^*).$$

For each $\alpha\in\mathbb{Z}_+^n$ we define the numbers $g_\alpha=\frac{\overline{d_\alpha}\alpha!}{c_\alpha(\varphi)}$ and consider the convergence of the series $\sum\limits_{|\alpha|\geqslant 0}|g_\alpha|^2c_\alpha(\varphi)$. We have

$$\sum_{|\alpha|\geqslant 0}|g_\alpha|^2c_\alpha(\varphi)=\sum_{|\alpha|\geqslant 0}\left|\frac{\overline{d_\alpha}\alpha!}{c_\alpha(\varphi)}\right|^2c_\alpha(\varphi)=\sum_{|\alpha|\geqslant 0}\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)}|d_\alpha|^2c_\alpha(\varphi^*).$$

By Lemma 4,

$$c_\alpha(\varphi)\geqslant e^{-1}V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac12\right)\right)e^{2(\varphi[e])^*(\tilde\alpha)}, \quad c_\alpha(\varphi^*)\geqslant e^{-1}V\left(D_{\tilde\alpha}^{\varphi^*[e]}\left(\frac12\right)\right)e^{2(\varphi^*[e])^*(\tilde\alpha)}$$

for each $\alpha\in\mathbb{Z}_+^n$. Therefore,

$$c_\alpha(\varphi)c_\alpha(\varphi^*)\geqslant e^{-2}V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac12\right)\right)V\left(D_{\tilde\alpha}^{\varphi^*[e]}\left(\frac12\right)\right)e^{2((\varphi[e])^*(\tilde\alpha)+(\varphi^*[e])^*(\tilde\alpha))}$$

for each $\alpha\in\mathbb{Z}_+^n$. By identity (2) this implies

$$\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)}\leqslant\frac{e^2(2e\pi)^n}{V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac12\right)\right)V\left(D_{\tilde\alpha}^{\varphi^*[e]}\left(\frac12\right)\right)\prod\limits_{j=1}^n(\alpha_j+1)}$$

for each $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$.

Employing the condition for φ, we obtain that $\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} \leqslant Ke^2(2e\pi)^n$, $\forall\alpha \in \mathbb{Z}_+^n$. Therefore, for the considered series we have

$$\sum_{|\alpha|\geqslant 0} |g_\alpha|^2 c_\alpha(\varphi) \leqslant Ke^2(2e\pi)^n \sum_{|\alpha|\geqslant 0} |d_\alpha|^2 c_\alpha(\varphi^*) = Ke^2(2e\pi)^n \|G\|^2_{\varphi^*}. \tag{3}$$

Thus, the series $\sum\limits_{|\alpha|\geqslant 0} |g_\alpha|^2 c_\alpha(\varphi)$ converges. But by Lemma 3 the function

$$g(\lambda) = \sum_{|\alpha|\geqslant 0} g_\alpha \lambda^\alpha, \quad \lambda \in \mathbb{C}^n,$$

is entire and by (3), g belongs to F^2_φ and

$$\|g\|^2_\varphi \leqslant Ke^2(2e\pi)^n \|G\|^2_{\varphi^*}. \tag{4}$$

We define a functional S on F^2_φ by the formula

$$S(f) = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(\mathrm{abs} z)}\, d\mu_n(z), \quad f \in F^2_\varphi.$$

It is clear that S is a linear continuous functional on F^2_φ. At that, $\hat{S} = G$. Since $\|S\| = \|g\|_\varphi$, estimate (4) shows that the inverse mapping $\mathcal{L}^{-1}$ is continuous. Thus, $\mathcal{L}$ makes an isomorphism between the spaces $(F^2_\varphi)^*$ and $F^2_{\varphi^*}$. The proof is complete.

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Il'dar Khamitovich Musin
Institute of Mathematics, Ufa Scientific Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: musin_ildar@mail.ru