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PAULI OPERATORS AND THE $\overline{\partial}$-NEUMANN PROBLEM

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Abstract. We apply methods from complex analysis, in particular the $\overline{\partial}$-Neumann operator, to study spectral properties of Pauli operators. For this purpose we consider the weighted $\overline{\partial}$-complex on $\mathbb{C}^n$ with a plurisubharmonic weight function. The Pauli operators appear at the beginning and at the end of the weighted $\overline{\partial}$-complex. We use the spectral properties of the corresponding $\overline{\partial}$-Neumann operator to answer the question when the Pauli operators are with compact resolvent. It is also of importance to know whether the related Bergman space of entire functions is of infinite dimension. The main results are formulated in terms of the properties of the Levi matrix of the weight function. If the weight function is decoupled, one gets additional informations. Finally, we point out that a corresponding Dirac operator fails to be with compact resolvent.

Keywords: $\overline{\partial}$-Neumann problem, Pauli operators, Schrödinger operators, compactness.

Mathematics Subject Classification: 32W05, 32W10, 30H20, 35J10, 35P10

1. Introduction

Let $\varphi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-function. We consider the Schrödinger operators with magnetic field of the form

$$P_\pm = -\Delta_A \pm V,$$

also called Pauli operators, where

$$A = \frac{1}{2}\left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \dots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n}\right)$$

is the magnetic potential and

$$\Delta_A = \sum_{j=1}^{n}\left[\left(-\frac{\partial}{\partial x_j} - \frac{i}{2}\frac{\partial\varphi}{\partial y_j}\right)^2 + \left(-\frac{\partial}{\partial y_j} + \frac{i}{2}\frac{\partial\varphi}{\partial x_j}\right)^2\right],$$

and $V = \frac{1}{2}\Delta\varphi$. We write elements of $\mathbb{R}^{2n}$ in the form $(x_1, y_1, \dots, x_n, y_n)$. We shall identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, writing $(z_1, \dots, z_n) = (x_1, y_1, \dots, x_n, y_n)$, this is mainly because we will use methods of complex analysis to analyze spectral properties of the above Schrödinger operators with magnetic field.

For $n = 1$, there is an interesting connection to Dirac and Pauli operators: recall the definition of A in this case and define the Dirac operator $\mathcal{D}$ by

$$\mathcal{D} = (-i\frac{\partial}{\partial x} - A_1)\,\sigma_1 + (-i\frac{\partial}{\partial y} - A_2)\,\sigma_2 = \mathcal{A}_1\sigma_1 + \mathcal{A}_2\sigma_2, \tag{1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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Hence we can write

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{A}_1 - i\mathcal{A}_2 \\ \mathcal{A}_1 + i\mathcal{A}_2 & 0 \end{pmatrix}.$$

We remark that $i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) = V$ and hence it turns out that the square of $\mathcal{D}$ is diagonal with the Pauli operators $P_\pm$ on the diagonal:

$$\begin{aligned} \mathcal{D}^2 &= \begin{pmatrix} \mathcal{A}_1^2 - i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) + \mathcal{A}_2^2 & 0 \\ 0 & \mathcal{A}_1^2 + i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) + \mathcal{A}_2^2 \end{pmatrix} \\ &= \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}, \end{aligned}$$

where

$$P_\pm = \left(-i\,\frac{\partial}{\partial x} - A_1\right)^2 + \left(-i\,\frac{\partial}{\partial y} - A_2\right)^2 \pm V = -\Delta_A \pm V,$$

see [3] and [10].

Our aim is to investigate spectral properties of the Pauli operators $P_\pm$. For this purpose we shall use methods from complex analysis, the weighted $\overline{\partial}$-complex. We suppose that $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ is a plurisubharmonic $\mathcal{C}^2$-function.

Let

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \{g : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ measurable} : \|g\|_\varphi^2 = (g,g)_\varphi = \int_{\mathbb{C}^n} |g|^2 e^{-\varphi}\, d\lambda < \infty\}.$$

Let $1 \leqslant q \leqslant n$ and

$$f = \sum_{|J|=q}{}' f_J \, d\overline{z}_J,$$

where the sum is taken only over increasing multiindices $J = (j_1, \dots, j_q)$ and $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ and $f_J \in L^2(\mathbb{C}^n, e^{-\varphi})$.

We write $f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$ and define

$$\overline{\partial} f = \sum_{|J|=q}{}' \sum_{j=1}^n \frac{\partial f_J}{\partial \overline{z}_j}\, d\overline{z}_j \wedge d\overline{z}_J$$

for $1 \leqslant q \leqslant n-1$ and

$$\operatorname{dom}(\overline{\partial}) = \{f \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \,:\, \overline{\partial} f \in L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi})\},$$

where the derivatives are taken in the sense of distributions.

In this way $\overline{\partial}$ becomes a densely defined closed operator and its adjoint $\overline{\partial}^*_\varphi$ depends on the weight φ.

We consider the weighted $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi})$$

and we set

$$\Box_\varphi^{(0,q)} = \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

where

$$\operatorname{dom}(\Box_\varphi^{(0,q)}) = \{u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*_\varphi) : \overline{\partial} u \in \operatorname{dom}(\overline{\partial}^*_\varphi), \overline{\partial}^*_\varphi u \in \operatorname{dom}(\overline{\partial})\}.$$

It turns out that $\Box_\varphi^{(0,q)}$ is a densely defined, non-negative self-adjoint operator, which has a uniquely determined self-adjoint square root $(\Box_\varphi^{(0,q)})^{1/2}$. The domain of $(\Box_\varphi^{(0,q)})^{1/2})$ coincides with $\operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*_\varphi)$, which is also the domain of the corresponding quadratic form

$$Q_\varphi(u,v) := (\overline{\partial} u, \overline{\partial} v)_\varphi + (\overline{\partial}^*_\varphi u, \overline{\partial}^*_\varphi v)_\varphi,$$

and $\operatorname{dom}(\Box_\varphi^{(0,q)})$ is a core of $(\Box_\varphi^{(0,q)})^{1/2}$, see for instance [4].

Next we consider the Levi matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{j,k=1}^n$$

and suppose that the lowest eigenvalue μ_φ of M_φ satisfies

$$\liminf_{|z|\to\infty} \mu_\varphi(z) > 0. \tag{2}$$

This inequality implies that $\Box_\varphi^{(0,1)}$ is injective and that the bottom of the essential spectrum $\sigma_e(\Box_\varphi^{(0,1)})$ is positive (Persson's Theorem), see [6]. Now this yields that $\Box_\varphi^{(0,1)}$ has a bounded inverse, which we denote by

$$N_\varphi^{(0,1)} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}).$$

Using the square root of $N_\varphi^{(0,1)}$ we get the basic estimates

$$\|u\|_\varphi^2 \leqslant C(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2), \tag{3}$$

for all $u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$, see [5] for more details.

In the following it will be important to know conditions on φ implying that the Bergman space of entire functions

$$A^2(\mathbb{C}^n, e^{-\varphi}) := L^2(\mathbb{C}^n, e^{-\varphi}) \cap \mathcal{O}(\mathbb{C}^n)$$

is infinite dimensional. This space coincides with $\ker\overline{\partial}$, where

$$\overline{\partial} : L^2(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}).$$

If $n = 1$, we can use the following concept. Let $D(z,r) = \{w : |z-w| < r\}$. A non-negative Borel measure μ on $\mathbb{C}$ is doubling, if there exists a constant $C > 0$ such that for any $z \in \mathbb{C}$ and any $r > 0$

$$\mu(D(z,r)) \leqslant C\mu(D(z,r/2)). \tag{4}$$

It can be shown that

$$\mu(D(z,2r)) \geq (1 + C^{-3})\mu(D(z,r)), \tag{5}$$

for each $z \in \mathbb{C}$ and for each $r > 0$; in particular $\mu(\mathbb{C}) = \infty$, unless $\mu(\mathbb{C}) = 0$ (see [9]).

Example: if $p(z,\overline{z})$ is a polynomial on $\mathbb{C}$ of degree d, then

$$d\mu(z) = |p(z,\overline{z})|^a \, d\lambda(z), \ a > -\frac{1}{d}$$

is a doubling measure on $\mathbb{C}$, see [9].

Theorem 1.1. [2], [7] *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{R}_+$ be a subharmonic $\mathcal{C}^2$-function. Suppose that $d\mu = \triangle\varphi \, d\lambda$ is a non-trivial doubling measure.*

Then the weighted space of entire functions

$$A^2(\mathbb{C}, e^{-\varphi}) = \{f \text{ entire} : \|f\|_\varphi^2 = \int_{\mathbb{C}} |f|^2 e^{-\varphi} \, d\lambda < \infty\}$$

is of infinite dimension.

More general, in $\mathbb{C}^n$, Hörmanders L^2-estimates for the solution of the inhomogeneous Cauchy-Riemann equations yield

Theorem 1.2. [8], [5] *Suppose that the lowest eigenvalue μ_φ satisfies*

$$\lim_{|z|\to\infty} |z|^2 \mu_\varphi(z) = +\infty. \tag{6}$$

Then the weighted space of entire functions

$$A^2(\mathbb{C}^n, e^{-\varphi}) = \{f \text{ entire} : \|f\|_\varphi^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda < \infty\}$$

is of infinite dimension.

Concerning compactness of the $\overline{\partial}$-Neumann operator we have the following result:

Theorem 1.3. [5] *Let $1 \leqslant q \leqslant n$. Suppose that the sum s_q of the smallest q eigenvalues of the Levi matrix M_φ satisfies*

$$\lim_{|z|\to\infty} s_q(z) = +\infty. \tag{7}$$

Then the $\overline{\partial}$-Neumann operator

$$N_\varphi^{(0,q)} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$$

is compact.

The next result asserts that compactness percolates up the $\overline{\partial}$-complex.

Theorem 1.4. [5] *Let $1 \leqslant q \leqslant n-1$. Suppose that $N_\varphi^{(0,q)}$ is compact. Then $N_\varphi^{(0,q+1)}$ is also compact.*

We will also consider special weight functions, the so-called decoupled weights, and, using the tensor product structure of the essential spectrum $\sigma_e(\Box_\varphi^{(0,q)})$ we get the following (see [1])

Theorem 1.5. *Let $\varphi_j \in \mathcal{C}^2(\mathbb{C}, \mathbb{R})$ for $1 \leqslant j \leqslant n$ with $n \geq 2$, and set*

$$\varphi(z_1, \dots, z_n) := \varphi_1(z_1) + \cdots + \varphi_n(z_n).$$

Assume that all φ_j are subharmonic and such that $\Delta\varphi_j$ defines a nontrivial doubling measure. Then

(i) $dim(ker(\Box_\varphi^{(0,0)}) = dim(A^2(\mathbb{C}^n, e^{-\varphi})) = \infty$, where $\Box_\varphi^{(0,0)} = \overline{\partial}_\varphi^* \,\overline{\partial}$,
(ii) $ker(\Box_\varphi^{(0,q)}) = \{0\}$, for $q \geq 1$,
(iii) $N_\varphi^{(0,q)}$ is bounded for $0 \leqslant q \leqslant n$,
(iv) $N_\varphi^{(0,q)}$ with $0 \leqslant q \leqslant n-1$ is not compact, and
(v) $N_\varphi^{(0,n)} = \overline{\partial}\,\overline{\partial}_\varphi^*$ is compact if and only if

$$\lim_{|z|\to\infty} \int_{B_1(z)} tr(M_\varphi)\, d\lambda = \infty,$$

where $B_1(z) = \{w \in \mathbb{C}^n : |w - z| < 1\}$.

2. Pauli operators

Now we apply the results on the weighted $\overline{\partial}$-Neumann operator to derive spectral properties of the Pauli operators and discuss some special examples.

Theorem 2.1. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a plurisubharmonic $\mathcal{C}^2$-function. Suppose that the smallest eigenvalue μ_φ of the Levi matrix M_φ satisfies*

$$\lim_{|z|\to\infty} \mu_\varphi(z) = \infty. \tag{8}$$

Let

$$A = \frac{1}{2}\left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \dots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n}\right)$$

and $V = \frac{1}{2}\Delta\varphi$. Then the Pauli operator $P_- = -\Delta_A - V$ fails to have a compact resolvent, whereas the Pauli operator $P_+ = -\Delta_A + V$ has a compact inverse operator acting on $L^2(\mathbb{R}^{2n})$.

Proof. For the proof we first consider the complex Laplacian $\Box_\varphi^{(0,0)} = \overline{\partial}_\varphi^* \,\overline{\partial}$, which acts on $L^2(\mathbb{C}^n, e^{-\varphi})$ at the beginning of the weighted $\overline{\partial}$-complex as a non-negative self-adjoint, densely defined operator, we take the maximal extension from $\mathcal{C}_0^\infty(\mathbb{C}^n)$, as $\Box_\varphi^{(0,0)}$ is essentially self-adjoint, there is only one self-adjoint extension. For $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ we get

$$\Box_\varphi^{(0,0)} f = \overline{\partial}_\varphi^* \,\overline{\partial} f = -\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\right)\frac{\partial f}{\partial \overline{z}_j}.$$

Now we apply the isometry

$$U_\varphi : L^2(\mathbb{C}^n) \longrightarrow L^2(\mathbb{C}^n, , e^{-\varphi})$$

defined by $U_\varphi(g) = e^{\varphi/2}g$, for $g \in L^2(\mathbb{C}^n)$, and afterwards the isometry

$$U_{-\varphi} : L^2(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2(\mathbb{C}^n)$$

defined by $U_{-\varphi}(f) = e^{-\varphi/2}f$, for $f \in L^2(\mathbb{C}^n, e^{-\varphi})$. Hence we get

$$e^{-\varphi/2}\Box_\varphi^{(0,0)}(e^{\varphi/2}g) = \sum_{j=1}^n \left(-\frac{\partial^2 g}{\partial z_j \partial \overline{z}_j} + \frac{1}{2}\frac{\partial\varphi}{\partial z_j}\frac{\partial g}{\partial \overline{z}_j} - \frac{1}{2}\frac{\partial\varphi}{\partial \overline{z}_j}\frac{\partial g}{\partial z_j} + \frac{1}{4}\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial \overline{z}_j} - \frac{1}{2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}g \right),$$

and separating into real and imaginary part

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial y_j}\right) \ , \ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{\partial}{\partial y_j}\right)$$

we obtain

$$e^{-\varphi/2}\Box_\varphi^{(0,0)}(e^{\varphi/2}g) = \frac{1}{4}(-\Delta_A - V)g, \tag{9}$$

where

$$A = \frac{1}{2}\left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \dots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n}\right)$$

and

$$V = 2\mathrm{tr}(M_\varphi) = \frac{1}{2}\Delta\varphi.$$

Since the kernel of $\overline{\partial} : L^2(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ coincides with the Bergman space $A^2(\mathbb{C}^n, e^{-\varphi})$ we get from (9) and the fact that (8) implies that $A^2(\mathbb{C}^n, e^{-\varphi})$ is infinite dimensional (see Theorem 1.2) that $0 \in \sigma_e(\Box_\varphi^{(0,0)})$. Hence $\Box_\varphi^{(0,0)}$ fails to be with compact resolvent.

In order to show that the Pauli operator P_+ has a compact inverse we look at the end of the weighted $\overline{\partial}$-complex.

Let $u = u\, d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ be a smooth $(0,n)$-form belonging to the domain of $\Box_\varphi^{(0,n)}$. For $1 \leqslant j \leqslant n$ denote by K_j the increasing multiindex $K_j := (1, \dots, j-1, j+1, \dots, n)$ of length $n-1$. Then

$$\overline{\partial}^*_\varphi u = \sum_{j=1}^n (-1)^{j+1}\left(\frac{\partial\varphi}{\partial z_j}u - \frac{\partial u}{\partial z_j}\right) d\overline{z}_{K_j}.$$

Hence

$$\begin{aligned}\overline{\partial}\overline{\partial}^*_\varphi u &= \left[\sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j}\left(\frac{\partial\varphi}{\partial z_j}u - \frac{\partial u}{\partial z_j}\right)\right] d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \\ &= \left[\sum_{j=1}^n \left(\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}u + \frac{\partial\varphi}{\partial z_j}\frac{\partial u}{\partial \overline{z}_j} - \frac{\partial^2 u}{\partial z_j \partial\overline{z}_j}\right)\right] d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.\end{aligned}$$

Conjugation with the unitary operator $U_{-\varphi} : L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2(\mathbb{C}^n)$ of multiplication by $e^{-\varphi/2}$ gives

$$e^{-\varphi/2}\Box_\varphi^{(0,n)}e^{\varphi/2}g = \sum_{j=1}^n \left(-\frac{\partial^2 g}{\partial z_j \partial \overline{z}_j} - \frac{1}{2}\frac{\partial\varphi}{\partial \overline{z}_j}\frac{\partial g}{\partial z_j} + \frac{1}{2}\frac{\partial\varphi}{\partial z_j}\frac{\partial g}{\partial \overline{z}_j} + \frac{1}{4}\frac{\partial\varphi}{\partial \overline{z}_j}\frac{\partial\varphi}{\partial z_j}g + \frac{1}{2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_j}g \right),$$

where $g \in L^2(\mathbb{C}^n)$ and we just wrote down the coefficient of the corresponding $(0,n)$-form. This operator can be expressed by real variables in the form

$$e^{-\varphi/2}\Box_\varphi^{(0,n)}e^{\varphi/2}g = \frac{1}{4}(-\Delta_A + V)g, \tag{10}$$

with

$$\Delta_A = \sum_{j=1}^n \left[\left(-\frac{\partial}{\partial x_j} - \frac{i}{2}\frac{\partial\varphi}{\partial y_j}\right)^2 + \left(-\frac{\partial}{\partial y_j} + \frac{i}{2}\frac{\partial\varphi}{\partial x_j}\right)^2\right],$$

and $V = 2\mathrm{tr}\,(M_\varphi)$. It follows that $-\Delta_A + V$ is a Schrödinger operator on $L^2(\mathbb{R}^{2n})$ with the magnetic vector potential

$$A = \frac{1}{2}\left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \dots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n}\right),$$

where $z_j = x_j + iy_j$, $j = 1, \dots, n$, and non-negative electric potential V in the case where φ is plurisubharmonic.

From (8) we get that $N_\varphi^{(0,1)}$ is compact (Theorem 1.3) and by Theorem 1.4 that $N_\varphi^{(0,n)}$ is compact. Finally (10) implies that the Pauli operator P_+ has a compact inverse.

□

For decoupled weights $\varphi(z_1, \dots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$ even more can be said.

Theorem 2.2. *Let $\varphi_j \in \mathcal{C}^2(\mathbb{C}, \mathbb{R})$ for $1 \leqslant j \leqslant n$ with $n \geq 1$, and set*

$$\varphi(z_1, \dots, z_n) := \varphi_1(z_1) + \cdots + \varphi_n(z_n).$$

Assume that all φ_j are subharmonic and such that $\Delta\varphi_j$ defines a nontrivial doubling measure.

Let

$$A = \frac{1}{2}\left(-\frac{\partial\varphi}{\partial y_1}, \frac{\partial\varphi}{\partial x_1}, \dots, -\frac{\partial\varphi}{\partial y_n}, \frac{\partial\varphi}{\partial x_n}\right)$$

and $V = \frac{1}{2}\Delta\varphi$. Then the Pauli operator $P_- = -\Delta_A - V$ fails to have a compact resolvent, the Pauli operator $P_+ = -\Delta_A + V$ has a compact inverse if and only if

$$\lim_{|z|\to\infty} \int_{B_1(z)} tr(M_\varphi)\, d\lambda = \infty,$$

where $B_1(z) = \{w \in \mathbb{C}^n : |w - z| < 1\}$.

Proof. By Theorem 1.1 we obtain that $A^2(\mathbb{C}^n, e^{-\varphi})$ is infinite dimensional. So, P_- fails to be with compact resolvent. The assertion about P_+ follows from Theorem 1.5. □

Example: For $\varphi(z_1, \dots, z_n) = |z_1|^2 + \cdots + |z_n|^2$ both Pauli operators P_- and P_+ fail to be with compact resolvent.

Finally, we get the following result for Dirac operators (1).

Theorem 2.3. *Let $n = 1$ and let φ be a subharmonic $\mathcal{C}^2$-function such that $\Delta\varphi$ defines a nontrivial doubling measure. Then the Dirac operator*

$$\mathcal{D} = \left(-i\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial\varphi}{\partial y}\right)\sigma_1 + \left(-i\frac{\partial}{\partial y} - \frac{1}{2}\frac{\partial\varphi}{\partial x}\right)\sigma_2,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

fails to be with compact resolvent.

Proof. By spectral analysis (see [5]) it follows that $\mathcal{D}^2$ has compact resolvent, if and only if $\mathcal{D}$ has compact resolvent. Suppose that $\mathcal{D}$ has compact resolvent. Since

$$\mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

this would imply that both Pauli operators P_- and P_+ have compact resolvent, contradicting Theorem 2.2.

□

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