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ESTIMATE FOR GROWTH AND DECAY OF FUNCTIONS IN MACINTYRE–EVGRAFOV KIND THEOREMS

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Dedicated to the centenary of corresponding member of AS USSR Alexei Fedorovich Leont'ev

Abstract. In the paper we obtain two results on the behavior of Dirichlet series on a real axis.

The first of them concerns the lower bound for the sum of the Dirichlet series on the system of segments $[\alpha, \alpha+\delta]$. Here the parameters $\alpha > 0$, $\delta > 0$ are such that $\alpha \uparrow +\infty$, $\delta \downarrow 0$. The needed asymptotic estimates is established by means of a method based on some inequalities for extremal functions in the appropriate non-quasi-analytic Carleman class. This approach turns out to be more effective than the known traditional ways for obtaining similar estimates.

The second result specifies essentially the known theorem by M.A. Evgrafov on existence of a bounded on $\mathbb{R}$ Dirichlet series. According to Macintyre, the sum of this series tends to zero on $\mathbb{R}$. We prove a spectific estimate for the decay rate of the function in an Macintyre-Evgrafov type example.

Keywords: Dirichlet series, gap-power series, asymptotic behavior.

Mathematics Subject Classification: 30D10

1. Introduction

We recall first the history of the issue. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (z = x + iy) \tag{1}$$

be an entire transcendental function with real coefficients and $\{p_n\}$ $(n \geqslant 1)$ be the sequence of sign changes of the coefficients; by the definition

$$p_n = \min_{k>p_{n-1}} \{k : a_{p_{n-1}} a_k < 0\}, \quad p_0 = \min\{k : a_k \neq 0\}.$$

For a long time the following problem coming back to work [1] by Pólya was topical: under which conditions for the sequence $\{p_n\}$ the identity

$$d(f;\, \mathbb{R}_+) = 1 \tag{2}$$

holds, where $\mathbb{R}_+$ is the positive ray $[0, \infty)$,

$$d(f;\, \mathbb{R}_+) = \varlimsup_{x\to+\infty} \frac{\ln|f(x)|}{\ln M_f(x)}, \quad M_f(x) = \max_{|z|=x} |f(z)|?$$

It should be noted that a more general analogue $d(f;\, \gamma) = 1$ of identity (2), where γ is an arbitrary curve going to infinity, was considered and established first by Pólya in work [1] for

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entire functions f of finite order represented by lacunary power series having, generally speaking, complex Taylor coefficients. Exactly this result initiated numerous studies, in which its various generalizations were obtained. But in the most general formulation, both this problem and the problem on identity (2) for entire functions with real Taylor coefficients turned out to be rather complicated.

Till the end of 90s–beginning of 2000s the following question remained open: what are the minimal restrictions for the sequence $\{p_n\}$, under which each entire function f given by series (1) with real Taylor coefficients satisfies identity (2)?

Already in work [2], M.N. Sheremeta formulated the conjecture on validity of identity (2) for any sequence $\{p_n\}$ obeying the only condition

$$\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty. \tag{3}$$

In [2] there had been even given the proof of this strong statement. However, later a gap was found in this proof. M.N. Sheremeta had not succeeded to cover this gap and he formulated it as an independent problem, which was given in various formulations in a series of issues of "Matematychni studii" (Lviv) and in other publications by Lviv mathematical society, see, for instance, [3], [4].

In [5] there was constructed a counterexample disproving the conjecture by M.N. Sheremeta. The main result of the paper gave an answer for the so-called Pólya problem.

The dual Pólya problem on behavior of entire transcendental functions of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^{p_n} \quad (0 < p_n \uparrow \infty, a_n \in \mathbb{C}) \tag{4}$$

on arbitrary curves γ going to infinity ∞ was completely solved in [6].

Let us provide this result. Let L be the class of all continuous on $\mathbb{R}_+$ functions $w = w(x)$, $0 < w(x) \uparrow \infty$ as $x \to \infty$. By W we denote the set of all functions w in L such that $w(x)x^{-2}$ is integrable on $[1, \infty)$. In [6] the following statement was proved; here we provide an equivalent formulation: *For each function f of form (4), for each curve γ going to infinity ∞ the identity $d(f;\gamma) = 1$ holds true if and only if the sequence $P = \{p_n\}$ satisfies the conditions:*

$$1)\ \sum_{n=1}^{\infty} \frac{1}{p_n} < \infty; \quad 2)\ I_P(p_n) = \int\limits_0^{p_n} \frac{\mu_P(p_n;\, t)}{t}\, dt \leqslant w(p_n)\, (n \geqslant 1),$$

where $\mu_P(p_n;\, t)$ is the amount of the points $p_k \neq p_n$ in the segment $\{h : |h - p_n| \leqslant t\}$, w is some function in W.

If $\gamma = \mathbb{R}_+$, Conditions 1) and 2) are the criterion for the validity of the identity

$$\ln M_f(x) = (1 + o(1)) \ln |f(x)|, \quad M_f(x) = \max_{|z|=x} |f(z)|. \tag{5}$$

as $x \to \infty$ outside some set $e \subset \mathbb{R}_+$ of zero logarithmic density [7][1] for each function (4).

Finally, we note that in [8] there was solved a more general problem related with the Pólya conjecture on the minimum of the absolute value: conditions 1) and 2) are the criterion for the validity of the identity

$$\ln M_f(x) = (1 + o(1)) \ln m_f(x)$$

as $x \to +\infty$ outside some set $e \subset \mathbb{R}_+$ of a finite logarithmic measure for each function f of form (4); here

$$m_f(x) = \min_{z \in C_x} |f(z)|,$$

[1]That is, outside e, $\int\limits_{e\cap[1,\,r)} \frac{dt}{t} = o(\ln r)$ as $r \to \infty$. If $\int\limits_{e} \frac{dt}{t} < \infty$, one says that the set e has a finite logarithmic measure.

and C_x is some closed contour obtained from the circumference $\{z : |z| = x\}$ by a "small deformation" [8].

Here we provide the formulations of the main results of works [5]–[8] but applied for lacunary power series (4). In these works, in fact, the corresponding problems are considered for more general series, Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad (0 < \lambda_n \uparrow \infty\ , s = \sigma + it), \tag{6}$$

converging absolutely in the entire plane; here we assume that not all the coefficients of the series vanish and the sequences of the exponents has a finite upper density.

As it is known, under the condition

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \tag{7}$$

the sum F of each series (6) is unbounded on $\mathbb{R}_+$ [9].

For natural λ_n the opposite statement is true; this follows the results by Macintyre [10]. For sequences $\Lambda = \{\lambda_n\}$ having a finite condensation index

$$\delta = \varlimsup_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|Q'(\lambda_n)|}, \quad Q(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$$

a similar statement was proven by N.N. Yusupova [11].

If $n(r) \sim cr^{\rho(r)}$ as $r \to \infty$, where $c \neq 0, \infty$, $\rho(r)$ is the specified order, $n(r)$ is the counting function of the sequence Λ, and $-\ln|Q^{'}(\lambda_n)| = O(n)$ as $n \to \infty$, the corresponding example was constructed in [9].

If only condition (7) holds, nothing is known about the asymptotics of the sum of series (6) even on $\mathbb{R}_+$. We can just state that $0 \leqslant d(F;\, \mathbb{R}_+) \leqslant 1$ [6], where

$$d(F;\, \mathbb{R}_+) = \varlimsup_{\sigma\to+\infty} \frac{|F(\sigma)|}{M_F(\sigma)}, \quad M_F(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)|.$$

As it was shown in [6], the estimates $0 \leqslant d(F;\, \mathbb{R}_+) \leqslant 1$ are sharp; $d(F;\, \mathbb{R}_+) = 1$ if I_Λ obeys the condition of type 2); otherwise there exists series (6), for which $d(F;\, \mathbb{R}_+) = 0$.

In connection with this, a question naturally arises: the logarithm of which unboundedly increasing "good" function preferably defined in terms of the coefficients and exponents of series (6) is the optimal minorant for the logarithm of the absolute value of the sum of this series at least on some dense sequence of points $\sigma_n \in \mathbb{R}_+$, $\sigma_n \to +\infty$?

A similar problem for the curves $\gamma = \{z = t + ig(t),\ 0 \leqslant t < \infty\}$ of a bounded slope was studied in [12]. In the case $\gamma = \mathbb{R}_+$ we can obtain a corresponding result but much simpler if we apply the properties of the extremal functions in a non-quasi-analytic Carleman class. As in [12], we assume that the sequence $\Lambda = \{\lambda_n\}$ obeys the condition: there exist numbers $\mu_n > 0$ such that

$$\lambda_n \geqslant \mu_n \ \ (n \geqslant 1), \quad \frac{n}{\mu_n} \downarrow, \quad \sum_{n=1}^{\infty} \frac{1}{\mu_n} < \infty. \tag{8}$$

As it is known (see [13]), the group of conditions (8) is stronger than condition (7).

2. Lower bound on $\mathbb{R}_+$ for the growth rate of the Dirichlet series

The following theorem holds.

Theorem 1. *Assume that conditions (8) hold. Then there exists a sequence $\{\sigma_n\}$, $0 < \sigma_n \uparrow \infty$, $\sigma_{n+1} - \sigma_n \to 0$ such that*

$$\ln \mu^*(\sigma_n) \leqslant (1 + o(1)) \ln |F(\sigma_n)|. \tag{9}$$

as $n \to \infty$. Here F is the sum of Dirichlet series (6), $\mu^*(\sigma) = \max_{n\geqslant 1}\{|a_n||Q'(\lambda_n)|e^{\lambda_n\sigma}\}$.

It is well-known that $\ln \mu^*(\sigma)$ is a convex function and $\ln \mu^*(\sigma) \uparrow \infty$ as $\sigma \to +\infty$ [14]; $\mu^*(\sigma)$ is the maximal term of the changed Dirichlet series.

The meaning of estimate (9) is that its left hand side is a convex function depending only the coefficients and the sequence Λ of the exponents of series (6) and it can be explicitly calculated. It was shown in [12] that under the assumptions of Theorem 1, estimate (9) can not be improved.

We note that Theorem 1 was proven in another way in [12], while the dual theorem on the Dirichlet series with real coefficients was established in [15]. The method of proving the corresponding theorem in [15] turns out to be applicable also in the case we consider here.

Proof of Theorem 1. Assume that condition (7) holds. Then $\Omega(M_n') \neq \{0\}$, where

$$\Omega(M_n') = \{\psi : \psi \in C^\infty[0,1], \psi(x) \geqslant 0, \psi^{(n)}(0) = 0, \sup_{0\leqslant t\leqslant 1} |\psi^{(n)}(t)| \leqslant M_n' \quad (n \geqslant 0)\},$$

$M_n' = M_{n-2}^c$, $(n \geqslant 3)$, $M_i' = 1$, $(i = 0, 1, 2)$. As in [15], we show that for each δ, $0 < \delta \leqslant 1$, each function $\psi \in \Omega(M_n')$ there exists an entire function of the form

$$Q_\delta(\lambda) = \int_0^\delta \varphi_\delta(t)e^{\lambda t}\,dt,$$

such that $Q_\delta(\lambda_n) = 0$, $Q_\delta'(\lambda_n) \neq 0$, $\sup_{0<\delta\leqslant 1}\max_{0\leqslant t\leqslant 1} |\varphi_\delta(t)| \leqslant c_0 < \infty$, and

$$|Q_\delta(\lambda)| \geqslant |Q(\lambda)| \int_0^{\delta/12} \psi(x)\,dx. \tag{10}$$

Now we introduce the Leont'ev interpolating function. Since the adjoint diagram of the function Q_δ is the segment $[0,\delta]$, we denote, for convenience, $\omega_{Q_\delta}(\mu,\alpha,F) = \omega(\mu,\alpha,F)$, $\varphi_\delta(t)\,dt = d\sigma(t)$, where α is a complex parameter, F is the sum of series (6), and we have [14]:

$$\omega(\mu,\alpha,F) = e^{-\alpha\mu}\int_C \gamma(t)\left(\int_0^t F(t+\alpha-\eta)e^{\mu\eta}\,d\eta\right)dt, \tag{11}$$

where C is a closed rectifiable contour enveloping the segment $[0,\delta]$, γ is the function associated with Q_δ in the Borel sense, which in the considered case is of the form:

$$\gamma(t) = \int_0^\delta \frac{d\sigma(\xi)}{t-\xi}.$$

Letting

$$f(t) = \int_0^t F(t+\alpha-\eta)e^{\mu\eta}\,d\eta,$$

we obtain

$$\omega(\mu,\alpha,F) = e^{-\alpha\mu}\frac{1}{2\pi i}\int_C \left(\int_0^\delta \frac{d\sigma(\xi)}{t-\xi}\right) f(t)\,dt = e^{-\alpha\mu}\int_0^\delta d\sigma(\xi)\left(\frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-\xi}\right).$$

The function f is regular on C and inside C since F is an entire function. This is why the inner integral is equal to $f(\xi)$ and hence

$$\omega(\mu,\alpha,F)=e^{-\alpha\mu}\int_0^{\delta}d\sigma(\xi)\left(\int_0^{\xi}F(\xi+\alpha-\eta)e^{\mu\eta}\,d\eta\right). \tag{12}$$

We employ the well-known formulae for coefficients [14]:

$$a_n=\frac{\omega(\lambda_n,\alpha,F)}{Q'_\delta(\lambda_n)}\quad (n\geqslant 1). \tag{13}$$

Dividing both sides of inequality (10) by $\lambda-\lambda_n$ and passing to the limit as λ tends to λ_n, we clearly obtain that

$$|Q'_\delta(\lambda_n)|\geqslant |Q'(\lambda_n)|\int_0^{\delta/12}\psi(x)\,dx\quad (n\geqslant 1). \tag{14}$$

Letting $\alpha=\sigma+\delta$ in formula (12), let us estimate $|\omega(\lambda_n,\alpha,F)|$. At that, we take into consideration that the variable $\xi+\alpha-\eta$ belongs to the segment $I_\sigma=[\sigma+\delta,\sigma+2\delta]$. We have

$$|\omega(\lambda_n,\alpha,F)|\leqslant c_0\delta^2e^{-\sigma\lambda_n}\max_{u\in I_\sigma}|F(u)|\quad (u\geqslant 1). \tag{15}$$

Therefore, taking into consideration (14), (15), by (13) we obtain

$$|a_n||Q'(\lambda_n)|e^{\lambda_n\sigma}\int_0^{\delta/12}\psi(x)\,dx\leqslant c_0\delta^2\max_{u\in I_\sigma}|F(u)|\quad (n\geqslant 1).$$

Taking the maximum in n in the right hand side, we get

$$\mu^*(\sigma)\int_0^{\delta/12}\psi(x)\,dx\leqslant c_0\delta^2|F(\sigma')|, \tag{16}$$

where σ' is some point in the segment I_σ, $0<\sigma\leqslant 1$.

Now we are going to estimate the functional

$$J_\delta(\psi)=\int_0^{\delta/12}\psi(x)\,dx$$

from below. In order to do it, we apply a mean theorem:

$$J_\delta(\psi)\geqslant\int_{\delta/24}^{\delta/12}\psi(x)\,dx=\frac{\delta}{24}\psi(\beta),$$

where β is a point in the segment $[\frac{\delta}{24},\frac{\delta}{12}]$. Therefore, by (16) we obtain

$$\psi(\beta)\mu^*(\sigma)\leqslant 24c_0|F(\sigma')|.$$

The right hand side of this inequality is independent of ψ. Hence, passing to the extremal function I in the left hand side, we finally obtain

$$I(\beta)\mu^*(\sigma)\leqslant 24c_0|F(\sigma')|,\quad \sigma'\in I_\sigma. \tag{17}$$

Let us employ a lemma similar to Lemma 9 in [15] (it can be proved exactly in the same way): *If conditions (8) hold, there exists a constant N depending on the sequence $\{M'_n\}$ such that*

$$I(\beta) \geqslant \frac{1}{NH^2(\frac{\beta}{4})}, \quad \beta \in \left[\frac{\delta}{24}, \frac{\delta}{12}\right] \quad (0 < \delta \leqslant 1),$$

where H is a function defined by the formula

$$H(y) = \sum_{n=0}^{\infty} \frac{n!}{M'_n y^{n+1}} \quad (0 < y < \infty).$$

As it is known, and we have used this above, conditions (8) are equivalent to (see [13])

$$\int_0^c \ln\ln H(\delta)\, d\delta < \infty, \tag{18}$$

where $c > 0$ is a sufficiently small constant such that $H(c) \geqslant e$.
Thus, by (17) we obtain

$$\mu^*(\sigma) \leqslant m(\delta)|F(\sigma')|, \tag{19}$$

where σ' is some point in the segment $[\sigma + \delta, \sigma + 2\delta]$ and

$$m(\delta) = 24c_0 NH^2\left(\frac{\delta}{96}\right).$$

The function $m(\delta)$ obviously satisfies bi-logarithmic Levinson condition (18). We choose δ as the unique solution to the equation

$$m(\delta) = e^{V(\sigma)}, \tag{20}$$

where $V(\sigma) = [\ln \mu^*(\sigma)]/[\ln\ln \mu^*(\sigma)]$. It is clear that the function $V(\sigma)$ is continuous as $\sigma \geqslant \sigma_0$ and $V(\sigma) \uparrow \infty$ as $\sigma \to \infty$. We rewrite equation (20) as

$$\ln\ln m(\delta) = \ln V(\sigma) \stackrel{def}{=} U(\sigma)$$

and by $K = K(t)$ we denote the inverse function for $t = \ln\ln m(\delta)$. Then $K(U(\sigma)) = \delta$. It is clear that the function $K(t)$ is continuous, $K(t) \downarrow 0$ as $t \to \infty$. Since $m(\delta)$ satisfies condition (18), then

$$\int_{u(\sigma_0)}^{\infty} K(u)\, du < \infty$$

that can checked straightforwardly. Therefore, applying Borel-Nevalinna lemma (see [16]), we obtain that for each $\varepsilon > 0$, for each $\sigma \geqslant \sigma_0$ but outside some set

$$F \subset \bigcup_{i=1}^{\infty} [a_i, a'_i], \quad mF \leqslant \sum_{i=1}^{\infty} (a'_i - a_i) < \infty,$$

the estimate

$$U[\sigma + 2K(U(\sigma))] < u(\sigma) + \varepsilon.$$

holds true [17]. This estimate can be improved, see, for instance, [16, Sect. 1], [18, Lm. 6]: *there exists an exceptional set $E \subset [\sigma_0, \infty)$ covered by a system of segments of a finite total length* [16] *such that outside this set*

$$u[\sigma + 2K(u(\sigma))] < u(\sigma) + o(1)$$

as $\sigma \to \infty$. Hence, taking into consideration the increasing of the function $V(\sigma)$, as $\sigma \to \infty$ outside the set E we have

$$\ln \mu^*(\sigma + 2\delta) < \ln \mu^*(\sigma) + o(1). \tag{21}$$

Therefore, taking into consideration (20), (21), by (19) we finally obtain that as $\delta \to \infty$, outside E,

$$(1 + o(1)) \ln \mu^*(\sigma') \leqslant \ln |F(\sigma')|, \tag{22}$$

where $\sigma + \delta \leqslant \sigma' \leqslant \sigma + 2\delta$, $\delta = K(u(\sigma))$. Since the set E is covered by a system of segments of a finite total length, estimate (22) holds on some sequence $\{\sigma_n\}$, $0 < \sigma_n \uparrow \infty$, $\sigma_{n+1} - \sigma_n \to 0$. The proof is complete. □

3. Existence of Dirichlet series fast decaying on $\mathbb{R}_+$[1]

Let $0 < \lambda_k \uparrow \infty$, $\sum\limits_{k=1}^{\infty} \lambda_k^{-2} < \infty$. Then the entire function

$$E(\lambda) = e^{b\lambda} \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) e^{\frac{\lambda}{\lambda_k}} \quad (b \in \mathbb{R}) \tag{23}$$

possesses the properties [19, Ch. I, Sect. 8; Ch. III, Sect. 6; Ch. V, Sect. 3]:

1) as $\operatorname{Re} \lambda < \lambda_1$

$$\frac{1}{E(\lambda)} = \int_{-\infty}^{\infty} G(t) e^{-\lambda t} dt,$$

where

$$G(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{E(\lambda)} e^{\lambda t} dt;$$

2) $G \in C^\infty(\mathbb{R})$ and $G(t) \geqslant 0$, $\int\limits_{-\infty}^{\infty} G(t) dt = 1$;

3) $G^{(n)}(t)$ $(n \geqslant 1)$ changes the sign n times on $\mathbb{R}$, while $G(t)$ is sign-definite;

4) $-\ln G(t)$ is a convex function on $\mathbb{R}$;

5) if $\sum\limits_{k=1}^{\infty} \lambda_k^{-1} < \infty$, $b = -\sum\limits_{k=1}^{\infty} \lambda_k^{-1}$, then $G(t) > 0$ as $t < 0$; $G(t) \equiv 0$ as $t \geqslant 0$;

6) if $\sum\limits_{k=1}^{\infty} \lambda_k^{-1} = \infty$, then the function $G(t)$ is the restriction of the entire function $G(z)$, $z = t + iy$ on $\mathbb{R}$ [19, Ch. I, Sect. 4].

It was shown in [19] that if $\sum\limits_{k=1}^{\infty} \lambda_k^{-2} < \infty$, but $\sum\limits_{k=1}^{\infty} \lambda_k^{-1} = \infty$, then

$$G^{(n)}[\lambda(r)] \sim \frac{1}{\sqrt{2\pi}} (-r)^n \Lambda(r) \quad (n = 0,\ 1,\ \ldots), \tag{24}$$

as $r \to +\infty$, where

$$\Lambda(r) = \frac{e^{-r\lambda(r)}}{\sigma(r) E(-r)},$$

and

$$\lambda(r) = \sum_{k=1}^{\infty} \frac{r}{\lambda_k(\lambda_k + r)}, \quad \sigma(r) = \left[\sum_{k=1}^{\infty} \frac{1}{(\lambda_k + r)^2}\right]^{\frac{1}{2}}.$$

We proceed to constructing the example. Let E be the function (23), $b = 0$,

$$L(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_k^2}\right).$$

[1] The results of this section belong to G.A. Gaisina.

Assume that the sequence $\{\lambda_n\}$ has a finite upper density τ and a finite condensation index δ, see [14]. We consider the Dirichlet series

$$F(z) = \sum_{k=1}^{\infty} \frac{e^{\lambda_k z}}{E'(\lambda_k)},\ z = t + iy. \tag{25}$$

We have

$$E'(\lambda_n) = -\frac{1}{\lambda_n} \prod_{k\neq n} \left(1 - \frac{\lambda_n}{\lambda_k}\right) e^{\frac{\lambda_n}{\lambda_k}}.$$

Hence,

$$\frac{1}{E'(\lambda_n)} = 2\frac{\prod\limits_{k\neq n} \left(1 + \frac{\lambda_n}{\lambda_k}\right) e^{-\frac{\lambda_n}{\lambda_k}}}{L'(\lambda_n)}.$$

Since $\delta < \infty$, then for some $C > 0$

$$\left|\frac{1}{E'(\lambda_n)}\right| \leqslant 2e^{C\lambda_n} E(-\lambda_n) \quad (n \geqslant 1).$$

And since $\sum\limits_{k=1}^{\infty} \lambda_k^{-1} = \infty$, then, in addition,

$$\lim_{n\to\infty} \frac{\ln|E(-\lambda_n)|}{\lambda_n} = -\infty \tag{26}$$

that can be checked in a usual way. This implies that series (25) converges absolutely in the entire plane and defines as entire function F.

Let us confirm that $F(t) \equiv G(t)$, $t \in \mathbb{R}$. We first observe that as $\frac{\pi}{4} \leqslant |\arg\lambda| \leqslant \frac{\pi}{2}$ and $a \geqslant 0$

$$I = \left|\left(1 - \frac{\lambda}{a}\right) e^{\frac{\lambda}{a}}\right| \geqslant 1. \tag{27}$$

Indeed, if $\lambda = |\lambda|e^{i\psi}$, then by letting $r = \frac{|\lambda|}{a}$ we have

$$\ln I = \frac{1}{2}[\ln\left(1 - 2r\cos\psi + r^2\right) + 2r\cos\psi] \quad (r \geqslant 0).$$

The quantity $\alpha = 2\cos\psi$, $\left(\frac{\pi}{4} \leqslant \psi \leqslant \frac{\pi}{2}\right)$ ranges in the segment $[0, \sqrt{2}]$. The function $g(\alpha) = \ln I$ at the end-points of this segment is non-negative, and $g'(\alpha) = 0$ at the point $\alpha = r$, where it attains a local maximum. Hence, $I \geqslant 1$.

In view of (27), in the angles

$$\Delta_\pm = \{\lambda = |\lambda|e^{i\psi} : |\lambda| > 0, \frac{\pi}{4} \leqslant |\psi| \leqslant \frac{\pi}{2}\},$$

Δ_+ is the upper angle, Δ_- is the lower one, we have

$$P = \left|\prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) e^{\frac{\lambda}{\lambda_k}}\right| \geqslant \left|\prod_{k=1}^{n} \left(1 - \frac{\lambda}{\lambda_k}\right) e^{\frac{\lambda}{\lambda_k}}\right| \quad (n = 1,\ 2,\ \ldots).$$

By this we see that for each $n \geqslant 1$

$$P \geqslant B\lambda^n, \quad \lambda \in \Delta_\pm.$$

Therefore, by the Cauchy theorem,

$$G(t) = \frac{1}{2\pi i} \int\limits_C \frac{e^{\lambda t}}{E(\lambda)} d\lambda \quad (t \geqslant 0), \tag{28}$$

where C is the boundary of the angle

$$\{\lambda : |\lambda| > 0, \arg\lambda = \pm\frac{\pi}{4}\}.$$

Since $E(\lambda)$ is an entire function of exponential type ($\tau < \infty$), we can make use of the following statement [14, Ch. I, Sect. 1, Thm. 1.19]:

Given $q > 1$, there exists a number $h > 0$ and circumferences $C_n = \{z : |z| = r_n\}$, $r_n \uparrow \infty$, $r_{n+1} < qr_n$, $(n = 1,\ 2,\ \ldots)$, on which

$$\ln|E(\lambda)| > -h|\lambda|,\ |\lambda| = r_n \quad (n \geqslant 1). \tag{29}$$

By (28), for $\lambda \in C$, $t \leqslant -\sqrt{2}(1+h)$ we have

$$\left|\frac{r^{\lambda t}}{E(\lambda)}\right| \leqslant e^{h|\lambda| + \frac{\sqrt{2}}{2}|\lambda|t} \leqslant e^{-|\lambda||t|}.$$

Hence, for such t integral (28) over the circumference C_n tends to zero $n \to \infty$. Therefore, as $t \leqslant -\sqrt{2}(1+h)$,

$$G(t) = \lim_{k\to\infty}\left(\frac{1}{2\pi i}\int\limits_{\Gamma_k}\frac{e^{\lambda t}}{E(\lambda)}d\lambda\right) = \lim_{k\to\infty}\sum_{\lambda_n < r_k}\frac{e^{\lambda_n t}}{E'(\lambda_n)},$$

where Γ_k is the boundary of the sector $\{\lambda : 0 \leqslant |\lambda| \leqslant r_k,\ |\arg\lambda| \leqslant \frac{\pi}{4}\}$ passed counterclockwise. But series (25) converges absolutely on the entire plane and its sum F is an entire function. As it has been said, $G(t)$ is the restriction of the entire function $G(z)$, $(z = t + iy)$ on $\mathbb{R}$. Thus, $G(t)$ is represented by series (25) as $t < -\sqrt{2}(1+h)$. This implies that $F(z) \equiv G(z)$ in the entire plane and

$$G(z) = \sum_{k=1}^{\infty}\frac{e^{\lambda_k z}}{E'(\lambda_k)}.$$

Let us find out a sharp asymptotics for $G(t)$ as $t \to +\infty$. In order to do it, we employ relation (24).

We obviously have:

$$E(\lambda) = \frac{L(\lambda)}{E(-\lambda)}.$$

This implies that

$$E'(\lambda_n) = \frac{L'(\lambda_n)}{E(-\lambda_n)}.$$

Thus, we can write

$$G(t) = \sum_{n=1}^{\infty}\frac{E(-\lambda_n)}{L'(-\lambda_n)}e^{\lambda_n t},\ t \in \mathbb{R}.$$

We consider the maximal term of the changed series $\sum\limits_{n=1}^{\infty} E(-\lambda_n)e^{\lambda_n t}$, that is,

$$\mu^*(t) = \max_{n\geqslant 1}[E(-\lambda_n)e^{\lambda_n t}].$$

This function is well-defined since the changed series, by (26), also converges absolutely in the entire plane.

We have

$$\ln\mu^*(t) = \max_{n\geqslant 1}\left\{\sum_{k=1}^{\infty}\left[\ln\left(1+\frac{\lambda_n}{\lambda_k}\right) - \frac{\lambda_n}{\lambda_k}\right] + \lambda_n t\right\} \leqslant \max_{r\geqslant 0}\varphi(r),$$

where

$$\varphi(r) = \sum_{k=1}^{\infty}\left[\ln\left(1+\frac{r}{\lambda_k}\right) - \frac{r}{\lambda_k}\right] + rt.$$

Since $\varphi(0) = 0$, $\varphi(+\infty) = -\infty$, then the maximum of this function is attained at the point, where

$$\varphi'(r) = -\sum_{k=1}^{\infty} \frac{r}{\lambda_k(\lambda_k + r)} + t = 0,$$

that is, at the point r, $\lambda(r) = t$; the function $\lambda = \lambda(r)$ was introduced above. In view of this relation and (24), we obtain that as $t = \lambda(r) \to +\infty$

$$a(r) = \frac{\ln G(\lambda(r))}{\ln \mu^*(t)} \sim \frac{\ln \Lambda(r)}{\ln \mu^*(t)}, \tag{30}$$

where

$$\ln \Lambda(r) = -r\lambda(r) - \ln \sigma(r) - \ln E(-r).$$

Taking into consideration that $t = \lambda(r)$,

$$\ln \mu^*(t) \leqslant \sum_{k=1}^{\infty} \left[\ln \left(1 + \frac{r}{\lambda_k}\right) - \frac{r}{\lambda_k} \right] + r\lambda(r),$$

by (30) we obtain the estimate $a(r) \leqslant b(r)$, and as $r \to +\infty$,

$$b(r) \sim -\frac{r\lambda'(r) + \frac{\sigma'(r)}{\sigma(r)}}{r\lambda'(r)} = -[1 + \varepsilon(r)],$$

where

$$\varepsilon(r) = \frac{\sigma'(r)}{r\lambda'(r)\sigma(r)}.$$

As one can check easily, $\lambda'(r) = \sigma^2(r)$. This yields that

$$2\sigma'(r)\sigma(r) = -\sum_{k=1}^{\infty} \frac{2}{(\lambda_k + r)^3},$$

that is,

$$\sigma'(r) = -\frac{1}{\sigma(r)} \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + r)^3}.$$

Therefore,

$$|\varepsilon(r)| \leqslant r^{-2} \left[\sum_{k=1}^{\infty} \frac{1}{(\lambda_k + r)^2} \right]^{-1} \leqslant r^{-2} \left[\sum_{\lambda_k \leqslant 2r} \frac{1}{(\lambda_k + r)^2} \right]^{-1}.$$

This yields that

$$|\varepsilon(r)| \leqslant \frac{9}{n(2r)} \to 0$$

as $r \to \infty$. Thus,

$$\varlimsup_{t \to +\infty} \frac{\ln G(t)}{\ln \mu^*(t)} \leqslant -1, \tag{31}$$

and hence, we have proved the following theorem.

Theorem 2. *Let $\{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ be an arbitrary sequence possessing a finite upper density τ and a finite condensation index δ. If $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$, then Dirichlet series (25) converges absolutely in the entire plane, $G(t) \to 0$ as $t \to +\infty$ and estimate (31) holds true.*

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