ISSN 2304-0122 Ufa Mathematical Journal. Vol. 9. No 2 (2017). P. 55-61.

doi:10.13108/2017-9-2-55 UDC 517.5

LOWER BOUNDS FOR THE AREA OF THE IMAGE OF A CIRCLE

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Abstract. In the work we consider Q-homeomorphisms w.r.t p-modulus on the complex plane as $p > 2$. We obtain a lower bound for the area of the image of a circle under such mappings. We solve the extremal problem on minimizing the functional of the area of the image of a circle.

Keywords: p-modulus of a family of curves, p-capacity of condenser, quasiconformal mappings, Q-homeomorphisms w.r.t. p-modulus.

Mathematics Subject Classification: 3065

1. Introduction

The problem on area deformations under quasi-conformal mappings originates from work by B. Bojarskii [1]. A series of results in this direction were obtained in works [2]–[4].

First an upper bound for the are of the image of a circle under quasi-conformal mappings was provided in monograph by M.A. Lavrent'ev, see [5]. In [6, Prop. 3.7], the Lavrentiev's inequality was specified in terms of the angular dilatation. Also earlier in works [7]–[8] there were obtained the upper bounds for the area deformation for annular and lower and Q-homeomorphisms. In the present work we obtain lower bounds for the area of the image of a circle under Q-homeomorphisms w.r.t. p-modulus as $p > 2$.

To simplify the presentation, we restrict ourselves by the planar case. We recall some definitions. Assume that we are given a family Γ of curves γ in the complex plane $\mathbb{C}$. A Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is called *admissible* for Γ, which is written as $\varrho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \varrho(z)\,|dz| \;\geqslant\; 1 \qquad \forall\, \gamma \in \Gamma. \tag{1}$$

Let $p \in (1, \infty)$. Then a *p-modulus* of the family Γ is the quantity

$$\mathcal{M}_p(\Gamma) \;=\; \inf_{\varrho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{C}} \varrho^p(z)\,dm(z)\,. \tag{2}$$

Assume that D is a domain in the complex plane $\mathbb{C}$, that is, an open connected subset $\mathbb{C}$ and $Q : D \to [0, \infty]$ is a measurable function. A homeomorphism $f : D \to \mathbb{C}$ is called a Q-homeomorphism w.r.t. p-modulus if

$$\mathcal{M}_p(f\Gamma) \;\leqslant\; \int\limits_{D} Q(z)\,\varrho^p(z)\,dm(z) \tag{3}$$

for each family Γ of curves in D and each admissible function ϱ for Γ.

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Submitted June 16, 2016.

The study of the inequalities of type (3) as $p = 2$ goes back to L. Ahlfors, see, for instance, [9, Ch. I, Sect. D, Thm. 3] as well as to O. Lehto and K. Virtanen [10, Ch. V, Sect. 6.3, Ineq. (6.6)]. In work [11] by C.J. Bishop, V.Ya. Gutlaynskii, O. Martio, M. Vuorinen, a multi-dimensional analogue of inequality (3) was proved for quasi-conformal mappings.

We also note that if the function Q in (3) is bounded almost everywhere by some constant $K \in [1, \infty)$ and $p = 2$, then we arrive at classical quasi-conformal mappings introduced originally in works by Grötzsch, Lavrentiev and Morrey.

Let $Q : D \to [0, \infty]$ be a measurable function. For each number $r > 0$ we denote by

$$q_{z_0}(r) = \frac{1}{2\pi\, r} \int\limits_{S(z_0,r)} Q(z)\,|dz|$$

the integral mean of the function Q over the circle $S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

Theorem 1. *Let D and D' be bounded domains in $\mathbb{C}$ and $f : D \to D'$ be a Q-homeomorphism w.r.t. p-modulus, $p > 2$, $Q \in L^1_{\mathrm{loc}}(D \setminus \{z_0\})$. Then for all $r \in (0, d_0)$, $d_0 = \mathrm{dist}(z_0, \partial D)$ the esimate*

$$|fB(z_0, r)| \geqslant \pi \left(\frac{p-2}{p-1}\right)^{\frac{2(p-1)}{p-2}} \left(\int\limits_0^r \frac{dt}{t^{\frac{1}{p-1}}\, q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{\frac{2(p-1)}{p-2}} \tag{4}$$

holds true, where $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$.

We note that as $p > 2$ and $Q(z) \leqslant K$, by Theorem 1 we arrive to the result for a circle in [12, Lm. 7].

2. Proof of main theorem

We provide some auxiliary information about the capacity of a condenser. Following work [13], the pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{C}$ is an open set and C is a non-empty compact set contained in A is called *condenser*. A condenser $\mathcal{E}$ is called an *annular condenser* if $\mathfrak{R} = A \setminus C$ is an annulus, that is, if $\mathfrak{R}$ is a domain whose complement $\overline{\mathbb{C}} \setminus \mathfrak{R}$ consists exactly of two components. A condenser $\mathcal{E}$ is called a *bounded condenser* if the set A is bounded. We also say that a condenser $\mathcal{E} = (A, C)$ lies in the domain D if $A \subset D$. It is obvious that if $f : D \to \mathbb{C}$ is a continuous open mapping and $\mathcal{E} = (A, C)$ is a condenser in D, then (fA, fC) is also a condenser in fD. We also have $f\mathcal{E} = (fA, fC)$.

Let $\mathcal{E} = (A, C)$ be a condenser. By $\mathcal{C}_0(A)$ we denote the set of continuous compactly supported functions $u : A \to \mathbb{R}^1$, by $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ we denote the family of non-negative functions $u : A \to \mathbb{R}^1$ such that

1) $u \in \mathcal{C}_0(A)$,
2) $u(x) \geqslant 1$ for $x \in C$,
3) u belongs to the class ACL.

As $p \geqslant 1$, the quantity

$$\mathrm{cap}_p\, \mathcal{E} = \mathrm{cap}_p\,(A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int\limits_A |\nabla u|^p\, dm(z)\,, \tag{5}$$

where

$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} \tag{6}$$

is called a *p-capacity* of the condenser $\mathcal{E}$. In what follows we shall make use the identity

$$\operatorname{cap}_p \mathcal{E} = \mathcal{M}_p(\Delta(\partial A, \partial C; A \setminus C)) \tag{7}$$

established in work [14], where for the sets $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ in $\mathbb{C}$, the symbol $\Delta(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F})$ stands for the family of all continuous curves connecting $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{F}$.

It is known [15, Prop. 5] that as $p \geqslant 1$,

$$\operatorname{cap}_p \mathcal{E} \geqslant \frac{[\inf l(\sigma)]^p}{|A \setminus C|^{p-1}}\,. \tag{8}$$

Here $l(\sigma)$ is the length of a smooth (infinitely differentiable) curve σ being the boundary $\sigma = \partial U$ of a bounded open set U containing C and contained together with its closure $\overline{U}$ in A and the infimum is taken over all such σ.

Proof of Theorem 1. Let $\mathcal{E} = (A, C)$ be a condenser, where $A = \{z \in D : |z - z_0| < t + \Delta t\}$, $C = \{z \in D : |z - z_0| \leqslant t\}$, $t + \Delta t < d_0$. Then $f\mathcal{E} = (fA, fC)$ is an annular condenser in D' and according to (7) we have the identity

$$\operatorname{cap}_p f\mathcal{E} = \mathcal{M}_p\left(\Delta(\partial fA, \partial fC; f(A \setminus C)\right)\,. \tag{9}$$

By inequality (8) we obtain

$$\operatorname{cap}_p f\mathcal{E} \geqslant \frac{[\inf\ l(\sigma)]^p}{|fA \setminus fC|^{p-1}}\,. \tag{10}$$

Here $l(\sigma)$ is the length of a smooth (infinitely differentiable) curve σ being the boundary $\sigma = \partial U$ of a bounded open set U containing C and contained together with its closure $\overline{U}$ in A and the infimum is taken over all such σ.

On the other hand, by the definition of Q-homemorphism w.r.t. p-modulus we have

$$\operatorname{cap}_p f\mathcal{E} \leqslant \int\limits_D Q(z)\,\varrho^p(z)\,dm(z) \tag{11}$$

for each $\varrho \in \operatorname{adm} \Delta(\partial A, \partial C; A \setminus C)$.

It is easy to check that the function

$$\varrho(z) = \begin{cases} \dfrac{1}{|z - z_0|\,\ln\frac{t+\Delta t}{t}}, & z \in A \setminus C \\ 0, & z \notin A \setminus C \end{cases}$$

is admissible for the family $\Delta(\partial A, \partial C; A \setminus C)$ and hence,

$$\operatorname{cap}_p f\mathcal{E} \leqslant \frac{1}{\ln^p\left(\frac{t+\Delta t}{t}\right)} \int\limits_R \frac{Q(z)}{|z - z_0|^p}\,dm(z), \tag{12}$$

where $R = \{z \in D : t \leqslant |z - z_0| \leqslant t + \Delta t\}$.

Combining inequalities (10) and (12), we get

$$\frac{[\inf\ l(\sigma)]^p}{|fA \setminus fC|^{p-1}} \leqslant \frac{1}{\ln^p\left(\frac{t+\Delta t}{t}\right)} \int\limits_R \frac{Q(z)}{|z - z_0|^p}\,dm(z). \tag{13}$$

By the Fubini theorem we have

$$\int\limits_R \frac{Q(z)}{|z - z_0|^p}\,dm(z) = \int\limits_t^{t+\Delta t} \frac{d\tau}{\tau^p} \int\limits_{S(z_0,\tau)} Q(z)\,|dz| = 2\pi \int\limits_t^{t+\Delta t} \tau^{1-p}\,q_{z_0}(\tau)\,d\tau, \tag{14}$$

where $q_{z_0}(\tau)=\frac{1}{2\pi\tau}\int\limits_{S(z_0,\tau)} Q(z)\,|dz|$ and $S(z_0,\tau)=\{z\in\mathbb{C}:|z-z_0|=\tau\}$. Thus,

$$\inf\ l(\sigma)\leqslant(2\pi)^{\frac{1}{p}}\,\frac{|fA\setminus fC|^{\frac{p-1}{p}}}{\ln\left(\frac{t+\Delta t}{t}\right)}\left[\int\limits_{t}^{t+\Delta t}\tau^{1-p}\,q_{z_0}(\tau)\,d\tau\right]^{\frac{1}{p}}. \tag{15}$$

Employing the isoperimetric inequality

$$\inf\ l(\sigma)\geqslant 2\sqrt{\pi|fC|}, \tag{16}$$

we obtain

$$2\sqrt{\pi\,|fC|}\leqslant(2\pi)^{\frac{1}{p}}\,\frac{|fA\setminus fC|^{\frac{p-1}{p}}}{\ln\left(\frac{t+\Delta t}{t}\right)}\left[\int\limits_{t}^{t+\Delta t}\tau^{1-p}\,q_{z_0}(\tau)\,d\tau\right]^{\frac{1}{p}}. \tag{17}$$

We introduce a function $\Phi(t)$ for this homeomorphism f as follows:

$$\Phi(t)=|fB(z_0,t)|, \tag{18}$$

where $B(z_0,t)=\{z\in\mathbb{C}:|z-z_0|\leqslant t\}$. Then it follows from (17) that

$$2\sqrt{\pi\,\Phi(t)}\leqslant(2\pi)^{\frac{1}{p}}\,\frac{\left[\frac{\Phi(t+\Delta t)-\Phi(t)}{\Delta t}\right]^{\frac{p-1}{p}}}{\frac{\ln(t+\Delta t)-\ln t}{\Delta t}}\left[\frac{1}{\Delta t}\int\limits_{t}^{t+\Delta t}\tau^{1-p}\,q_{z_0}(\tau)\,d\tau\right]^{\frac{1}{p}}. \tag{19}$$

Letting $\Delta t\to 0$ in inequality (19) and taking into consideration a monotonous increasing of the function Φ in $t\in(0,d_0)$, for almost all t we have:

$$\frac{2\pi^{\frac{p-2}{2(p-1)}}}{t^{\frac{1}{p-1}}\,q_{z_0}^{\frac{1}{p-1}}(t)}\leqslant\frac{\Phi'(t)}{\Phi^{\frac{p}{2(p-1)}}(t)}. \tag{20}$$

This implies easily the following inequality:

$$\frac{2\pi^{\frac{p-2}{2(p-1)}}}{t^{\frac{1}{p-1}}\,q_{z_0}^{\frac{1}{p-1}}(t)}\leqslant\left(\frac{\Phi^{\frac{p-2}{2(p-1)}}(t)}{\frac{p-2}{2(p-1)}}\right)'. \tag{21}$$

Since $p>2$, the function

$$g(t)=\frac{\Phi^{\frac{p-2}{2(p-1)}}(t)}{\frac{p-2}{2(p-1)}}$$

is non-decreasing on $(0,d_0)$, where $d_0=\operatorname{dist}(z_0,\partial D)$. Integrating both sides of the inequality in $t\in[\varepsilon,r]$ and taking into consideration that

$$\int\limits_{\varepsilon}^{r}\left(\frac{\Phi^{\frac{p-2}{2(p-1)}}(t)}{\frac{p-2}{2(p-1)}}\right)'dt=\int\limits_{\varepsilon}^{r}g'(t)\,dt\leqslant g(r)-g(\varepsilon)\leqslant\frac{\Phi^{\frac{p-2}{2(p-1)}}(r)-\Phi^{\frac{p-2}{2(p-1)}}(\varepsilon)}{\frac{p-2}{2(p-1)}}, \tag{22}$$

see, for instance, [16, Thm. IV.7.4], we obtain

$$2\pi^{\frac{p-2}{2(p-1)}}\int\limits_{\varepsilon}^{r}\frac{dt}{t^{\frac{1}{p-1}}q_{z_0}^{\frac{1}{p-1}}(t)}\leqslant\frac{\Phi^{\frac{p-2}{2(p-1)}}(r)-\Phi^{\frac{p-2}{2(p-1)}}(\varepsilon)}{\frac{p-2}{2(p-1)}}. \tag{23}$$

Letting $\varepsilon\to 0$ in inequality (23), we arrive at the estimate

$$\Phi(r)\geqslant\pi\left(\frac{p-2}{p-1}\right)^{\frac{2(p-1)}{p-2}}\left(\int\limits_{0}^{r}\frac{dt}{t^{\frac{1}{p-1}}\,q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{\frac{2(p-1)}{p-2}}. \tag{24}$$

Finally, denoting $\Phi(r)=|fB(z_0,r)|$ in the latter inequality, we get

$$|fB(z_0,r)|\geqslant \pi\left(\frac{p-2}{p-1}\right)^{\frac{2(p-1)}{p-2}}\left(\int\limits_0^r\frac{dt}{t^{\frac{1}{p-1}}\,q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{\frac{2(p-1)}{p-2}} \tag{25}$$

and this completes the proof of Theorem 1. □

3. Corollaries of Theorem 1

Theorem 1 implies the following statements.

Employing the condition $q_{z_0}(t)\leqslant q_0\,t^{-\alpha}$, we estimate the right hand side of inequality (4) and after elementary transformations we arrive at the following result.

Corollary 1. *Let D and D' be bounded domains in $\mathbb{C}$ and $f:D\to D'$ be a Q-homeomorphism w.r.t. p-modulus as $p>2$. Assume that the function Q satisfies the condition*

$$q_{z_0}(t)\leqslant q_0\,t^{-\alpha},\ q_0\in(0,\infty)\,,\ \alpha\in[0,\infty) \tag{26}$$

for $z_0\in D$ and almost all $t\in(0,d_0)$, $d_0=\operatorname{dist}(z_0,\partial D)$. Then for each $r\in(0,d_0)$ the estimate

$$|fB(z_0,r)|\geqslant \pi^{-\frac{\alpha}{p-2}}\left(\frac{p-2}{\alpha+p-2}\right)^{\frac{2(p-1)}{p-2}} q_0^{\frac{2}{2-p}}\,|B(z_0,r)|^{1+\frac{\alpha}{p-2}} \tag{27}$$

holds true.

In particular, letting here $\alpha=0$, we obtain the following conclusion.

Corollary 2. *Let D and D' be bounded domains in $\mathbb{C}$ and $f:D\to D'$ be a Q-homeomorphism w.r.t. p-modulus as $p>2$ and $q_{z_0}(t)\leqslant q_0<\infty$ for almost each $t\in(0,\,d_0)$, $d_0=\operatorname{dist}(z_0,\partial D)$. Then the estimate*

$$|fB(z_0,r)|\geqslant q_0^{\frac{2}{2-p}}\,|B(z_0,r)| \tag{28}$$

holds true for each $r\in(0,d_0)$.

Corollary 3. *Suppose that the assumptions of Theorem 1 are satisfied and $Q(z)\leqslant K<\infty$ for almost each $z\in D$. Then the estimate*

$$|fB(z_0,r)|\geqslant K^{\frac{2}{2-p}}\,|B(z_0,r)| \tag{29}$$

holds true for each $r\in(0,d_0)$.

Remark 1. *Corollary 3 is a particular result by Gehring for $E=B(z_0,r)$, see* [12, Lm. 7].

Corollary 4. *Let $f:\mathbb{B}\to\mathbb{B}$ be a Q-homeomorphism w.r.t. p-modulus as $p>2$. Assume that the function $Q(z)$ satisfies the condition*

$$q(t)\leqslant\frac{q_0}{t\,\ln^{p-1}\frac{1}{t}},\ q_0\in(0,\infty)\,, \tag{30}$$

for almost each $t\in(0,1)$, where $q(t)=\frac{1}{2\pi t}\int\limits_{S_t}Q(z)\,|dz|$ is the integral mean over the circumference $S_t=\{z\in\mathbb{C}:\ |z|=t\}$. Then for each $r\in(0,1)$ the estimate

$$|fB_r|\geqslant \pi\left(\frac{p-2}{p-1}\right)^{\frac{2(p-1)}{p-2}} q_0^{\frac{2}{2-p}}\left(r\ln\frac{e}{r}\right)^{\frac{2(p-1)}{p-2}} \tag{31}$$

holds true, where $B_r=\{z\in\mathbb{C}:|z|\leqslant r\}$.

4. Extremal problems for area functional

Let $Q : \mathbb{B} \to [0, \infty]$ be a measurable function satisfying the condition

$$q(t) \leqslant q_0 \,,\; q_0 \in (0, \infty) \tag{32}$$

for almost each $t \in (0, 1)$, where $q(t) = \frac{1}{2\pi t} \int\limits_{S_t} Q(z)\,|dz|$ is the integral mean over the circumference $S_t = \{z \in \mathbb{C} : |z| = t\}$.

Let $\mathcal{H} = \mathcal{H}(q_0, p, \mathbb{B})$ be the set of all Q-homeomorphisms $f : \mathbb{B} \to \mathbb{C}$ w.r.t. p-modulus as $p > 2$ obeying condition (32). On the class $\mathcal{H}$ we consider the area functional

$$\mathbf{S}_r(f) = |fB_r| \,. \tag{33}$$

Theorem 2. *For each $r \in [0, 1]$ the identity*

$$\min_{f \in \mathcal{H}} \mathbf{S}_r(f) = \pi\, q_0^{\frac{2}{2-p}}\, r^2 \tag{34}$$

holds true.

Proof. Corollary 2 implies immediately the estimate

$$\mathbf{S}_r(f) \geqslant \pi\, q_0^{\frac{2}{2-p}}\, r^2 \,. \tag{35}$$

Let us specify a homeomorphism $f \in \mathcal{H}$, at which the minimum of the functional $\mathbf{S}_r(f)$ is attained. Let $f_0 : \mathbb{B} \to \mathbb{C}$, where

$$f_0(z) = q_0^{\frac{1}{2-p}}\, z. \tag{36}$$

It is obvious that (35) becomes the identity at the mapping f_0. It remains to show that the mapping defined in such way is a Q-homemorphism w.r.t. p-modulus with $Q(z) = q_0$. Indeed,

$$l(z, f_0) = L(z, f_0) = q_0^{\frac{1}{2-p}} \,, \qquad J(z, f_0) = q_0^{\frac{2}{2-p}} \tag{37}$$

and

$$K_{I,p}(z, f_0) = \frac{J(z, f_0)}{l^p(z, f_0)} = q_0 \,. \tag{38}$$

By Theorem 1.1 in work [17], the mapping f_0 is a Q-homeomorphism w.r.t. p-modulus with $Q(z) = K_{I,p}(z, f_0) = q_0$. □

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