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ANALOGUE OF BOHL THEOREM FOR A CLASS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We study the existence and uniqueness of a solution bounded in the entire space for a class of higher order linear partial differential equations. We prove the theorem on the necessary and sufficient condition for the existence and uniqueness of a bounded solution for a studied class of equations. This theorem is an analogue of the Bohl theorem known in the theory of ordinary differential equations. In a partial case the unique solvability conditions are expressed in terms of the coefficients of the equation and we provide the integral representation for the bounded solution.

Keywords: Bohl theorem, bounded solution, symbol of equation, representation of a bounded solution.

Mathematics Subject Classification: 35G05, 35E99, 35A01, 35A24, 35C15

1. Introduction

In the theory of ordinary differential equations, the Bohl theorem is known [1] on the unique solvability on the entire real axis $\mathbb{R} = (-\infty, +\infty)$ of the linear ordinary differential equation

$$y^{(m)} + c_1 y^{(m-1)} + \ldots + c_{m-1} y' + c_m y = f(x), \quad x \in \mathbb{R}, \tag{1.1}$$

with constant coefficients $c_1, \ldots, c_m$ and the right hand side $f(x)$ continuous and bounded on $\mathbb{R}$. In accordance with Bohl theorem, for each continuous and bounded on $\mathbb{R}$ function $f(x)$, equation (1.1) has the unique bounded solution only in the case, when the symbol (characteristic polynomial) of the equation

$$s^m + c_1 s^{m-1} + \ldots + c_{m-1} s + c_m,$$

where $s = \sigma + i\tau$ is a complex variable, has no pure imaginary roots $i\tau$, $\tau \in \mathbb{R}$.

In the present work we formulate and prove an analogue of Bohl theorem for linear partial differential equations of the following form:

$$\begin{aligned} P_1\left(\frac{\partial}{\partial x_1}\right) \ldots P_n\left(\frac{\partial}{\partial x_n}\right) u + \sum_{k_1=0}^{m_1-1} \ldots \sum_{k_n=0}^{m_n-1} b_{k_1\ldots k_n} \frac{\partial^{k_1+\ldots+k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \\ = f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \end{aligned} \tag{1.2}$$

Here we assume that we are given the natural numbers $n \geqslant 2$, $m_1, \ldots, m_n$, the complex numbers $b_{k_1\ldots k_n}$, $k_j = \overline{0, m_j - 1}$, $j = \overline{1, n}$ and the polynomials

$$P_j(s) = s^{m_j} + a_{j1} s^{m_j - 1} + \ldots + a_{jm_j}, \quad j = \overline{1, n},$$

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with constant complex coefficients a_{jk}, $k = \overline{1, m_j}$, $j = \overline{1, n}$. The function $f(x_1, \ldots, x_n)$ is assumed to be complex-valued, continuous and bounded in $\mathbb{R}^n$.

The issue on existence of bounded solutions to linear partial differential equations with constant coefficients was studied in work [2]. It was proved in this work that an arbitrary differential work of the form

$$\sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} c_{k_1 \ldots k_n} \frac{\partial^{k_1+\ldots+k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} = f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

with constant coefficients $c_{k_1 \ldots k_n}$ is uniquely solvable in the space of bounded generalized functions if and only if the symbol of the equation

$$\sum_{k_1=0}^{m_1} \cdots \sum_{k_n=0}^{m_n} c_{k_1 \ldots k_n} s_1^{k_1} \ldots s_n^{k_n},$$

where $s_1, \ldots, s_n$ are complex variables, has no pure imaginary roots $(i\tau_1, \ldots, i\tau_n)$, $\tau_j \in \mathbb{R}$, $j = \overline{1, n}$. At that, if f is a continuous and bounded in $\mathbb{R}^n$ function, then the solution u is not necessarily continuous and bounded together with all its derivatives involved in the equation. As the theorems on hypoellipticity show [3], the smoothness of the solution depends on the behavior of the symbol of the equation at infinity. This is why it is an interesting issue on finding additional conditions for the symbol, apart from the absence of pure imaginary root, ensuring the smoothness of the solutions in the classical sense.

2. Main results

For equations (1.2) we can formulate and prove the conditions for the unique solvability in the classical sense.

A bounded solution to equation (1.2) is a complex-valued function $u(x_1, \ldots, x_n)$ continuous and bounded in $\mathbb{R}^n$ together with all partial derivatives

$$\frac{\partial^{k_1+\ldots+k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \quad \text{where} \quad k_j = \overline{0, m_j}, \quad j = \overline{1, n},$$

which solves equation (1.2).

In the present work we prove the following theorem.

Theorem 2.1. *For each continuous and bounded in $\mathbb{R}^n$ function $f(x_1, \ldots, x_n)$, equation (1.2) has the unique bounded solution if and only if the polynomials P_1, …, P_n and the symbol of the equation*

$$P(s_1, \ldots, s_n) \equiv P_1(s_1) \cdot \ldots \cdot P_n(s_n) + P_0(s_1, \ldots, s_n),$$

where

$$P_0(s_1, \ldots, s_n) = \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_n=0}^{m_n-1} b_{k_1 \ldots k_n} s_1^{k_1} \ldots s_n^{k_n},$$

have no pure imaginary roots, that is, for all $\tau_1, \ldots, \tau_n \in \mathbb{R}$, the conditions

$$P(i\tau_1, \ldots, i\tau_n) \neq 0, \tag{2.1}$$

$$P_1(i\tau_1) \neq 0, \quad \ldots, \quad P_n(i\tau_n) \neq 0, \tag{2.2}$$

hold true.

We note that in the case, when in equation (1.2) all coefficients $b_{k_1\dots k_n}$ are zero and conditions (2.2) are satisfied, the existence and the uniqueness of a bounded solution is implied by the Bohl theorem. One can confirm this by inverting one by one the differential operators

$$P_1\left(\frac{\partial}{\partial x_1}\right), \quad \dots, \quad P_n\left(\frac{\partial}{\partial x_n}\right).$$

Theorem 2.1 implies

Corollary 1. *For each continuous and bounded in $\mathbb{R}^2$ function $f(x_1, x_2)$ the equation*

$$\begin{aligned} &\frac{\partial^{m_1+m_2}u}{\partial x_1^{m_1}\partial x_2^{m_2}} + \sum_{k_1=1}^{m_1} a_{1k_1}\frac{\partial^{m_1-k_1+m_2}u}{\partial x_1^{m_1-k_1}\partial x_2^{m_2}} + \sum_{k_2=1}^{m_2} a_{2k_2}\frac{\partial^{m_1+m_2-k_2}u}{\partial x_1^{m_1}\partial x_2^{m_2-k_2}} \\ &\qquad + \sum_{k_1=0}^{m_1-1}\sum_{k_2=0}^{m_2-1} b_{k_1k_2}\frac{\partial^{k_1+k_2}u}{\partial x_1^{k_1}\partial x_2^{k_2}} = f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{2.3}$$

with constant coefficients a_{1k_1}, a_{2k_2}, $b_{k_1k_2}$ has the unique bounded solution if and only if the symbol of the equation and the following two polynomials

$$Q_1(s) = s^{m_1} + \sum_{k_1=1}^{m_1} a_{1k_1}s^{m_1-k_1}, \tag{2.4}$$

$$Q_2(s) = s^{m_2} + \sum_{k_2=1}^{m_2} a_{2k_2}s^{m_2-k_2} \tag{2.5}$$

have no pure imaginary roots.

We consider a particular case of equation (1.2), namely, the equation of the form

$$\left(\frac{\partial}{\partial x_1} + a_1\right)^{m_1} \dots \left(\frac{\partial}{\partial x_n} + a_n\right)^{m_n} u - bu = f(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{2.6}$$

where the coefficients $a_1 = \alpha_1 + i\beta_1$, $\dots$, $a_n = \alpha_n + i\beta_n$ and b are complex numbers. The following theorems are true.

Theorem 2.2. *For each continuous and bounded in $\mathbb{R}^n$ function $f(x_1, \dots, x_n)$ equation (2.6) has the unique bounded solution if and only if the numbers $a_1 = \alpha_1 + i\beta_1$, $\dots$, $a_n = \alpha_n + i\beta_n$ and b satisfy the conditions*

$$\alpha_1 \neq 0, \quad \dots, \quad \alpha_n \neq 0, \tag{2.7}$$

$$R_{|}m|\left(b\alpha_1^{-m_1}\dots\alpha_n^{-m_n}\right) < 1, \tag{2.8}$$

where $|m| = m_1 + \dots + m_n$,

$$R_{|m|}(c) = \max_{k=0,\dots,|m|-1} \operatorname{Re}\left(\sqrt[|m|]{c}\right)_k = |c|^{1/|m|} \max_{k=0,\dots,|m|-1} \cos\left(\frac{\arg(c) + 2\pi k}{|m|}\right). \tag{2.9}$$

Theorem 2.3. *Assume that the numbers α_1, $\dots$, α_n are positive and condition (2.8) is satisfied. Then the unique bounded solution of equation (2.6) can be represented as*

$$u(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} G(x_1 - \xi_1, \dots, x_n - \xi_n) f(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n, \tag{2.10}$$

where the function $G(x_1, \dots, x_n)$ is defined by the formula

$$G(x_1, \dots, x_n) = e^{-a_1x_1 - \dots - a_nx_n} \sum_{k=0}^{\infty} \frac{x_1^{m_1-1}\dots x_n^{m_n-1}\left(bx_1^{m_1}\dots x_n^{m_n}\right)^k}{(m_1(k+1)-1)!\dots(m_n(k+1)-1)!} \tag{2.11}$$

and is absolutely integrable in the domain $x_1 > 0$, …, $x_n > 0$:

$$\int_0^{+\infty} \dots \int_0^{+\infty} |G(x_1, \dots, x_n)| \, dx_1 \dots dx_n < \infty. \tag{2.12}$$

Remark 1. *Under the change of x_j by $y_j = -x_j$, the bracket $(\partial/\partial x_j + a_j)^{m_j}$ is transformed into the bracket $(-1)^{m_j} (\partial/\partial y_j - a_j)^{m_j}$. Therefore, equation (2.6) with nonzero α_1, …, α_n can be always reduced to the case when α_1, …, α_n are positive.*

Theorems 2.2 and 2.3 for $n = 2$ and $m_1 = m_2 = 1$ were proved in works [4], [5]. The monograph [6] provides results on exponential representation of generalized solutions to linear differential equations with constant coefficients. The results of this monograph do not imply Theorems 2.1–2.3.

In our opinion, the obtained results can be extended for partial differential equations with variable coefficients by proving and applying Carleman type inequalities ([7]).

3. Existence and uniqueness of bounded solution

In this section we prove Theorem 2.1. First we introduce the notations: $m = (m_1, \dots, m_n)$ is the vector formed by the degrees $m_1, \dots, m_n$ of the polynomials $P_1, \dots, P_n$, $|m| = m_1 + \dots + m_n$, C_0 is the Banach space of continuous and bounded in $\mathbb{R}^n$ functions $v(x_1, \dots, x_n)$ with the norm

$$\|v\| = \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} |v(x_1, \dots, x_n)|,$$

C_m is the Banach space of functions $v(x_1, \dots, x_n)$, belonging to C_0 together with all its derivatives

$$\frac{\partial^{k_1+\dots+k_n} v}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \quad k_j = \overline{0, m_j}, \quad j = \overline{1, n},$$

where the norm is defined by the formula

$$\|v\|_m = \sum_{k_j=\overline{0,m_j}, j=\overline{1,n}} \left\| \frac{\partial^{k_1+\dots+k_n} v}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right\|,$$

S is the space of functions $v(x_1, \dots, x_n)$ infinitely differentiable in $\mathbb{R}^n$ and fast decaying at infinity [8], S' is the space of tempered distributions [8]. The embeddings $S \subset C_m \subset C_0 \subset S'$ hold true [8].

Before proving Theorem 2.1, let us check three lemmata.

Lemma 3.1. *If condition (2.1) is satisfied, then a solution of equation (1.2) is unique in the space S'.*

Proof. Let $f = 0$ and $u \in S'$ in equation (1.2). We apply the Fourier transform of the generalized functions [8] to the both sides of (1.2) and obtain the identity

$$(\widehat{u}, P'\psi) = 0 \quad \text{for each} \quad \psi \in S. \tag{3.1}$$

Here $\widehat{u}$ is the Fourier image of u, $\widehat{u} \in S'$, $P' = P(-i\tau_1, \dots, -i\tau_n)$. It follows from condition (2.1) that $\varphi/P' \in S$ for each compactly supported function $\varphi \in S$. Letting $\psi = \varphi/P'$ in the identity (3.1), we obtain $(\widehat{u}, \varphi) = 0$ for each compactly supported function $\varphi \in S$. The set of compactly supported functions is dense in S [8] and therefore, $\widehat{u} = 0$ and $u = 0$. The proof is complete. □

Lemma 3.2. *Assume that conditions (2.1), (2.2) are satisfied and for some $f \in C_0$ there exists a solution u to equation (1.2) in S' such that $P_0(\partial/\partial x_1, \dots, \partial/\partial x_n)u \in C_0$. Then $u \in C_m$.*

Proof. Given f and u, we consider the equation

$$P_1\left(\frac{\partial}{\partial x_1}\right)\dots P_n\left(\frac{\partial}{\partial x_n}\right)v=g, \tag{3.2}$$

where

$$g=f-P_0\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)u\in C_0.$$

On one hand, u is a generalized solution to equation (3.2). On the other hand, we can apply n times Bohl theorem [1] to equation (3.2) and invert the differential operators

$$P_1\left(\frac{\partial}{\partial x_1}\right),\quad\dots,\quad P_n\left(\frac{\partial}{\partial x_n}\right).$$

While inverting each differential operator $P_j(\partial/\partial x_j)$, we keep the property of being bounded and continuous for partial derivatives w.r.t. other variables. As a result we obtain a solution $v\in C_m$ of equation (3.2). Equation (3.2) is a particular case of equation (1.2) and by Lemma 3.1 it can have only the unique solution in S'. Therefore, $u=v$. The proof is complete. □

Lemma 3.3. *If conditions (2.1) and (2.2) are satisfied, there exists a positive number γ such that for all $\tau_1,\dots,\tau_n\in\mathbb{R}$ the estimate*

$$|P(i\tau_1,\dots,i\tau_n)|\geqslant\gamma\,(1+|\tau_1|)^{m_1}\dots(1+|\tau_n|)^{m_n} \tag{3.3}$$

holds true.

Proof. For each $j=\overline{1,n}$ we expand the polynomial $P_j(s)$ over its roots

$$P_j(s)=(s-\lambda_{j1})\dots(s-\lambda_{jm_j}).$$

We have

$$\frac{|P_j(i\tau_j)|}{(1+|\tau_j|)^{m_j}}=\prod_{k=1}^{m_j}\left|\frac{i\tau_j-\lambda_{jk}}{1+|\tau_j|}\right|\quad\to\quad 1\quad\text{as}\quad\tau_j\to\infty,$$

and by conditions (2.2) we obtain:

$$\inf_{\tau_j\in\mathbb{R}}\frac{|P_j(i\tau_j)|}{(1+|\tau_j|)^{m_j}}=\gamma_j>0.$$

Therefore, for all $\tau_1,\dots,\tau_n\in\mathbb{R}$ we have the estimate

$$\frac{|P_1(i\tau_1)\dots P_n(i\tau_n)|}{(1+|\tau_1|)^{m_1}\dots(1+|\tau_n|)^{m_n}}\geqslant\gamma_1\dots\gamma_n=2\widetilde{\gamma}_1.$$

For the polynomial P_0 we get

$$|P_0(i\tau_1,\dots,i\tau_n)|\leqslant c_0\,(1+|\tau_1|)^{m_1-1}\dots(1+|\tau_n|)^{m_n-1},$$

where c_0 is independent of $\tau_1,\dots,\tau_n$. This yields

$$\frac{|P(i\tau_1,\dots,i\tau_n)|}{(1+|\tau_1|)^{m_1}\dots(1+|\tau_n|)^{m_n}}\geqslant 2\widetilde{\gamma}_1-\frac{c_0}{(1+|\tau_1|)\dots(1+|\tau_n|)}>\widetilde{\gamma}_1$$

as $|\tau_1|+\ldots+|\tau_n|>c_0/\widetilde{\gamma}_1$. By condition (2.1) we obtain:

$$\min_{|\tau_1|+\ldots+|\tau_n|\leqslant c_0/\widetilde{\gamma}_1}\frac{|P(i\tau_1,\dots,i\tau_n)|}{(1+|\tau_1|)^{m_1}\dots(1+|\tau_n|)^{m_n}}=\widetilde{\gamma}_2>0.$$

Hence, for all $\tau_1,\dots,\tau_n\in\mathbb{R}$ estimate (3.3) holds true, where γ is the smallest of the numbers $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$. The proof is complete. □

We proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Necessity. Assume that for each $f \in C_0$ equation (1.2) has the unique solution $u \in C_m$. Then condition (2.1) should be satisfied. Indeed, if $P(i\tau_1^0, \ldots, i\tau_n^0) = 0$ for some $\tau_1^0, \ldots, \tau_n^0 \in \mathbb{R}$, then the function $u + \exp(i\tau_1^0 x_1 + \ldots + i\tau_n^0 x_n)$ also is a solution of equation (1.2) in C_m.

Assume that one of conditions (2.2) fails, for instance, $P_1(i\tau_1^0) = 0$ for some $\tau_1^0 \in \mathbb{R}$. We take some function$v^0(x_2, \ldots, x_n) \in C_{m''} \backslash C_{m'}$, where $m' = (m_2, m_3, \ldots, m_n)$, $m'' = (m_2 - 1, m_3, \ldots, m_n)$, and consider the function $u^0(x_1, \ldots, x_n) = \exp(i\tau_1^0 x_1) v^0(x_2, \ldots, x_n)$. It is obvious that $u^0 \in S' \backslash C_m$ and

$$P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u^0 = P_0\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u^0 \equiv f^0 \in C_0.$$

Therefore, the function u^0 is a generalized solution to equation (1.2) in S' as $f = f^0$. Since condition (2.1) is satisfied, by Lemma 3.1, equation (1.2) can have only the unique solution in the space S'. Thus, as $f = f^0$, equation (1.2) has no solutions in C_m, which is a contradiction. Hence, conditions (2.2) are necessary.

Sufficiency. Assume that conditions (2.1) and (2.2) hold true. We first prove the solvability of equation (1.2) for periodic functions f in C_0. We shall call function f periodic with a period ω, where ω is a fixed positive number, if for each $j = \overline{1, n}$ the identity $f(x_1, \ldots, x_j + \omega, \ldots, x_n) \equiv f(x_1, \ldots, x_j, \ldots, x_n)$ holds true.

Lemma 3.4. *For each ω-periodic function f in C_0 there exists the unique solution u of equation (1.2) in C_m, which ω-periodic together with all its derivatives of order up to m. At that, the estimate*

$$\|u\|_m \leqslant M_0 \|f\| \tag{3.4}$$

holds true, where M_0 is a positive number independent of f and ω for all $\omega \geqslant 1$.

Proof. We shall seek a periodic solution as the Fourier series of the exponentials [9]. In order to do it, we expand ω-periodic function f into the Fourier series

$$f = \sum_{(l_1, \ldots, l_n)} c_{l_1 \ldots l_n}(f) e^{i\frac{2\pi}{\omega}(l_1 x_1 + \ldots + l_n x_n)}.$$

The series converges in the norm of the space $L_2(D_\omega)$ ([9])

$$\|g\|^2_{L_2(D_\omega)} = \frac{1}{\omega^n} \int_0^\omega \cdots \int_0^\omega |g(\xi_1, \ldots, \xi_n)|^2 d\xi_1 \ldots d\xi_n,$$

where $D_\omega = \{(\xi_1, \ldots, \xi_n) : 0 < \xi_j < \omega, j = \overline{1, n}\}$, and the Parseval identity

$$\sum_{(l_1, \ldots, l_n)} |c_{l_1 \ldots l_n}(f)|^2 = \|f\|^2_{L_2(D_\omega)}$$

holds true. A periodic solution to equation (1.2) can be defined by the formulae

$$u = \sum_{(l_1, \ldots, l_n)} c_{l_1 \ldots l_n}(u) e^{i\frac{2\pi}{\omega}(l_1 x_1 + \ldots + l_n x_n)}, \tag{3.5}$$

$$c_{l_1 \ldots l_n}(u) = \frac{c_{l_1 \ldots l_n}(f)}{P\left(i\frac{2\pi}{\omega} l_1, \ldots, i\frac{2\pi}{\omega} l_n\right)}. \tag{3.6}$$

Taking into consideration Lemma 3.3, we get the following estimates as $0 \leqslant k_j < m_j$, $j = \overline{1, n}$:

$$\left|\left(i\frac{2\pi}{\omega} l_1\right)^{k_1} \cdots \left(i\frac{2\pi}{\omega} l_n\right)^{k_n} c_{l_1 \ldots l_n}(u)\right| \leqslant \frac{\left|\frac{2\pi}{\omega} l_1\right|^{k_1} \cdots \left|\frac{2\pi}{\omega} l_n\right|^{k_n} |c_{l_1 \ldots l_n}(f)|}{\gamma\left(1 + \left|\frac{2\pi}{\omega} l_1\right|\right)^{m_1} \cdots \left(1 + \left|\frac{2\pi}{\omega} l_n\right|\right)^{m_n}}$$

$$\leqslant \frac{1}{\gamma}\left(1+\left|\frac{2\pi}{\omega}l_1\right|\right)^{-1}\dots\left(1+\left|\frac{2\pi}{\omega}l_n\right|\right)^{-1}|c_{l_1\dots l_n}(f)|$$

and by Cauchy-Schwarz inequality [10]

$$\sum_{(l_1,\dots,l_n)}\left|\left(i\frac{2\pi}{\omega}l_1\right)^{k_1}\dots\left(i\frac{2\pi}{\omega}l_n\right)^{k_n}c_{l_1\dots l_n}(u)\right|$$
$$\leqslant \frac{1}{\gamma}\left(\sum_{(l_1,\dots,l_n)}\left(1+\left|\frac{2\pi}{\omega}l_1\right|\right)^{-2}\dots\left(1+\left|\frac{2\pi}{\omega}l_n\right|\right)^{-2}\right)^{1/2}\left(\sum_{(l_1,\dots,l_n)}|c_{l_1\dots l_n}(f)|^2\right)^{1/2}.$$

This implies, first, that $u\in C_{m'}$, where $m'=(m_1-1,\dots,m_n-1)$, and second, that the function u and all its partial derivatives $\partial^{k_1+\dots+k_n}u/\partial x_1^{k_1}\dots\partial x_n^{k_n}$, where $k_j=\overline{0,m_j-1}$, $j=\overline{1,n}$, are ω-periodic, and third, the estimate

$$\|u\|_{m'}\leqslant M_1\|f\|_{L_2(D_\omega)}\leqslant M_1\|f\| \tag{3.7}$$

holds true. Here $M_1>0$ and M_1 is independent of f and ω for all $\omega\geqslant 1$. It follows from estimate (3.7) and formulae (3.5), (3.6) that $u\in S'$ and u is a generalized solution of equation (1.2) and $P_0(\partial/\partial x_1,\dots,\partial/\partial x_n)u\in C_0$. Applying Lemma 3.2, we get that $u\in C_m$ and the estimate

$$\|u\|_m\leqslant M_2\|\widetilde{f}\| \tag{3.8}$$

is valid, where $\widetilde{f}=f-P_0(\partial/\partial x_1,\dots,\partial/\partial x_n)u$. By (3.7) and (3.8) we obtain estimate (3.4). The proof is complete. □

Let f be an arbitrary function in C_0. We construct the following sequence of periodic functions:

$$f_q(x_1,\dots,x_n)=f(x_1,\dots,x_n)\quad\text{as}\quad\max(|x_1|,\dots,|x_n|)\leqslant q,$$
$$f_q(x_1,\dots,x_n)=\eta_q(\max(|x_1|,\dots,|x_n|))\,f(x_1,\dots,x_n)\quad\text{as}\quad q<\max(|x_1|,\dots,|x_n|)\leqslant q+1,$$

where $q=1,2,\dots$, $(x_1,\dots,x_n)\in D_q$, $D_q=\left\{(\xi_1,\dots,\xi_n):|\xi_j|\leqslant q+1, j=\overline{1,n}\right\}$, the function $\eta_q(t)$ is continuous on $\mathbb{R}$, $0\leqslant\eta_q(t)\leqslant 1$ for all t, $\eta_q(t)=1$ as $|t|\leqslant q$ and $\eta_q(t)=0$ as $|t|>q+1$. We continue each function f_q outside D_q periodically with the period $2(q+1)$. It is obvious that $\|f_q\|\leqslant\|f\|$ for all $q=1,2,\dots$ and $f_q\to f$ as $q\to\infty$ uniformly in each bounded set $D\in\mathbb{R}^n$. In accordance with Lemma 3.4, for each q there exists the unique solution u_q in C_m of equation (1.2) and the estimate $\|u_q\|_m\leqslant M_0\|f\|$ holds true. This estimate, by Arzelà-Ascoli theorem [10], there exists a subsequence $u_{q_1},u_{q_2},\dots$ converging uniformly in each bounded set $D\in\mathbb{R}^n$ to some function $\widetilde{u}\in C_{m'}$ ($m'=(m_1-1,\dots,m_n-1)$) together with all partial derivatives $\partial^{k_1+\dots+k_n}/\partial x_1^{k_1}\dots\partial x_n^{k_n}$, where $k_j=\overline{0,m_j-1}$, $j=\overline{1,n}$.

In equation (1.2) we let $f=f_{q_j}$, $u=u_{q_j}$ and pass to the limit as $j\to\infty$ in the space of generalized functions. Then we obtain that $\widetilde{u}$ is a generalized solution to equation (1.2) and satisfy assumptions of Lemma 3.2. According to Lemma 3.2, we have $\widetilde{u}\in C_m$. The proof of Theorem 2.1 is complete. □

Proof Corollary 1. Let us show that Theorem 2.1 implies Corollary 1. In order to prove this, it is sufficient to confirm that equation (2.3) can be represented as (1.2). As $P_1(s_1)$ and $P_2(s_2)$, we consider the following polynomials

$$P_1(s_1)=(s_1+z_{11})\cdot\ldots\cdot(s_1+z_{1m_1})\equiv(-1)^{m_1}Q_1(s_1),$$
$$P_2(s_2)=(s_2+z_{21})\cdot\ldots\cdot(s_2+z_{2m_2})\equiv(-1)^{m_2}Q_2(s_2),$$

where $-z_{11}$, …, $-z_{1m_1}$ are the roots of the polynomial $Q_1(s)$, $-z_{21}$, …, $-z_{2m_2}$ are the roots of the polynomial $Q_2(s)$. Multiplying the polynomials $P_1(s_1)$ and $P_2(s_2)$ and employing Vieta's formulae expressing the coefficients of a polynomial in terms of its roots [11], we get

$$\begin{aligned} P_1(s_1)P_2(s_2) = {} & s_1^{m_1}s_2^{m_2} + \sum_{k_1=0}^{m_1-1}\sum_{k_2=0}^{m_2-1}\widetilde{b}_{k_1k_2}s_1^{k_1}s_2^{k_2} \\ & + s_2^{m_2}\big((z_{11}+\ldots+z_{1m_1})s_1^{m_1-1}+(z_{11}z_{12}+\ldots+z_{1m_1-1}z_{1m_1})s_1^{m_1-2}\ldots+z_{11}\ldots z_{1m_1}\big) \\ & + s_1^{m_1}\big((z_{21}+\ldots+z_{2m_2})s_2^{m_2-1}+(z_{21}z_{22}+\ldots+z_{2m_2-1}z_{2m_2})s_2^{m_2-2}+\ldots+z_{21}\ldots z_{2m_2}\big) \\ = {} & s_1^{m_1}s_2^{m_2}+s_2^{m_2}\sum_{k_1=1}^{m_1}a_{1k_1}s_1^{m_1-k_1}+s_1^{m_1}\sum_{k_2=1}^{m_2}a_{2k_2}s_2^{m_2-k_2}+\sum_{k_1=0}^{m_1-1}\sum_{k_2=0}^{m_2-1}\widetilde{b}_{k_1k_2}s_1^{k_1}s_2^{k_2}. \end{aligned}$$

Therefore, the symbol $P(s_1,s_2)$ of equation (2.3) can be represented as

$$P(s_1,s_2) = P_1(s_1)P_2(s_2) + \sum_{k_1=0}^{m_1-1}\sum_{k_2=0}^{m_2-1}\left(b_{k_1k_2}-\widetilde{b}_{k_1k_2}\right)s_1^{k_1}s_2^{k_2}.$$

The polynomials P_1 and P_2 have no pure imaginary roots simultaneously with the polynomials Q_1 and Q_2. Hence, equation (2.3) can be represented as (1.2) and the assumptions of Theorem 2.1 are satisfied only in the case, when the symbol of equation (2.3) and the polynomials Q_1, Q_2 have no pure imaginary roots. The proof is complete. □

4. Proof of Theorem 2.2

Theorem 2.2 is implied by Theorem 2.1 and the following lemma.

Lemma 4.1. *If conditions (2.7) are satisfied, then the symbol of equation (2.6) has no pure imaginary roots only in the case, when condition (2.8) holds true.*

Proof. It is easy to check that the condition on the absence of pure imaginary roots for the symbol of equation (2.6) is equivalent to the condition

$$(i\tau_1+1)^{m_1}\ldots(i\tau_n+1)^{m_n} \neq \widetilde{b} \quad \text{for all} \quad \tau_1,\ldots,\tau_n\in\mathbb{R}, \tag{4.1}$$

where $\widetilde{b} = b\alpha_1^{-m_1}\ldots\alpha_n^{-m_n}$. Let us show that condition (4.1) is equivalent to the condition

$$R_{|m|}(\widetilde{b}) < 1, \tag{4.2}$$

where $|m| = m_1+\ldots+m_n$, and $R_{|m|}(\widetilde{b})$ is determined by the formula (2.9). This will prove Lemma 4.1.

We let $\tau_j = tg\varphi_j$, where $\varphi_j\in(-\pi/2,\pi/2)$ for all $j=\overline{1,n}$. Then condition (4.1) becomes

$$e^{i(m_1\varphi_1+\ldots+m_n\varphi_n)} \neq \widetilde{b}\cos^{m_1}\varphi_1\ldots\cos^{m_n}\varphi_n \quad \text{for all} \quad \varphi_1,\ldots,\varphi_n\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

In its turn, this is equivalent to the condition:

$$|\widetilde{b}|\cos^{m_1}\varphi_1\ldots\cos^{m_n}\varphi_n \neq 1 \tag{4.3}$$

for all $\varphi_1,\ldots,\varphi_n\in(-\pi/2,\pi/2)$ satisfying the condition $m_1\varphi_1+\ldots+m_n\varphi_n=\theta+2\pi l$ for some integer l and $\theta=\arg\widetilde{b}$, which is the argument of $\widetilde{b}$, $|\theta|\leqslant\pi$. It is obvious that the integer number l should satisfy the inequality $|\theta+2\pi l|<|m|\pi/2$.

Let $(\varphi_1^0,\ldots,\varphi_n^0)$ be a set satisfying the aforementioned conditions for some integer l_0, on which the extremum of the function is attained $F(\varphi_1,\ldots,\varphi_n) = \cos^{m_1}\varphi_1\ldots\cos^{m_n}$. Let us check that the inequalities

$$|\widetilde{b}|\cos^{m_1}\varphi_1^0\ldots\cos^{m_n}\varphi_n^0 < 1 \tag{4.4}$$

and the identities

$$\varphi_1^0 = \ldots = \varphi_n^0 = \frac{\theta + 2\pi l_0}{|m|} \tag{4.5}$$

hold true. If so, then condition (4.3) would be equivalent the inequality

$$|\widetilde{b}|^{1/|m|} \cos \frac{\theta + 2\pi l_0}{|m|} < 1,$$

which in notations (2.9) become (4.2) and this would complete the proof of the lemma.

Before we check inequality (4.4), we construct the set $(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ satisfying the conditions

$$(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad m_1\widetilde{\varphi}_1 + \ldots + m_n\widetilde{\varphi}_n = \theta + 2\pi l_0, \quad |\widetilde{b}|F(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) < 1.$$

At that, without loss of generality, we can assume that $\theta + 2\pi l_0 \geqslant 0$ $m_1 \leqslant |m|/2$. We choose $\delta \in (0, \pi/2)$ so that the inequalities

$$|\widetilde{b}| \cos^{m_1}\left(\frac{\pi}{2} - \delta\right) < 1, \quad 0 < \theta + 2\pi l_0 + m_1\delta < \frac{|m|\pi}{2}$$

are satisfied. We let $\widetilde{\varphi}_1 = \pi/2 - \delta$. For $\widetilde{\varphi}_1$ we have the belonging

$$\theta + 2\pi l_0 - m_1\widetilde{\varphi}_1 \in \left(-(|m| - m_1)\frac{\pi}{2} \quad , \quad (|m| - m_1)\frac{\pi}{2}\right). \tag{4.6}$$

Indeed, by the assumptions $\theta + 2\pi l_0 \geqslant 0$, $m_1 \leqslant |m|/2$ and the choice of δ, we have

$$\theta + 2\pi l_0 - m_1\widetilde{\varphi}_1 \geqslant -m_1\frac{\pi}{2} + m_1\delta > -(|m| - m_1)\frac{\pi}{2},$$
$$\theta + 2\pi l_0 - m_1\widetilde{\varphi}_1 = \theta + 2\pi l_0 + m_1\delta - m_1\frac{\pi}{2} < (|m| - m_1)\frac{\pi}{2}.$$

By (4.6) we obtain the existence of $\widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_n \in (-\pi/2, \pi/2)$ such that

$$m_2\widetilde{\varphi}_2 + \ldots + m_n\widetilde{\varphi}_n = \theta + 2\pi l_0 - m_1\widetilde{\varphi}_1.$$

For the set $(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ we have

$$|\widetilde{b}|F(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) \leqslant |\widetilde{b}| \cos^{m_1}\left(\frac{\pi}{2} - \delta\right) < 1.$$

By two sets $(\varphi_1^0, \ldots, \varphi_n^0)$ and $(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ we introduce the function

$$g(t) = |\widetilde{b}|F((1-t)\widetilde{\varphi}_1 + t\varphi_1^0, \ldots, (1-t)\widetilde{\varphi}_n + t\varphi_n^0), \quad t \in [0, 1].$$

The function $g(t)$ is continuous on the segment $[0, 1]$ and $g(0) < 1$. If (4.4) fails, then $g(1) > 1$ and therefore, for some $t_0 \in (0, 1)$ we should have $g(t_0) = 1$. For this value t_0 we get

$$(1-t_0)\widetilde{\varphi}_1 + t_0\varphi_1^0, \ldots, (1-t_0)\widetilde{\varphi}_n + t_0\varphi_n^0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$m_1\left[(1-t_0)\widetilde{\varphi}_1 + t_0\varphi_1^0\right] + \ldots + m_n\left[(1-t_0)\widetilde{\varphi}_n + t_0\varphi_n^0\right] = \theta + 2\pi l_0,$$
$$|\widetilde{b}|F((1-t_0)\widetilde{\varphi}_1 + t_0\varphi_1^0, \ldots, (1-t_0)\widetilde{\varphi}_n + t_0\varphi_n^0) = 1.$$

This contradicts condition (4.3). Therefore, (4.4) indeed holds true.

The set $(\varphi_1^0, \ldots, \varphi_{n-1}^0)$ is the point of the maximum of the function

$$F_1(\varphi_1, \ldots, \varphi_{n-1}) \equiv F\left(\varphi_1, \ldots, \varphi_{n-1}, \frac{1}{m_n}(\theta + 2\pi l_0 - m_1\varphi_1 - \ldots m_{n-1}\varphi_{n-1})\right)$$

and it lies inside the domain, in which the function F_1 is considered. This is why by the extremum necessary condition we obtain:

$$\frac{\partial F_1}{\partial \varphi_j}(\varphi_1^0, \ldots, \varphi_{n-1}^0) = 0, \quad j = \overline{1, n-1},$$

$$\sin\left(\frac{1}{m_n}\left(\theta+2\pi l_0-m_1\varphi_1^0-\ldots m_{n-1}\varphi_{n-1}^0\right)-\varphi_j^0\right)=0,\quad j=\overline{1,n-1},$$

where

$$\frac{1}{m_n}\left(\theta+2\pi l_0-m_1\varphi_1^0-\ldots m_{n-1}\varphi_{n-1}^0\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),$$
$$\varphi_j^0\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\quad\text{for all}\quad j=\overline{1,n-1}.$$

Hence,

$$\frac{1}{m_n}\left(\theta+2\pi l_0-m_1\varphi_1^0-\ldots m_{n-1}\varphi_{n-1}^0\right)-\varphi_j^0=0,\quad j=\overline{1,n-1}.$$

We have obtained a system of linear algebraic equations for $\varphi_1^0,\ldots,\varphi_{n-1}^0$. This system has the unique solution

$$\varphi_1^0=\ldots=\varphi_{n-1}^0=\frac{\theta+2\pi l_0}{|m|}.$$

Now we find φ_n^0:

$$\varphi_n^0=\frac{1}{m_n}\left(\theta+2\pi l_0-(|m|-m_n)\frac{\theta+2\pi l_0}{|m|}\right)=\frac{\theta+2\pi l_0}{|m|}.$$

Hence, identities (4.5) hold true. The proof is complete. □

5. Representation of bounded solution

Here prove Theorem 2.3 on representing a bounded solution to equation (2.6) by formula (2.10). Let $f\in C_0$, the numbers $\alpha_1,\ldots,\alpha_n$ be positive and condition (2.8) is satisfied. Let us check that if the function G defined by formula (2.11) is absolutely integrable in the domain $x_1>0,\ldots,x_n>0$, then the function u defined by formula (2.10) is a bounded solution to equation (2.6).

It is obvious that the absolute integrability of G implies that $u\in C_0$. If, in addition, u is a generalized solution of equation (2.6), then by Lemma 3.2 we obtain that $u\in C_m$ and u is a bounded solution of equation (2.6). In the case, when f has a compact support, we can straightforwardly check that u is indeed a generalized solution. In the case, when f is an arbitrary function in C_0, the same is checked as follows:

1) As in the proof of Theorem 2.1, we construct a sequence of functions $f_q\in C_0$, $q=1,2,\ldots$ with compact supports converging uniformly to f in each bounded set $D\in\mathbb{R}^n$;

2) Letting $f=f_q$ in formula (2.10), we get a generalized solution $u_q\in C_0$ of equation (2.6);

3) It follows from representation (2.10) that the sequence u_q, $q=1,2,\ldots$ converges to u uniformly in each bounded set $D\in\mathbb{R}^n$;

4) Letting $f=f_q$, $u=u_q$ in equation (2.6) and passing to the limit as $q\to\infty$, we obtain that u is a generalized solution of equation (2.6). Thus, the proof of Theorem 2.3 is reduced to checking the absolute integrability of the function G in the domain $x_1>0,\ldots,x_n>0$.

We write the function G as

$$G(x_1,\ldots,x_n)=e^{-a_1x_1-\ldots-a_nx_n}x_1^{m_1-1}\ldots x_n^{m_n-1}z\left(x_1^{m_1}\ldots x_n^{m_n}\right),\tag{5.1}$$

where

$$z(t)=\sum_{k=0}^{\infty}\frac{(b\cdot t)^k}{(m_1(k+1)-1)!\ldots(m_n(k+1)-1)!}.\tag{5.2}$$

It is obvious that the function $z(t)$ is well-defined and infinitely differentiable on the interval $(-\infty,+\infty)$. Let us estimate the growth rate of $|z(t)|$ for large positive t. The following lemma is true.

Lemma 5.1. *There exist positive numbers M and β depending only on n and $m_1,\dots,m_n$ such that for all $t\geqslant 1$ and $l=0,1,\dots,|m|-1$ the estimate*

$$|z^{(l)}(t)|\leqslant Mt^{(\beta-l(|m|-1))/|m|}e^{\lambda_m\sqrt[|m|]{t}} \tag{5.3}$$

holds true, where

$$\lambda_m=\frac{|m|R_{|m|}(b)}{\sqrt[|m|]{m_1^{m_1}\dots m_n^{m_n}}}. \tag{5.4}$$

We employ Lemma 5.1 to estimate $|G(x_1,\dots,x_n)|$ from above for $x_1>0,\dots,x_n>0$. It follows from Lemma 5.1 that

$$e^{-\lambda_m\sqrt[|m|]{t}}|z(t)|\leqslant M_3\left(1+t^{\beta/|m|}\right)\qquad\text{for all}\quad t\geqslant 0, \tag{5.5}$$

where $M_3>0$ and M_3 is independent of t.

Taking into consideration condition (2.8), we choose $\varepsilon>0$ so that the inequality

$$R_{|m|}(b)<((\alpha_1-\varepsilon)^{m_1}\dots(\alpha_n-\varepsilon)^{m_n})^{1/|m|}$$

holds true. Then for all $x_1>0,\dots,x_n>0$ we have

$$\begin{aligned}(\alpha_1-\varepsilon)x_1+\ldots+(\alpha_n-\varepsilon)x_n&=m_1\left(\frac{\alpha_1-\varepsilon}{m_1}x_1\right)+\ldots+m_n\left(\frac{\alpha_n-\varepsilon}{m_n}x_n\right)\\ &\geqslant|m|\left(\left(\frac{\alpha_1-\varepsilon}{m_1}x_1\right)^{m_1}\cdot\ldots\cdot\left(\frac{\alpha_n-\varepsilon}{m_n}x_n\right)^{m_n}\right)^{1/|m|}\\ &>\frac{|m|R_{|m|}(b)}{\sqrt[|m|]{m_1^{m_1}\dots m_n^{m_n}}}\left(x_1^{m_1}\cdot\ldots\cdot x_n^{m_n}\right)^{1/|m|}=\lambda_m\left(x_1^{m_1}\cdot\ldots\cdot x_n^{m_n}\right)^{1/|m|}.\end{aligned}$$

We estimate $|G(x_1,\dots,x_n)|$ by employing inequality (5.5):

$$\begin{aligned}|G(x_1,\dots,x_n)|&=e^{-\varepsilon(x_1+\ldots+x_n)}e^{-(\alpha_1-\varepsilon)x_1-\ldots-(\alpha_n-\varepsilon)x_n}x_1^{m_1-1}\dots x_n^{m_n-1}\left|z\left(x_1^{m_1}\dots x_n^{m_n}\right)\right|\\ &<e^{-\varepsilon x_1}x_1^{m_1-1}\dots e^{-\varepsilon x_n}x_n^{m_n-1}e^{-\lambda_m\sqrt[|m|]{x_1^{m_1}\dots x_n^{m_n}}}\left|z\left(x_1^{m_1}\dots x_n^{m_n}\right)\right|\\ &\leqslant M_3e^{-\varepsilon x_1}x_1^{m_1-1}\dots e^{-\varepsilon x_n}x_n^{m_n-1}\left(1+\left(x_1^{m_1}\dots x_n^{m_n}\right)^{\beta/|m|}\right).\end{aligned}$$

This implies the absolute integrability of the function G in the domain $x_1>0,\dots,x_n>0$.

The proof of Lemma 5.1 will follow the next lemma.

Lemma 5.2. *The function $z(t)$ defined by formula (5.2) satisfies the identity*

$$\sum_{l=1}^{|m|}p_lt^{l-1}z^{(l)}(t)\equiv bz(t), \tag{5.6}$$

where $p_0=0$, $p_1,\dots,p_{|m|}$ are the coefficients of the expansion of the polynomial

$$Q(z)=\prod_{j=1}^{n}(m_jz+m_j)(m_jz+m_j+1)\dots(m_jz+2m_j-1)$$

into the interpolation Newton polynomial [12] *by the nodes $-1,0,1,\dots,|m|-1:$*

$$Q(z)=\sum_{l=1}^{|m|}p_l(z+1)z\dots(z-l+2). \tag{5.7}$$

Proof. Expansion (5.7) implies the following identities:

$$\sum_{l=1}^{k+1} p_l(k+1)k\dots(k-l+2) = Q(k), \quad k = 0,1,\dots,|m|-1,$$
$$\sum_{l=1}^{|m|} p_l(k+1)k\dots(k-l+2) = Q(k), \quad k = |m|,|m|+1,\dots \quad .$$

This implies

$$\begin{aligned}
\sum_{l=1}^{|m|} p_l t^{l-1} z^{(l)}(t) &= b\sum_{l=1}^{|m|} p_l \sum_{k=l-1}^{\infty} \frac{(bt)^k(k+1)k\dots(k-l+2)}{(m_1(k+2)-1)!\dots(m_n(k+2)-1)!} \\
&= b\sum_{l=1}^{|m|} p_l \left(\sum_{k=l-1}^{|m|-1} + \sum_{k=|m|}^{\infty}\right) \frac{(bt)^k(k+1)k\dots(k-l+2)}{(m_1(k+1)-1)!\dots(m_n(k+1)-1)!\cdot Q(k)} \\
&= b\sum_{k=0}^{|m|-1} \frac{(bt)^k(k+1)k\dots(k-l+2)}{(m_1(k+1)-1)!\dots(m_n(k+1)-1)!\cdot Q(k)} \sum_{l=1}^{k+1} p_l(k+1)k\dots(k-l+2) \\
&\quad + b\sum_{k=|m|}^{\infty} \frac{(bt)^k(k+1)k\dots(k-l+2)}{(m_1(k+1)-1)!\dots(m_n(k+1)-1)!\cdot Q(k)} \sum_{l=1}^{|m|} p_l(k+1)k\dots(k-l+2) \\
&= bz(t).
\end{aligned}$$

The proof is complete. □

Proof of Lemma 5.1. It follows from identity (5.6) that the vector function $y(t) = (z(t), z'(t), \dots, z^{(|m|-1)}(t))^\top$ is a solution to the system of differential equations

$$y'(t) = B(t)y(t), \quad t > 0, \tag{5.8}$$

where

$$B(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \frac{b}{p_{|m|}t^{|m|-1}} & \frac{-p_1}{p_{|m|}t^{|m|-1}} & \frac{-p_2}{p_{|m|}t^{|m|-2}} & \dots & \frac{-p_{|m|-2}}{p_{|m|}t^2} & \frac{-p_{|m|-1}}{p_{|m|}t} \end{pmatrix}, \quad p_{|m|} = m_1^{m_1}\dots m_n^{m_n}.$$

In system (5.8) we make the change

$$y(t) = C(\tau)u(\tau)\big|_{\tau=t^{1/|m|}}, \tag{5.9}$$

where

$$C(\tau) = \operatorname{diag}(1, c_1(\tau), \dots, c_{|m|-1}(\tau)), \quad c_l(\tau) = \left(|m|\tau^{|m|-1}\right)^{-l}, \quad l = 1,\dots,|m|-1. \tag{5.10}$$

As a result, we obtain the system of differential equations

$$u'(\tau) = D(\tau)u(\tau), \quad \tau > 0, \tag{5.11}$$

where

$$D(\tau) = |m|\tau^{|m|-1}C^{-1}(\tau)B(\tau^{|m|})C(\tau) - C^{-1}(\tau)C'(\tau).$$

We calculate $D(\tau)$:

$$D(\tau) = D_0 + D_1(\tau),$$

where the matrix $D_1(\tau)$ satisfies the condition $|D_1(\tau)| \leqslant M_4\tau^{-1}$ as $\tau \geqslant 1$, and the matrix D_0 is determined by the formula

$$D_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \frac{b|m|^{|m|}}{p_{|m|}} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix D_0 are $|m|$-th roots of the number $b|m|^{|m|}/p_{|m|}$:

$$w_k = |m|\left(\frac{b}{p_{|m|}}\right)^{1/|m|} e^{i(\theta+2\pi(k-1))/|m|}, \quad k = 1, \dots, |m|,$$

where θ is the argument of the complex number b. To each eigenvalue w_k, the eigenvector $(1, w_k, \dots, w_k^{|m|-1})^\top$ is associated. By these eigenvectors we form the matrix

$$W = \begin{pmatrix} 1 & \dots & 1 \\ w_1 & \dots & w_{|m|} \\ . & \dots & . \\ w_1^{|m|-1} & \dots & w_{|m|}^{|m|-1} \end{pmatrix}.$$

It is easy to check the identity $W^{-1}D_0W = \Lambda$, where $\Lambda = diag(w_1, \dots, w_{|m|})$. Making the change

$$u(\tau) = Wv(\tau) \tag{5.12}$$

in system (5.11), we obtain the system

$$v'(\tau) = \left(\Lambda + W^{-1}D_1(\tau)W\right)v(\tau), \quad \tau > 0. \tag{5.13}$$

For the coordinates $v_j(\tau)$, $j = 1, \dots, |m|$, of the vector function $v(\tau)$ we have

$$v_j'(\tau) - w_jv_j(\tau) = e_{j1}(\tau)v_1(\tau) + \ldots + e_{j|m|}(\tau)v_{|m|}(\tau),$$

where $|e_{jl}(\tau)| \leqslant \beta_0\tau^{-1}$ for all $\tau \geqslant 1$, $j, l = 1, \dots, |m|$. We multiply each differential equation by $\exp(-w_j\tau)$ and then we integrate from 1 to τ:

$$v_j(\tau)e^{-w_j\tau} = v_j(1)e^{-w_j} + \int_1^\tau \left(e_{j1}(\xi)v_1(\xi) + \ldots + e_{j|m|}(\xi)v_{|m|}(\xi)\right)e^{-w_j\xi}d\xi.$$

We estimate $|v_j(\tau)|$ as $\tau \geqslant 1$:

$$|v_j(\tau)| \leqslant M_5e^{\lambda_m\tau} + \beta_0\int_1^\tau \left(|v_1(\xi)| + \ldots + |v_{|m|}(\xi)|\right)e^{\lambda_m(\tau-\xi)}\xi^{-1}d\xi,$$

where $\lambda_m = \max\left(\mathrm{Re}\,(w_1), \dots, \mathrm{Re}\,(w_{|m|})\right)$. This implies:

$$e^{-\lambda_m\tau}\sum_{j=1}^{|m|}|v_j(\tau)| \leqslant M_5|m| + \beta_0|m|\int_1^\tau \xi^{-1}e^{-\lambda_m\xi}\sum_{j=1}^{|m|}|v_j(\xi)|d\xi \quad \text{as} \quad \tau \geqslant 1.$$

By Grönwall lemma [1] this inequality yields the estimate

$$e^{-\lambda_m\tau}\sum_{j=1}^{|m|}|v_j(\tau)| \leqslant M_5|m|\tau^{\beta_0|m|} \quad \text{as} \quad \tau \geqslant 1.$$

In view of the obtained estimate, changes (5.12), (5.9) and formula (5.10), it is easy to get estimates (5.3). The proof is complete. □

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