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DETERMINATION OF PARAMETERS IN TELEGRAPH EQUATION

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Abstract. We study the solvability of the inverse problems on finding a solution $u(x,t)$ and an unknown coefficient c for a telegraph equation

$$u_{tt} - \Delta u + cu = f(x,t).$$

We prove the theorems on the existence of the regular solutions. The feature of the problems is a presence of new overdetermination conditions for the considered class of equations.

Keywords: telegraph equation, unknown coefficient, inverse problems, special type integral overdetermination, regular solutions, existence.

Mathematics Subject Classification: 35R30, 35L20

Introduction

Mathematical modelling of oscillating processes (propagation of electromagnetic waves [1], [2], of acoustic waves [3], [4], etc.) leads one to studying the solvability of various boundary value problems and the properties of the solutions of the equation

$$u_{tt} - a^2\Delta u + cu = f(x,t)$$

called a telegraph equation. Here the coefficients a and c are positive and are determined by the properties of a media. In the case of a media with apriori unknown properties these coefficients, both or one of them, are unknown and determination of them allow one to analyze a physical process employing the telegraph equation with all known data.

In the present work we study the case of the unknown coefficient c, for simplicity we assume that $a = 1$.

The problems in which one need to determine both a solution and a coefficient or coefficients of a differential equation or the right hand side of the equation are called inverse problem in mathematics and mathematical modelling. As a rule, in such problems, apart of boundary and initial conditions specific for some class of differential equations, other additional conditions are imposed, which are usually called overdetermination conditions. The theory of inverse problems for differential equations is an actively developing direction in the modern mathematics. In particular, the solvability of the inverse problems in various formulations with various overdetermination conditions for hyperbolic equations was studied in in many works, see monographs [4]–[9], papers [10]–[23] and others. At the same time we note, that the inverse problems for the telegraph equation with overdetermination conditions in the present paper were not studied before.

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1. Formulation of the problems

Let Ω be a bounded domain in the space $\mathbb{R}^n$ with a smooth (infinitely differentiable) boundary Γ, Q be a cylinder $\Omega \times (0,T)$ of a finite height T, $S = \Gamma \times (0,T)$, $f(x,t)$, $u_0(x)$, $u_1(x)$ be given functions defined as $x \in \overline{\Omega}$ and as $t \in [0,T]$, A be a given positive number.

Inverse problem I: find a function $u(x,t)$ and a number c such that in the cylinder Q the equation

$$u_{tt} - \Delta u + cu = f(x,t) \tag{1}$$

is satisfied and at that, for the function $u(x,t)$ the conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t)|_S = 0, \tag{3}$$

$$\int\limits_\Omega u^2(x,T)\,dx = A. \tag{4}$$

Inverse problem II: find a function $u(x,t)$ and a number c such that in the cylinder Q equation (1) is satisfied and at that, for the function $u(x,t)$ conditions (2) and (4) hold as well as the condition

$$\frac{\partial u(x,t)}{\partial \nu_x}|_S = 0, \tag{5}$$

$\nu_x = (\nu_1, \dots, \nu_n)$ is the inner normal for Γ at a point x.

In Inverse problems I–II conditions (2) and (3), (2) and (5) are the conditions for the Dirichlet or Neumann initial boundary value problems for a second order hyperbolic equation, while condition (4) is an overdetermination condition. The inverse problems for the hyperbolic equations with such overdetermination condition was not studies before. At the same time, we note that in works [24]–[26], inverse problems of close formulation for hyperbolic equations were studied, but the methods of these works of are not applied directly to Inverse problems I, II.

And one more remark before the substantial part of the work. The solvability of Inverse problems I and II can be established formally by means of the Fourier method. Namely, regarding the coefficient c as known, representing a solution to the Dirichlet and Neumann initial boundary value problem for equation (1) by a Fourier series and employing then overdetermination condition (4), we obtain an algebraic condition for the coefficient c. But it is obvious that this equation has a rather complicated structure and it is not easy to give relatively simple solvability conditions of this equation. The method we propose below gives conditions, which can be checked easily in particular problems and this method can be also applied in situations more general than considered here.

2. Solvability of Inverse problem I

The solvability of Inverse problem I is based on studying the solvability of the Dirichlet initial boundary value problem for some auxiliary nonlinear integral differential ("loaded" [27], [28]) equation. In its turn, the proof of solvability of the direct Dirichlet initial boundary value problem for the auxiliary equation is based on the cut-off functions method, regularization method and the fixed point method.

For the sake of brevity we denote

$$A_0 = \int\limits_\Omega u_0^2(x)\,dx$$

and in what follows we assume the condition

$$A_0 < A. \tag{6}$$

We introduce extra notations:

$$B_0 = \sum_{i=1}^{n} \int\limits_{\Omega} u_{0x_i}^2(x)\,dx, \quad B_1 = \int\limits_{\Omega} u_1^2(x)\,dx, \quad b_0 = \frac{B_0 + B_1}{A - A_0}, \quad b_1 = \frac{1}{A - A_0}, \quad \beta = \frac{b_0}{b_1}.$$

Given a fixed positive number N, we define a cut-off function $G_N(\xi)$:

$$G_N(\xi) = \begin{cases} \xi, & \text{as} \quad 0 \leqslant \xi \leqslant N, \\ N, & \text{as} \quad \xi > N, \\ 0, & \text{as} \quad \xi < 0. \end{cases}$$

Given a function $v(x,t)$ in the space $W_2^2(Q)$, we denote by $\Phi(v)$ the number

$$\Phi(v) = \int\limits_{\Omega} v_t^2(x,T)\,dx + \sum_{i=1}^{n} \int\limits_{\Omega} v_{x_i}^2(x,T)\,dx - 2\int\limits_{Q} f v_t\,dxdt.$$

We consider the problem: find a function $u(x,t)$ solving the equation

$$u_{tt} - \Delta u + [b_0 - b_1 G_\beta(\Phi(u))]u = f(x,t) \tag{7}$$

in the cylinder Q and obeying conditions (2) and (3).

Exactly problem (7), (2), (3) is auxiliary. The solvability of this problem will allow us to establish the solvability of Inverse problem I.

Theorem 1. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$f(x,t) \in L_2(Q), \quad f_t(x,t) \in L_2(Q), \quad u_0(x) \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad u_1(x) \in \overset{\circ}{W}{}_2^1(\Omega).$$

Moreover, assume that condition (6) is satisfied. Then boundary value problem (7), (2), (3) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, $u_t(x,t) \in L_\infty(0,T; \overset{\circ}{W}{}_2^1(\Omega))$, $u_{tt}(x,t) \in L_\infty(0,T; L_2(\Omega))$.

Proof. We employ the regularization method and the fixed point method .

Let ε be a positive number. We consider the following problem: find a function $u(x,t)$ solving the equation

$$u_{tt} - \Delta u + [b_0 - b_1 G_\beta(\Phi(u))]u - \varepsilon \Delta u_t = f(x,t) \tag{7_ε}$$

in the cylinder Q and obeying conditions (2) and (3).

We introduce the linear space V:

$$V = \big\{v(x,t) : v(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)), v_t(x,t) \in L_\infty(0,T; \overset{\circ}{W}{}_2^1(\Omega)), \\ v_{tt}(x,t) \in L_2(Q), \Delta v_t(x,t) \in L_2(Q)\big\}.$$

We equip the space V by the norm

$$\|v\|_V = \Big[\|v\|^2_{L_\infty(0,T;W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega))} + \|v_t\|^2_{L_\infty(0,T;\overset{\circ}{W}{}_2^1(\Omega))} + \|v_{tt}\|^2_{L_2(Q)} + \|\Delta v_t\|^2_{L_2(Q)}\Big]^{\frac{1}{2}}.$$

The space V equipped with this norm is obviously a Banach space.

By using the fixed point method, let us show that boundary value problem (7_ε), (2), (3) is solvable in the space V for a fixed ε and if the function $f(x,t)$ is an element of the space $L_2(Q)$.

Let $w(x,t)$ be a function in the space V. We consider the problem: find a function $u(x,t)$ solving the equation

$$u_{tt} - \Delta u + [b_0 - b_1 G_\beta(\Phi(w))]u - \varepsilon \Delta u_t = f(x,t) \tag{$7_{\varepsilon,w}$}$$

in the cylinder Q and obeying conditions (2) and (3).

Boundary value problem ($7_{\varepsilon,w}$), (2), (3) is the Dirichlet initial boundary value problem for a linear pseudo-hyperbolic equation with constant coefficients and its solvability in the space V for a function $f(x,t)$ in the space $L_2(Q)$ is known, see [29], [30]. Therefore, this problem generates the operator $\Re$ mapping the space V into itself: $\Re(w) = u$. Let us show that the operator $\Re$ has fixed points in the space V.

We consider the identity

$$\int_0^t\int_\Omega \{u_{\tau\tau} - \Delta u + [b_0 - b_1 G_\beta(\Phi(w))]u - \varepsilon\Delta u_\tau\}u_\tau\,dxd\tau = \int_0^t\int_\Omega f u_\tau\,dxd\tau.$$

Integrating by parts, it is easy to pass from this identity to the following one:

$$\begin{aligned}\frac{1}{2}\int_\Omega u_t^2(x,t)\,dx + \frac{1}{2}\sum_{i=1}^n\int_\Omega u_{x_i}^2(x,t)\,dx + \varepsilon\sum_{i=1}^n\int_0^t\int_\Omega u_{x_i\tau}^2\,dxd\tau&\\ + \frac{1}{2}[b_0 - b_1 G_\beta(\Phi(w))]\int_\Omega u^2(x,t)\,dx&\\ =\frac{1}{2}(B_0 + B_1 + b_0A_0) - \frac{1}{2}b_1A_0G_\beta(\Phi(w)) + \int_0^t\int_\Omega f u_\tau\,dxd\tau.&\end{aligned}\tag{8}$$

We observe that the inequalities

$$0 \leqslant b_0 - b_1 G_\beta(\Phi(w)) \leqslant b_0 \tag{9}$$

hold true. We denote

$$N_0 = B_0 + B_1 + b_0A_0 + \int_Q f^2\,dxdt.$$

Employing inequalities (9) and applying Grönwall lemma, we obtain by (8) the estimates

$$\int_0^t\int_\Omega u_\tau^2\,dxd\tau \leqslant N_0(e^t - 1), \tag{10}$$

$$\int_\Omega u_t^2(x,t)\,dx + \sum_{i=1}^n\int_\Omega u_{x_i}^2(x,t)\,dx + \varepsilon\sum_{i=1}^n\int_0^t\int_\Omega u_{x_i\tau}^2\,dxd\tau \leqslant N_0e^t, \tag{11}$$

which are valid for all t in the segment $[0,T]$.

At the next step we consider the identity

$$\int_0^t\int_\Omega \{u_{\tau\tau} - \Delta u + [b_0 - b_1G_\beta(\Phi(w))]u - \varepsilon\Delta u_\tau\}(u_{\tau\tau} - \Delta u_\tau)\,dxd\tau = \int_0^t\int_\Omega f(u_{\tau\tau} - \Delta u_\tau)\,dxd\tau. \tag{12}$$

We integrate by parts, employ estimates (10) and (11) and apply Young inequality to obtain that for all possible solutions $u(x,t)$ of problem ($7_{\varepsilon,w}$), (2), (3) the estimate

$$\|u\|_V \leqslant R_0 \tag{13}$$

holds true, where the number R_0 is determined just by the functions $f(x,t)$, $u_0(x)$, $u_1(x)$ and the numbers ε, A, T.

Estimate (13) means that the operator $\Re$ maps a closed ball of radius R_0 in the space V into itself.

Let us show that the operator $\Re$ is continuous on the space V. Let $\{w_m(x,t)\}_{m=1}^{\infty}$ be a sequence of functions in the space V converging in this space to the function $w_0(x,t)$, $\{v_m(x,t)\}_{m=1}^{\infty}$ be a sequence of images of the functions $w_m(x,t)$ under the action of the operator $\Re$, $v_0(x,t)$ be the image of the function $w_0(x,t)$ under the action of the operator $\Re$. We denote $\overline{w}_m(x,t) = w_m(x,t) - w_0(x,t)$, $\overline{v}_m(x,t) = v_m(x,t) - v_0(x,t)$. The identities

$$\begin{aligned}
&\overline{v}_{mtt} - \Delta\overline{v}_m + [b_0 - b_1 G_\beta(\Phi(w_m))]\overline{v}_m - \varepsilon\Delta\overline{v}_{mt} = b_1[G_\beta(\Phi(w_m)) - G_\beta(\Phi(w_0))]v_0, \quad (x,t)\in Q,\\
&\overline{v}_m(x,0) = \overline{v}_{mt}(x,0) = 0, \quad x\in\Omega, \quad \overline{v}_m(x,t)|_S = 0
\end{aligned}$$

hold true. These identities imply the estimate

$$\begin{aligned}
\int\limits_\Omega \overline{v}_{mt}^2(x,t)\,dx + \sum_{i=1}^n \int\limits_\Omega \overline{v}_{mx_i}^2(x,t)\,dx + \varepsilon\sum_{i=1}^n \int\limits_0^t\int\limits_\Omega \overline{v}_{mx_i\tau}^2\,dxd\tau&\\
\leqslant \int\limits_0^t\int\limits_\Omega v_{m\tau}^2\,dxd\tau + b_1^2[G_\beta(\Phi(w_m)) - G_\beta(\Phi(w_0))]^2 \int\limits_Q v_0^2\,dxdt.&
\end{aligned} \tag{14}$$

We observe that the function $G_\beta(\xi)$ satisfies the Lipschitz condition and the convergence $\Phi(w_m) - \Phi(w_0) \to 0$ holds true as $m\to\infty$. The latter is implied by the continuity of the norm and by the fact that the strong convergence implies the weak one. Taking into consideration these facts and applying Grönwall lemma, by (14) we obtain the convergence

$$\|\overline{v}_m\|_{L_\infty(0,T;W_2^1(\Omega))} \to 0$$

as $m\to\infty$.

We consider the identity

$$\begin{aligned}
\int\limits_0^t\int\limits_\Omega &\overline{v}_{m\tau\tau} - \Delta\overline{v}_m + [b_0 - b_1G_\beta(\Phi(w_m))]\overline{v}_m - \varepsilon\Delta\overline{v}_{m\tau}\}(\overline{v}_{m\tau\tau} - \Delta\overline{v}_{m\tau})\,dxd\tau\\
&= b_1\int\limits_0^t\int\limits_\Omega [G_\beta(\Phi(w_m)) - G_\beta(\Phi(w_0))](\overline{v}_{m\tau\tau} - \Delta\overline{v}_{m\tau})\,dxd\tau.
\end{aligned}$$

Integrating by parts and taking into consider the Lipschitz property of the function $G_\beta(\xi)$ once again as well as the established convergence, it is easy to show that this identity implies an apriori estimate for the family $\{\overline{v}_m(x,t)\}_{m=1}^{\infty}$ in the space V and the convergence

$$\|\overline{v}_m\|_V \to 0 \quad \text{as} \quad m\to\infty.$$

This convergence does mean that the operator $\Re$ is continuous in the space V.

Let us show that the operator $\Re$ is compact.

Let $\{w_m(x,t)\}_{m=1}^{\infty}$ is a bounded family in the space V. Since the families $\{w_{mt}(x,t)\}_{m=1}^{\infty}$, $\{w_{mx_i}(x,t)\}_{m=1}^{\infty}$, $i=1,\dots,n$, are bounded in the space $W_2^1(Q)$ and since the embeddings $W_2^1(Q)\subset L_2(Q)$, $W_2^1(Q)\subset L_2(\partial Q)$ are compact (see [31], [32]), there exits a subsequence

$\{w_{m_k}(x,t)\}_{m=1}^{\infty}$ and a function $w_0(x,t)$ in the space V such that, as $k\to\infty$, we have

$$\begin{array}{lll} w_{m_k}(x,t)\to w_0(x,t) & \text{weakly in} & W_2^2(Q), \\ \Delta w_{m_k t}(x,t)\to \Delta w_{0t}(x,t) & \text{weakly in} & L_2(Q), \\ w_{m_k t}(x,T)\to w_{0t}(x,T) & \text{strongly in} & L_2(\Omega), \\ w_{m_k x_i}(x,T)\to w_{0x_i}(x,T) & \text{strongly in} & L_2(\Omega) \quad \text{for} \quad i=1,\dots,n. \end{array}$$

We again define the functions $v_m(x,t)$, $v_0(x,t)$ as the images of the functions $w_m(x,t)$, $w_0(x,t)$ under the action of the operator $\Re$. Reproducing the proof of the continuity of the operator $\Re$, we obtain that the convergence

$$\|\overline{v}_{m_k}\|_V\to 0 \quad \text{as} \quad k\to\infty.$$

holds true. In other words, each bounded sequence $\{w_m(x,t)\}_{m=1}^{\infty}$ of elements in the space V contains a subsequence $\{w_{m_k}(x,t)\}_{k=1}^{\infty}$ such that the sequence $\{\Re(w_{m_k})\}_{k=1}^{\infty}$ converges strongly in V. This means that the operator $\Re$ is compact.

Thus, the operator $\Re$ maps a closed ball of radius R_0 in the space V into itself, is continuous and compact in the space V. By the Schauder theorem [33], the operator $\Re$ has at least one fixed point in this ball. For this fixed point, the equation (7_ε) holds true and conditions (2) and (3) are satisfied. We denote the sought fixed point by $u^{\varepsilon}(x,t)$. Let us show that under additional assumption $f_t(x,t)\in L_2(Q)$, the family of the functions $\{u^{\varepsilon}(x,t)\}$ contains a sequence converging to a solution of boundary value problem (7), (2), (3).

We choose a sequence $\{\varepsilon_m\}_{m=1}^{\infty}$ of positive numbers such that $\varepsilon_m\to 0$ as $m\to\infty$.

Reproducing the proofs of estimates (10), (11) and (13) but integrating by parts w.r.t. the variable τ in the term $f\Delta u_{\tau}^{\varepsilon_m}$ in the right hand side of (12), it is easy to obtain the estimate

$$\varepsilon_m\int\limits_Q (\Delta u_{\tau}^{\varepsilon_m})^2\,dxd\tau+\|\Delta u^{\varepsilon_m}\|^2_{L_\infty(0,T;L_2(\Omega))}+\|u_{tt}^{\varepsilon_m}\|^2_{L_2(Q)}\leqslant M_1$$

with a constant M_1 independent of ε. Employing the compactness of the embeddings $W_2^1(Q)\subset L_2(Q)$, $W_2^1(Q)\subset L_2(\partial Q)$, we obtain that there exists a sequence $\{m_k\}_{k=1}^{\infty}$ of natural numbers and a function $u(x,t)$ such that as $k\to\infty$ the convergences

$$\begin{array}{lll} \varepsilon_{m_k}\to 0, & & \\ u^{\varepsilon_{m_k}}(x,t)\to u(x,t) & \text{weakly in} & W_2^2(Q), \\ \varepsilon_{m_k}\Delta u_t^{\varepsilon_{m_k}}(x,t)\to 0 & \text{weakly in} & L_2(Q), \\ u_t^{\varepsilon_{m_k}}(x,T)\to u_t(x,T) & \text{strongly in} & L_2(\Omega), \\ u_{x_i}^{\varepsilon_{m_k}}(x,T)\to u_{x_i}(x,T) & \text{strongly in} & L_2(\Omega) \quad \text{for} \quad i=1,\dots,n, \\ \Phi(u^{\varepsilon_{m_k}})\to\Phi(u). & & \end{array}$$

These convergences imply that the limiting function $u(x,t)$ solves equation (7) and obeys conditions (2) and (3). The limiting function $u(x,t)$ belongs to $L_\infty(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$. This belonging and the belonging of the function $f(x,t)$ to the space $L_\infty(0,T;L_2(\Omega))$ imply that the function $u_{tt}(x,t)$ belongs to the same space. In other words, the limiting function $u(x,t)$ belongs to the sought class and in this way it is the required solution to boundary value problem (7), (2), (3). □

We proceed to studying the solvability of Inverse problem I.

The most simple case is when in Problem I we have $f(x,t)\equiv 0$.

Theorem 2. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$f(x,t) \equiv 0, \quad u_0(x) \in W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega), \quad u_1(x) \in \mathring{W}_2^1(\Omega)$$

and condition (6) be satisfied. Then Inverse problem I has the solution $\{u(x,t), c\}$ such that $u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))$, $u_t(x,t) \in L_\infty(0,T; \mathring{W}_2^1(\Omega))$, $u_{tt}(x,t) \in L_\infty(0,T; L_2(\Omega))$, $c \geqslant 0$.

Proof. Under the assumptions of the theorem boundary value problem (7), (2), (3) has the solution $u(x,t)$ belonging to class mentioned in the theorem. We observe that in the case $f(x,t) \equiv 0$ we have $\Phi(u) \geqslant 0$. We multiply equation (7) by the function $u_t(x,t)$ and integrate over the cylinder Q. We obtain the identity

$$\begin{gathered}\frac{1}{2}\int\limits_\Omega u_t^2(x,T)\,dx + \frac{1}{2}\sum_{i=1}^n \int\limits_\Omega u_{x_i}^2(x,T)\,dx + \frac{b_0 - b_1 G_\beta(\Phi(u))}{2}\int\limits_\Omega u^2(x,T)\,dx \\ = \frac{1}{2}(B_0 + B_1 + b_0 A_0) - \frac{b_1 A_0 G_\beta(\Phi(u))}{2}.\end{gathered} \tag{15}$$

This identity implies the estimate

$$\Phi(u) + b_1 A_0 G_\beta(\Phi(u)) \leqslant B_0 + B_1 + b_0 A_0;$$

here we have employed the left inequality in (9). The inequality $G_\beta(\Phi(u)) \leqslant \Phi(u)$ holds. This implies

$$(1 + b_1 A_0) G_\beta(\Phi(u)) \leqslant B_0 + B_1 + b_0 A_0,$$

or

$$G_\beta(\Phi(u)) \leqslant B_0 + B_1.$$

Then the identity $G_\beta(\Phi(u)) = \Phi(u)$ holds. It means that the solution $u(x,t)$ of boundary value problem (7), (2), (3) solves the equation

$$u_{tt} - \Delta u + [b_0 - b_1 \Phi(u)]u = 0. \tag{16}$$

We denote $c = b_0 - b_1\Phi(u)$. Then the function $u(x,t)$ and the number c are related by equation (1) in the cylinder Q, the function $u(x,t)$ satisfies the required belongings, conditions (2) and (3) are satisfied and the number c is positive. Let us show that the function $u(x,t)$ satisfies condition (4).

We denote

$$\alpha = \int\limits_\Omega u^2(x,T)\,dx.$$

The identities

$$c(A - A_0) = b_0 - b_1\Phi(u), \quad c(\alpha - A_0) = b_0 - b_1\Phi(u)$$

hold true. This implies

$$c(\alpha - A) = 0. \tag{17}$$

We denote by $U_0(x,t)$ the solution to boundary value problem (1)–(3) in the case $c = 0$. If the identity

$$A = \int\limits_\Omega U_0^2(x,T)\,dx \tag{18}$$

holds, then the pair $\{U_0(x,t), 0\}$ solves Inverse problem I and this implies immediately condition (4). If identity (18) does not hold, then the number c is non-zero and thus, (17) implies $\alpha = A$. And this means that the solution $u(x,t)$ to boundary value problem (7), (2), (3) satisfies all required belongings and conditions (2)–(4) hold. Hence, the pair $\{u(x,t), b_0 - b_1\Phi(u)\}$ is the sought solution to Inverse problem I. The proof is complete. $\square$

While studying the solvability of Inverse problem I with a nonzero function $f(x,t)$, an important role is played by the inequality $\Phi(u) \geqslant 0$ for solutions $u(x,t)$ to boundary value problem (7), (2). (3). We provide a simple statement giving sufficient conditions ensuring this inequality.

Let $\psi(x)$ be a function in the space $\overset{\circ}{W}{}_2^1(\Omega)$. The identity

$$\int\limits_{\Omega} \psi^2(x)\,dx \leqslant m_0 \sum_{i=1}^{n} \int\limits_{\Omega} \psi_{x_i}^2(x)\,dx \tag{19}$$

holds true with some number m_0 determined just by the domain Ω.

We let

$$N_1 = [m_0 N_0 (e^T - 1)]^{\frac{1}{2}}, \quad N_2 = (T^2 N_0 e^T + 2TA_0)^{\frac{1}{2}},$$

where the number N_0 was determined in the proof of apriori estimates (10) and (11) for solutions of boundary value problems ($7_{\varepsilon,w}$), (2), (3).

Proposition 1. *Let the assumptions of Theorem 1 be satisfied as well as the condition*

$$2\left(\int\limits_{Q} f_t^2\,dxdt\right)^{\frac{1}{2}} \min(N_1, N_2) \leqslant 2\int\limits_{\Omega} f(x,0)u_0(x)\,dx - m_0 \int\limits_{\Omega} f^2(x,T)\,dx. \tag{20}$$

Then for the solutions $u(x,t)$ of boundary value problem (7), (2), (3) the inequality

$$\Phi(u) \geqslant 0 \tag{21}$$

holds true.

Proof. We first of all observe that there exists a solution $u(x,t)$ of boundary value problem (7), (2), (3) in the class mentioned in Theorem 1. For the sake of brevity we denote

$$I(u) = \int\limits_{\Omega} u_t^2(x,T)\,dx + \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i}^2(x,T)\,dx.$$

The identity

$$\Phi(u) = I(u) + 2\int\limits_{\Omega} f(x,0)u_0(x)\,dx - 2\int\limits_{\Omega} f(x,T)u(x,T)\,dx + 2\int\limits_{Q} f_t u\,dxdt \tag{22}$$

holds true. The inequalities

$$
\begin{aligned}
2\left|\int\limits_\Omega f(x,T)u(x,T)\,dx\right| \leqslant& \delta^2\int\limits_\Omega u^2(x,T)\,dx+\frac{1}{\delta^2}\int\limits_\Omega f^2(x,T)\,dx\\
\leqslant& \delta^2 m_0\sum_{i=1}^n\int\limits_\Omega u^2_{x_i}(x,T)\,dx+\frac{1}{\delta^2}\int\limits_\Omega f^2(x,T)\,dx\\
\leqslant& \delta^2 m_0 I+\frac{1}{\delta^2}\int\limits_\Omega f^2(x,T)\,dx,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
2\left|\int\limits_Q f_t u\,dxdt\right| \leqslant& 2\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12}\left(\int\limits_Q u^2\,dxdt\right)^{\frac12}\\
\leqslant& 2\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12}\left(m_0\sum_{i=1}^n\int\limits_Q u^2_{x_i}\,dxdt\right)^{\frac12}\leqslant 2N_1\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
2\left|\int\limits_Q f_t u\,dxdt\right| \leqslant& 2\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12}\left(\int\limits_Q u^2\,dxdt\right)^{\frac12}\\
\leqslant& 2\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12}\left[T^2\int\limits_Q u_t^2\,dxdt+2A_0T\right]^{\frac12}\leqslant 2N_2\left(\int\limits_Q f_t^2\,dxdt\right)^{\frac12}.
\end{aligned}
\tag{25}
$$

are valid.

In inequality (23) we fix a number δ so that $\delta^2 m_0=1$. Then representation (22), inequality (23)–(25) and condition (20) imply inequality (21). The proof is complete. $\square$

Theorem 3. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$f(x,t)\in L_2(Q),\quad f_t\in L_2(Q),\quad u_0(x)\in W_2^2(\Omega)\cap \mathring{W}_2^1(\Omega),\quad u_1(x)\in \mathring{W}_2^1(\Omega),$$

and conditions (6) and (20) be satisfied. Then Inverse problem I has a solution $\{u(x,t),c\}$ satisfying

$$
\begin{aligned}
&u(x,t)\in L_\infty(0,T;W_2^2(\Omega)\cap\mathring{W}_2^1(\Omega)),\quad u_t(x,t)\in L_\infty(0,T;\mathring{W}_2^1(\Omega)),\\
&u_{tt}(x,t)\in L_\infty(0,T;L_2(\Omega)),\quad c\geqslant 0.
\end{aligned}
$$

As for Theorem 2, the proof of this theorem is based on proving the estimate

$$G_\beta(\Phi(u))\leqslant B_0+B_1;$$

at that, one should use inequality $\Phi(u)\geqslant 0$. We just mention that instead of the function $U_0(x,t)$ we should use the function $U_1(x,t)$ solving initial boundary value problem (1)–(3) in the case $c=0$.

3. Solvability of inverse problem II

The study of the solvability of Inverse problem II is made similar to that for Inverse problem I. First we establish the solvability of the auxiliary problem (7), (2), (5) and this can be done similar to the proof of the solvability of problem (7), (2), (3). Then we prove the solvability of Inverse problem II in the case $f(x,t)\equiv 0$ that can be done similar to the proof of Theorem 2.

At the next step we prove the inequality $\Phi(u) \geqslant 0$ for the solutions $u(x,t)$ of boundary value problem (7), (2), (5). Finally, at the last step we prove the solvability of Inverse problem II for a nonzero function $f(x,t)$. The differences between this scheme and that for Inverse problem I are just in the conditions ensuring $\Phi(u) \geqslant 0$ for the solutions to boundary value problem (7), (2), (5). This is why we provide completely an appropriate statement.

We let

$$N_3 = \left(2TN_0(e^T - 1) + 2A_0\right)^{\frac{1}{2}}.$$

Proposition 2. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$f(x,t) \in L_2(Q), \quad f_t(x,t) \in L_2(Q), \quad u_0(x) \in W_2^2(\Omega), \quad \frac{\partial u_0(x)}{\partial \nu} = 0$$

as $x \in \Gamma$, $u_1(x) \in W_2^1(\Omega)$. Moreover, assume that conditions (6) are satisfied as well as the condition

$$N_3 \left(\int\limits_\Omega f^2(x,T)\,dx \right)^{\frac{1}{2}} + N_2 \left(\int\limits_Q f_t^2\,dxdt \right)^{\frac{1}{2}} \leqslant \int\limits_\Omega f(x,0)u_0(x)\,dx. \tag{26}$$

Then for the solutions $u(x,t)$ of boundary value problem (7), (2), (5) inequality (21) holds true.

Proof. Under the belongings for the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ and condition (6) boundary value problem (7), (2), (5) has a solution $u(x,t)$ such that $u(x,t) \in L_\infty(0,T;W_2^2(\Omega))$, $u_t(x,t) \in L_\infty(0,T;W_2^1(\Omega))$, $u_{tt}(x,t) \in L_\infty(0,T;L_2(\Omega))$; as it has been said above, the appropriate theorem can be proved exactly in the same way as Theorem 1. Function $\Phi(u)$ satisfies representation (22), for the function $u(x,t)$ the inequalities

$$\begin{aligned} 2\left| \int\limits_\Omega f(x,T)u(x,T)\,dx \right| &\leqslant 2\left(\int\limits_\Omega f^2(x,T)\,dx \right)^{\frac{1}{2}} \left(\int\limits_\Omega u^2(x,T)\,dx \right)^{\frac{1}{2}} \\ &\leqslant 2\left(\int\limits_\Omega f^2(x,T)\,dx \right)^{\frac{1}{2}} \left[2T\int\limits_Q u_t^2\,dxdt + 2A_0 \right]^{\frac{1}{2}} \leqslant 2N_3 \left(\int\limits_\Omega f^2(x,T)\,dx \right)^{\frac{1}{2}}. \end{aligned} \tag{27}$$

hold true. Employing (27), (25), condition (26) and representation (22), we obtain inequality (21) for the function $u(x,t)$. The proof is complete. □

In conclusion we formulate the solvability theorem for Inverse problem II.

Theorem 4. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$f(x,t) \equiv 0, \quad u_0(x) \in W_2^2(\Omega), \quad \frac{\partial u_0(x)}{\partial \nu} = 0 \quad \textit{as} \quad x \in \Gamma, \quad u_1(x) \in W_2^1(\Omega),$$

and condition (6) be satisfied. Then Inverse problem II has a solution $\{u(x,t), c\}$ such that

$$\begin{aligned} &u(x,t) \in L_\infty(0,T;W_2^2(\Omega)), \quad u_t(x,t) \in L_\infty(0,T;W_2^1(\Omega)), \\ &u_{tt}(x,t) \in L_\infty(0,T;L_2(\Omega)), \quad c \geqslant 0. \end{aligned}$$

Theorem 5. *Let the functions $f(x,t)$, $u_0(x)$ and $u_1(x)$ be such that*

$$\begin{aligned} &f(x,t) \in L_2(Q), \quad f_t(x,t) \in L_2(Q), \quad u_0(x) \in W_2^2(\Omega), \\ &\frac{\partial u_0(x)}{\partial \nu} = 0 \quad \textit{as} \quad x \in \Gamma, \quad u_1(x) \in W_2^1(\Omega), \end{aligned}$$

and conditions (6) and (26) be satisfied. Then Inverse problem II has a solution $\{u(x,t), c\}$ *such that*

$$u(x,t) \in L_\infty(0,T; W_2^2(\Omega)), \quad u_t(x,t) \in L_\infty(0,T; W_2^1(\Omega)),$$
$$u_{tt}(x,t) \in L_\infty(0,T; L_2(\Omega)), \quad c \geqslant 0.$$

The proofs of these theorems are obvious.

4. Comments and additions

1. The technique used in the present work can be employed in many other situations. For instance, by means of this technique one can study the solvability of inverse problems with overdetermination condition (4), the unknown solution $u(x,t)$ and the unknown constant coefficient c

a) for Barenblatt-Zheltova-Kochina equation

[34]

$$u_t - a\Delta u - b\Delta u_t + cu = f(x,t) \tag{28}$$

$(a > 0,\ b > 0)$;

b) for pseudo-hyperbolic equations

$$u_{tt} - a\Delta u - b\Delta u_t + cu = f(x,t) \tag{29}$$

$(a > 0,\ b > 0)$;

c) for non-stationary high order equations

$$u_{tt} + (-1)^m \Delta^m u + cu = f(x,t), \tag{30}$$

$m > 0$ is integer.

Together with inverse problems for equations (28)–(30) with constants coefficients, it is easy to study the inverse problems similar to the studied above for some non-stationary equations with variable coefficients. For instance, employing the technique of the present work, it is easy to study the solvability of the inverse problem with conditions (2)–(4) for the equations

$$u_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{ij}(x,t)u_{x_j}\right) + b(x,t)u_t + cu = f(x,t)$$

under the conditions

$$a^{ij}(x,t) \in C^2(\overline{Q}), \quad a^{ij}(x,t) = a^{ji}(x,t), \quad i,j = 1,\dots,n, \quad (x,t) \in \overline{Q};$$
$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geqslant a_0|\xi|^2, \quad a_0 > 0, \quad (x,t) \in \overline{Q}, \quad \xi \in \mathbb{R}^n;$$
$$\sum_{i,j=1}^{n} a_t^{ij}(x,t)\xi_i\xi_j \leqslant 0, \quad (x,t) \in \overline{Q}, \quad \xi \in \mathbb{R}^n;$$
$$b(x,t) \in C(\overline{Q}), \quad b(x,t) \geqslant b_0 > 0, \quad (x,t) \in \overline{Q}.$$

It is easy to provide other examples of equations with variable coefficients, for which Inverse problems I and II can be studied by the technique of the present work.

2. Conditions (20) and (26) are obviously satisfied for the identically vanishing function $f(x,t)$, at that, the function $u_0(x)$ can be both zero or non-zero. If the function $f(x,t)$ is not identically zero, then the function $u_0(x)$ is not identically zero as well. Moreover, the number

$$\int_\Omega f(x,0)u_0(x)\,dx$$

should be positive. To satisfy conditions (20) or (26), it is sufficient, for instance, if the number T is small enough.

3. In the present work we do not study the uniqueness of solutions to Inverse problems I and II.

4. The results on solvability of boundary value problems (7), (2), (3) and (7), (2), (5) can be of independent interest for the theory of "loaded" equations. We note that in equation (7), the function $\Phi(u)$ can be replaced by the function $\widetilde{\Phi}(u)$ having the form

$$\widetilde{\Phi}(u) = \int\limits_{\Omega} u_t^2(x,T)\,dx + \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i}^2(x,T)\,dx - 2\int\limits_{Q} g u_t\,dxdt$$

with the function $g(x,t)$ such that $g(x,t) \in L_2(Q)$, $g_t(x,t) \in L_2(Q)$.

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