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GROWTH REGULARITY FOR THE ARGUMENTS OF MEROMORPHIC IN $\mathbb{C} \setminus \{0\}$ FUNCTIONS OF COMPLETELY REGULAR GROWTH

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Abstract. We study the asymptotic behaviour for the arguments of meromorphic function in $\mathbb{C} \setminus \{0\}$ of completely regular growth with respect to a growth function λ. We find that that the key role in the description of this behaviour is played by the function $\lambda_1(r) = \int_1^r \lambda(t)/t\,dt$.

Keywords: meromorphic function, function of moderate growth, completely regular growth, growth indicator, Fourier coefficients.

Mathematics Subject Classification: 30D15, 30D35

1. Introduction

The theory of entire functions of completely regular growth with respect to the function λ close to a power function was created in late 30's of the last century by B. Levin and A. Pfluger. This theory has many applications in various areas of modern complex analysis. A full exposition this theory as well as its applications can be found in [1].

Using the Fourier series method developed by L. A. Rubel and B. A. Taylor [2], A. A. Kondratyuk generalized Levin-Pfluger theory of entire functions of completely regular growth. The growth of a function was measured with respect to an arbitrary non-decreasing continuous function λ satisfying the condition $\lambda(2r) \leqslant M\lambda(r)$ for some $M > 0$ and all $r > 0$. This generalization made it possible to describe asymptotic behaviour of entire functions of completely regular growth in L_p-metrics. He also introduced the classes of meromorphic functions of completely regular growth [3], [4], [5]. One can find a thorough description of this theory in [6].

The next possible step in this field is to extend and generalize this theory for multiply connected domains. Many authors studied meromorphic functions in multiply connected domains. One of the recent approaches was proposed in [7], [8], [9]. Using a Nevanlinna type characteristic introduced in these works and the notion of finite λ-density [9], the notion of holomorphic function of completely regular growth in the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ was introduced in [10] as well as its growth indicators. This work was concerned mainly with the properties of the growth indicators. Another work in this direction is [11], where using the Fourier series method under general assumptions, the problem of description of the sets of holomorphic functions in $\mathbb{C}^*$ possessing the property of simultaneous regular growth of the logarithm of modulus and argument was solved.

In the present work we make further studies in this direction. Namely, we study the asymptotic behaviour of the arguments of meromorphic functions of completely regular growth in $\mathbb{C}^*$.

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2. Definitions, notations and main results

Definition 1 ([9]). *A positive nondecreasing continuous unbounded function λ in $[1,+\infty)$ is said to be a growth function.*

We say that a growth function is a function of *moderate growth* if there exists positive M such that $\lambda(2r) \leqslant M\lambda(r)$ for all $r \geqslant 1$. Let λ be a function of moderate growth. We denote

$$\lambda_1(r) := \int_1^r \frac{\lambda(t)}{t} dt. \tag{1}$$

Let f be a meromorphic function in $\mathbb{C}^*$ not vanishing identically. By A^* we denote $\mathbb{C}^*$ without the intervals $\{z = \tau a : \tau \geqslant 1\}$ if $|a| > 1$, and $\{z = \tau a : 0 \leqslant \tau \leqslant 1\}$ if $|a| < 1$, where a is a zero or pole of f. Let $\{a_j\}$ be the zeros and $\{b_j\}$ be the poles of f,

$$\tilde{f}(z) = f(z) \prod_{|a_j|=1} (z - a_j)^{-1} \prod_{|b_j|=1} (z - b_j).$$

Then [9, Lemma 4.1] there exists $m \in \mathbb{Z}$ such that for the function $F(z) = z^{-m}\tilde{f}(z)$ and for any given closed path $\gamma \in A^*$ we have

$$\int_\gamma \frac{F'(z)}{F(z)} dz = 0.$$

This allows us to determine a branch of the logarithm of $F(z)$ in A^*. We observe that

$$m = \frac{1}{2\pi i} \int_{|z|=1} \frac{\tilde{f}'(z)}{\tilde{f}(z)} dz,$$

see [9, Lemma 4.1].

We use the following notations for the Fourier coefficients

$$l_k(t, F) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \log F(te^{i\theta}) d\theta, \quad t > 0, \quad k \in \mathbb{Z}, \tag{2}$$

$$c_k(t, F) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \log |F(te^{i\theta})| d\theta, \quad t > 0, \quad k \in \mathbb{Z}, \tag{3}$$

$$a_k(t, F) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \arg F(te^{i\theta}) d\theta, \quad t > 0, \quad k \in \mathbb{Z}. \tag{4}$$

Remark 1. *Note that*

$$c_k(t, F) = \frac{1}{2}(l_k(t, F) + \overline{l_{-k}(t, F)}), \quad a_k(t, F) = \frac{1}{2i}(l_k(t, F) - \overline{l_{-k}(t, F)}),$$

for $t > 0$ and $k \in \mathbb{Z}$.

The Nevanlinna type characteristic $T_0(r, f)$ for a function f meromorphic on the annulus $\{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leqslant +\infty$, was introduced in [7]. Namely,

$$T_0(r, f) = m_0(r, f) + N_0(r, f), \qquad 1 < r < R_0,$$

where

$$m_0(r,f) = m(r,f) + m\left(\frac{1}{r}, f\right) - 2m(1,f),$$

$$m(t,f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ |f(te^{i\theta})|\, d\theta, \qquad \frac{1}{R_0} < t < R_0,$$

$$N_0(r,f) = \int_1^r \frac{n_0(t,f)}{t} dt,$$

$n_0(t,f)$ is the counting function of the poles of f in the annulus $1/t \leqslant |z| \leqslant t$, $t \geqslant 1$. This characteristic possesses the properties (see [7], [8], [9]) similar to the properties of the classical Nevanlinna characteristic $T(r,f)$ [12].

Definition 2 ([9])**.** *Let λ be a growth function and f be a meromorphic function in $\mathbb{C}^*$. We say that f is of finite λ-type if $T_0(r,f) \leqslant A\lambda(Br)$, for some positive constants A, B and for all r, $r \geqslant 1$.*

Definition 3. *A meromorphic function f in $\mathbb{C}^*$ is called a function of the first type completely regular growth (c.r.g.1) if f is of finite λ-type and for all $k \in \mathbb{Z}$ there exist*

$$\lim_{r\to+\infty} \frac{c_k(r,f)}{\lambda(r)} =: c_k^1 \quad \textit{and} \quad \lim_{r\to+\infty} \frac{c_k(\frac{1}{r}, f)}{\lambda(r)} =: c_k^2.$$

Definition 4. *A meromorphic function f in $\mathbb{C}^*$ is called a function of the second type completely regular growth (c.r.g.2) if f is of finite λ-type and for all $k \in \mathbb{Z}$ there exist*

$$\lim_{r\to+\infty} \frac{c_k(r,f) + c_k(\frac{1}{r}, f)}{\lambda(r)} =: c_k^*.$$

We denote by $\Lambda^{\circ,1}$, $\Lambda^{\circ,2}$ the classes of meromorphic functions of c.r.g.1 and c.r.g.2 in $\mathbb{C}^*$ respectively. If $f \in \Lambda^{\circ,1}$ or $f \in \Lambda^{\circ,2}$ we say that f is of completely regular growth (c.r.g.) in $\mathbb{C}^*$.

Remark 2. *It is obvious that $\Lambda^{\circ,1} \subset \Lambda^{\circ,2}$. However, these classes do not coincide.*

For example, take a growth function λ and an entire function g of finite λ-type, which is not of c.r.g. with respect to λ in the entire complex plane $\mathbb{C}$. Without loss of generality we can assume $g(0) \neq 0$ and g has no zeros on the unit circle $|z| = 1$. For $z \in \mathbb{C}^*$ put $f(z) = g(z)/\overline{g(1/\overline{z})}$. Then f is obviously meromorphic in $\mathbb{C}^*$. We have that $\log |g(1/\overline{z})|$ is bounded as $z \to \infty$. And because g is not of c.r.g. in $\mathbb{C}$, there exists $k \in \mathbb{Z}$ such that the limit $\lim\limits_{r\to+\infty} \frac{c_k(r,g)}{\lambda(r)}$ does not exist or is infinite (see [3] or [6]). Therefore, there is no finite limit of $\frac{c_k(r,f)}{\lambda(r)}$ as $r \to +\infty$ for that same k. Hence, $f \notin \Lambda^{\circ,1}$. On the other hand, $c_k(r,f) + c_k(\frac{1}{r}, f) = 0$ for all $k \in \mathbb{Z}$. Thus, $f \in \Lambda^{\circ,2}$.

Definition 5 ([10])**.** *If f is of c.r.g.1 then the functions*

$$h_1(\theta, f) = \sum_{k\in\mathbb{Z}} c_k^1 e^{ik\theta}, \quad h_2(\theta, f) = \sum_{k\in\mathbb{Z}} c_k^2 e^{ik\theta}$$

are called the growth indicators of f; in case of c.r.g.2 the growth indicator of f is

$$h(\theta, f) = \sum_{k\in\mathbb{Z}} c_k^* e^{ik\theta},$$

where c_k^1, c_k^2, c_k^* are given by Definitions 3 and 4.

Our main results are the following theorems.

Theorem 1. *Let λ be a function of moderate growth, λ_1 be defined by (1), f be a meromorphic in $\mathbb{C}^*$ function of c.r.g.1 with respect to λ, and h_1, h_2 be the growth indicators of f. Then for all $p \in [1, +\infty)$*

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\arg F(re^{i\theta}) + \lambda_1(r)h_1'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \qquad r \to +\infty, \tag{5}$$

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\arg F\left(\frac{1}{r}e^{i\theta}\right) - \lambda_1(r)h_2'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \quad r \to +\infty. \tag{6}$$

Theorem 2. *Let λ be a function of moderate growth, λ_1 be defined by (1), f be a meromorphic in $\mathbb{C}^*$ function of c.r.g.2 with respect to λ, and h be the growth indicator of f. Then for all $p \in [1, +\infty)$*

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\arg F(re^{i\theta}) - \arg F\left(\frac{1}{r}e^{i\theta}\right) + \lambda_1(r)h'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \; r \to +\infty. \tag{7}$$

Theorem 3. *Let λ be a function of moderate growth, λ_1 be defined by (1), f be a meromorphic in $\mathbb{C}^*$ function of c.r.g.1 with respect to λ, and h_1, h_2 be the growth indicators of f. Then for all $p \in [1, +\infty)$*

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\log F(re^{i\theta}) - \lambda(r)h_1(\theta, f) + \lambda_1(r)h_1'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \quad r \to +\infty, \tag{8}$$

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\log F(\frac{1}{r}e^{i\theta}) - \lambda(r)h_2(\theta, f) - \lambda_1(r)h_2'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \quad r \to +\infty. \tag{9}$$

Theorem 4. *Let λ be a function of moderate growth, λ_1 be defined by (1), f be a meromorphic in $\mathbb{C}^*$ function of c.r.g.2 with respect to λ, and h be the growth indicator of f. Then for all $p \in [1, +\infty)$*

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\log F(re^{i\theta}) + \overline{\log F\left(\frac{1}{r}e^{i\theta}\right)} - \lambda(r)h(\theta, f) + \lambda_1(r)h'(\theta, f)\right|^p d\theta\right\}^{\frac{1}{p}} = o(\lambda_1(r)), \tag{10}$$

as $r \to +\infty$.

3. Auxiliary results

Let f be a meromorphic in $\mathbb{C}^*$ function not vanishing identically, $F(z) = z^{-m}\tilde{f}(z)$, where $\tilde{f}$ and m are determined as above. Let $\{b_j\}$ be the poles of f, $\sigma_j = \arg b_j$. For $k \in \mathbb{Z}$ we denote ([10])

$$n_k^1(r, f) = \sum_{1<|b_j|\leqslant r} e^{-ik\sigma_j}, \qquad n_k^2(r, f) = \sum_{\frac{1}{r}\leqslant|b_j|<1} e^{-ik\sigma_j}, \qquad r > 1,$$

and

$$n_k(r, f) = \sum_{\frac{1}{r}\leqslant|b_j|\leqslant r} e^{-ik\sigma_j}, \qquad r \geqslant 1,$$

where every pole b_j is counted according to its multiplicity. In particular, $n_0(r, f)$ is the counting function that appears in the definition of $T_0(r, f)$. We assume $n_k^1(1, f) = n_k^2(1, f) = 0$ for all $k \in \mathbb{Z}$. Thus,

$$n_k(r, f) = n_k^1(r, f) + n_k^2(r, f) + n_k(\mathbb{T}, f), \quad k \in \mathbb{Z}, \quad r \geqslant 1,$$

where $\mathbb{T} = \{z : |z| = 1\}$ and $n_k(\mathbb{T}, f) = n_k(1, f) = \sum_{|b_j|=1} e^{-ik\sigma_j}$.

Remark 3. *Note that* $|n_k^i(r, f)| \leqslant n_0^i(r, f)$, $i = 1, 2$, $|n_k(\mathbb{T}, f)| \leqslant n_0(\mathbb{T}, f)$, *and consequently* $|n_k(r, f)| \leqslant n_0(r, f)$ *for all* $k \in \mathbb{Z}$ *and* $r \geqslant 1$.

Let

$$N_k^i(r, f) = \int_1^r \frac{n_k^i(t, f)}{t} dt, \ \ i = 1, 2, \ \ N_k(r, f) = \int_1^r \frac{n_k(t, f)}{t} dt, \ \ k \in \mathbb{Z}, \ \ r \geqslant 1. \tag{11}$$

Since F does not have zeros and poles on the unit circle $\mathbb{T}$, we have that $\log F$ is holomorphic in some annular neighbourhood of the unit circle, and therefore admits a Laurent series expansion

$$\log F(z) = \sum_{k \in \mathbb{Z}} \alpha_k z^k \tag{12}$$

in that neighbourhood.

To prove our main results, we need following auxiliary lemmas.

Lemma 1. *The identities*

$$l_k(r, F) = \alpha_k r^k + r^k \int_1^r \frac{n_k^1(t, 1/f) - n_k^1(t, f)}{t^{k+1}} dt, \quad k \neq 0, \quad r \geqslant 1, \tag{13}$$

$$l_0(r, F) - l_0(1, F) = N_0^1(r, 1/f) - N_0^1(r, f), \quad r \geqslant 1, \tag{14}$$

hold true.

Lemma 2. *The identities*

$$l_k\left(\frac{1}{r}, F\right) = \alpha_k r^{-k} + r^{-k} \int_1^r \frac{n_k^2\left(t, \frac{1}{f}\right) - n_k^2(t, f)}{t^{-k+1}} dt, \quad k \neq 0, \quad r \geqslant 1, \tag{15}$$

$$l_0\left(\frac{1}{r}, F\right) - l_0(1, F) = N_0^2\left(r, \frac{1}{f}\right) - N_0^2(r, f), \quad r \geqslant 1, \tag{16}$$

hold true.

Lemma 3. *The identities*

$$\begin{aligned} a_k(r, F) = -ik \int_1^r \frac{c_k(t, f)}{t} dt + \\ + \frac{\alpha_k - \overline{\alpha_{-k}}}{2i} - \frac{r^{-k} - 1}{2ki} \left(n_k\left(\mathbb{T}, \frac{1}{f}\right) - n_k(\mathbb{T}, f)\right), \quad k \neq 0, \quad r \geqslant 1, \end{aligned} \tag{17}$$

$$\begin{aligned} a_k\left(\frac{1}{r}, F\right) = ik \int_1^r \frac{c_k\left(\frac{1}{t}, f\right)}{t} dt + \\ + \frac{\alpha_k - \overline{\alpha_{-k}}}{2i} + \frac{r^{-k} - 1}{2ki} \left(n_k\left(\mathbb{T}, \frac{1}{f}\right) - n_k(\mathbb{T}, f)\right), \quad k \neq 0, \quad r \geqslant 1. \end{aligned} \tag{18}$$

hold true.

Lemma 4. *The identities*

$$\begin{gathered}c_k(r,f)=ik\int\limits_1^r\frac{a_k(t,F)}{t}dt+\frac{\alpha_k+\overline{\alpha_{-k}}}{2}+\\+N_k^1\left(r,\frac{1}{f}\right)-N_k^1(r,f)-\frac{n_k\left(\mathbb{T},\frac{1}{f}\right)-n_k(\mathbb{T},f)}{2kr^k},\quad k\neq 0,\quad r\geqslant 1,\end{gathered}\tag{19}$$

$$\begin{gathered}c_k\left(\frac{1}{r},F\right)=-ik\int\limits_1^r\frac{a_k\left(\frac{1}{t},F\right)}{t}dt+\frac{\alpha_k+\overline{\alpha_{-k}}}{2}+\\+N_k^2\left(r,\frac{1}{f}\right)-N_k^2(r,f)-\frac{n_k\left(\mathbb{T},\frac{1}{f}\right)-n_k(\mathbb{T})}{2kr^k},\quad k\neq 0,\quad r\geqslant 1,\end{gathered}\tag{20}$$

hold true.

For a holomorphic function f, Lemmata 1 - 4 were proved in [11]. The presence of poles does not complicate the proof essentially.

Lemma 5. *The identities*

$$\begin{aligned}N_k^1(r,f)=&c_k(r,f)-k^2\int\limits_1^r\frac{dt}{t}\int\limits_1^t\frac{c_k(\tau,f)}{\tau}d\tau-c_k(1,f)-\\&-ik\cdot a_k(1,F)\log r-\frac{n_k(\mathbb{T},1/f)-n_k(\mathbb{T},f)}{2}\log r,\quad k\neq 0,\quad r\geqslant 1,\end{aligned}\tag{21}$$

$$\begin{aligned}N_k^2(r,f)=&c_k\left(\frac{1}{r},f\right)-k^2\int\limits_1^r\frac{dt}{t}\int\limits_1^t\frac{c_k\left(\frac{1}{\tau},f\right)}{\tau}d\tau-c_k(1,f)+\\&+ik\cdot a_k(1,F)\log r-\frac{n_k\left(\mathbb{T},\frac{1}{f}\right)-n_k(\mathbb{T},f)}{2}\log r,\quad k\neq 0,\quad r\geqslant 1.\end{aligned}\tag{22}$$

hold true.

Proof. In view of (12) we have $\alpha_k+\overline{\alpha_{-k}}=2c_k(1,F)$ and $\alpha_k-\overline{\alpha_{-k}}=2ia_k(1,F)$ for all $k\in\mathbb{Z}$. Note that

$$c_k(1,F)=c_k(1,f)-\sum_{|a_j|=1}c_k\left(1,1-\frac{z}{a_j}\right)+\sum_{|b_j|=1}c_k\left(1,1-\frac{z}{b_j}\right),\quad k\in\mathbb{Z}.$$

Bearing in mind that $|w|=1$, one can easily compute $c_k(t,1-\frac{z}{w})$ for $t\neq 1$. Then by using the continuity of the Fourier coefficients we get that

$$c_k(1,F)=c_k(1,f)+\frac{1}{2k}\left(n_k\left(\mathbb{T},\frac{1}{f}\right)-n_k(\mathbb{T},f)\right).$$

Replacing $a_k(t,F)$ in (17) by its representation (19), we arrive at obtain (21). Similarly, using (18) in (20), one gets (22). □

Lemma 6. *Let λ be a function of moderate growth, λ_1 be defined by (1), f be a function of c.r.g. with respect to λ, and c_k^1, c_k^2, c_k^* be given by Definitions 3, 4. Then*

(i) *if $f\in\Lambda_H^{\circ,1}$, then for every $k\in\mathbb{Z}$ there exist limits*

$$\lim_{r\to+\infty}\frac{a_k(r,F)}{\lambda_1(r)}=-ikc_k^1\quad\textit{and}\quad\lim_{r\to+\infty}\frac{a_k(\frac{1}{r},F)}{\lambda_1(r)}=ikc_k^2;$$

(ii) *if* $f \in \Lambda_H^{\circ,2}$, *then for every* $k \in \mathbb{Z}$ *there exists*

$$\lim_{r\to+\infty} \frac{a_k(r,F) - a_k(\frac{1}{r},F)}{\lambda_1(r)} = -ikc_k^*\,.$$

Proof. *(i)* Let f be a function of c.r.g.1 with respect to λ.

If $k=0$ then from (14), (16) we obtain $a_0(r,F) = a_0(\frac{1}{r},F) = a_0(1,F)$, and obviously

$$\lim_{r\to+\infty} \frac{a_0(r,F)}{\lambda_1(r)} = \lim_{r\to+\infty} \frac{a_0(\frac{1}{r},F)}{\lambda_1(r)} = 0.$$

Since $f \in \Lambda_H^{\circ,1}$, we have

$$c_k(r,f) = c_k^1\lambda(r) + o(\lambda(r)), \quad c_k\left(\frac{1}{r}, f\right) = c_k^2\lambda(r) + o(\lambda(r)), \quad r\to+\infty, \quad k\in\mathbb{Z}.$$

By (17) we get

$$a_k(r,F) = -ikc_k^1\lambda_1(r) + o(\lambda_1(r)) + \frac{\alpha_k - \overline{\alpha_{-k}}}{2i} - \frac{r^{-k}-1}{2ki}\left(n_k\left(\mathbb{T},\frac{1}{f}\right) - n_k(\mathbb{T},f)\right),\ k>0.$$

Hence,

$$\lim_{r\to+\infty} \frac{a_k(r,F)}{\lambda_1(r)} = -ikc_k^1$$

for all integer $k>0$. Using the properties $a_{-k}(r,F) = \overline{a_k(r,F)}$ and $c_{-k}^1 = \overline{c_k^1}$, we obtain that

$$\lim_{r\to+\infty} \frac{a_k(r,F)}{\lambda_1(r)} = -ikc_k^1, \quad k\in\mathbb{Z}.$$

Similarly, using (18), we have

$$\lim_{r\to+\infty} \frac{a_k(\frac{1}{r},F)}{\lambda_1(r)} = ikc_k^2, \quad k\in\mathbb{Z}.$$

(ii) Let now f be a function of c.r.g.2 with respect to λ. By (14) and (16) we obtain that $a_0(r,F) - a_0(\frac{1}{r},F) = 0$. It follows from (17), (18) that

$$a_k(r,F) - a_k\left(\frac{1}{r},F\right) = -ikc_k^*\lambda_1(r) + o(\lambda_1(r)) - \frac{r^{-k}-1}{ik}\left(n_k\left(\mathbb{T},\frac{1}{f}\right) - n_k(\mathbb{T},f)\right),\ k>0.$$

Similarly as in case *(i)* this implies that

$$\lim_{r\to+\infty} \frac{a_k(r,F) - a_k(\frac{1}{r},F)}{\lambda_1(r)} = -ikc_k^*$$

for all integer k. □

Lemma 7. *Let f be a meromorphic function in $\mathbb{C}^*$ with zeros $\{a_j\}$ and poles $\{b_j\}$, and $\{\alpha_k\}$ be defined by* (12)*. Then for $k \in \mathbb{Z}\setminus\{0\}$ and $r \geqslant 1$*

$$\begin{aligned} a_k(r,F) =& \frac{1}{2i}\left(\alpha_k r^k - \overline{\alpha_{-k}} r^{-k}\right) - \frac{n_k^1\left(r,\frac{1}{f}\right) - n_k^1(r,f)}{ik} + \\ &+ \frac{1}{2ik}\sum_{1<|a_j|\leqslant r}\left(\frac{r^k}{a_j^k} + \frac{\overline{a_j^k}}{r^k}\right) - \frac{1}{2ik}\sum_{1<|b_j|\leqslant r}\left(\frac{r^k}{b_j^k} + \frac{\overline{b_j^k}}{r^k}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} a_k\left(\frac{1}{r},F\right) =& \frac{1}{2i}\left(\alpha_k r^{-k} - \overline{\alpha_{-k}} r^{k}\right) + \frac{n_k^2\left(r,\frac{1}{f}\right) - n_k^2(r,f)}{ik} + \\ &+ \frac{1}{2ik}\sum_{\frac{1}{r}\leqslant|a_j|<1}\left((\overline{a_j}r)^k + \frac{1}{(a_j r)^k}\right) - \frac{1}{2ik}\sum_{\frac{1}{r}\leqslant|b_j|<1}\left((\overline{b_j}r)^k + \frac{1}{(b_j r)^k}\right). \end{aligned} \tag{24}$$

Proof. It follows from Remark 1 and Lemma 1 that

$$\begin{aligned} a_k(r,F) =& \frac{1}{2i}(\alpha_k r^k - \overline{\alpha_{-k}} r^{-k}) + \\ &+ \frac{1}{2i}\int_1^r \left(\frac{r^k}{t^{k+1}} - \frac{t^{k-1}}{r^k}\right)\left(n_k^1\left(t,\frac{1}{f}\right) - n_k^1(t,f)\right)dt, \quad k\neq 0, \quad r>1. \end{aligned} \tag{25}$$

We denote the integral in (25) by I_1. Integrating by parts, we obtain

$$\begin{aligned} I_1 =& -\frac{1}{2ik}\int_1^r \left(n_k^1\left(t,\frac{1}{f}\right) - n_k^1(t,f)\right) d\left(\frac{r^k}{t^k} + \frac{t^k}{r^k}\right) = \\ =& -\left(\frac{r^k}{t^k} + \frac{t^k}{r^k}\right)\frac{n_k^1\left(t,\frac{1}{f}\right) - n_k^1(t,f)}{2ik}\Bigg|_{t=1}^{t=r} + \frac{1}{2ik}\int_1^r \left(\frac{r^k}{t^k} + \frac{t^k}{r^k}\right) d\left(n_k^1\left(t,\frac{1}{f}\right) - n_k^1(t,f)\right) = \\ =& -\frac{n_k^1\left(r,\frac{1}{f}\right) - n_k^1(r,f)}{ik} + \frac{1}{2ik}\int_1^r \left(\frac{r^k}{t^k} + \frac{t^k}{r^k}\right) d\left(n_k^1\left(t,\frac{1}{f}\right) - n_k^1(t,f)\right), \quad k\neq 0, \quad r>1. \end{aligned}$$

Now, using a property of the Stieltjes integral [13], we represent the last integral as a sum and get (23) for $r > 1$. If $r = 1$ then (23) is implied by Remark 1 and (12).

Similarly, in view of (15) and Remark 1, we get

$$\begin{aligned} a_k(\frac{1}{r},F) =& \frac{1}{2i}(\alpha_k r^{-k} - \overline{\alpha_{-k}} r^{k}) + \\ &+ \frac{1}{2i}\int_1^r \left(\frac{t^{k-1}}{r^k} - \frac{r^k}{t^{k+1}}\right)\left(n_k^2\left(t,\frac{1}{f}\right) - n_k^2(t,f)\right)dt, \quad k\neq 0, \quad r>1. \end{aligned} \tag{26}$$

Again, denoting the integral in (26) by I_2 and integrating by parts we obtain

$$\begin{aligned} I_2 =& \frac{1}{2ik}\int_1^r (n_k^2(t,\frac{1}{f}) - n_k^2(t,f))\, d\left(\frac{t^k}{r^k} + \frac{r^k}{t^k}\right) = \\ =& \frac{n_k^2\left(r,\frac{1}{f}\right) - n_k^2(r,f)}{ik} - \frac{1}{2ik}\int_1^r \left(\frac{t^k}{r^k} + \frac{r^k}{t^k}\right) d\left(n_k^2\left(t,\frac{1}{f}\right) - n_k^2(t,f)\right), \quad k\neq 0, \quad r>1. \end{aligned}$$

Using the same property of the Stieltjes integral as before, we have (24) for $r > 1$. If $r = 1$ it follows from Remark 1 and (12). □

Remark 4. *Note that $\lambda(r) = O(\lambda_1(r))$ as $r \to +\infty$. Indeed,*

$$\lambda_1(er) = \int_1^{er} \frac{\lambda(t)}{t} dt \geqslant \int_r^{er} \frac{\lambda(t)}{t} dt \geqslant \lambda(r), \quad r > 1$$

and, taking into account that λ is a function of moderate growth,

$$\lambda_1(2r) = \int_1^{2r} \frac{\lambda(t)}{t} dt = \int_1^{2} \frac{\lambda(t)}{t} dt + \int_2^{2r} \frac{\lambda(t)}{t} dt = \int_1^{2} \frac{\lambda(t)}{t} dt + \int_1^{r} \frac{\lambda(2t)}{t} dt \leqslant$$
$$\leqslant \int_1^{2} \frac{\lambda(t)}{t} dt + M \int_1^{r} \frac{\lambda(t)}{t} dt \leqslant M' \lambda_1(r), \quad r > 1.$$

Lemma 8. *Let λ be a function of moderate growth, λ_1 be defined by (1), and f be a holomorphic function of finite λ-type. Then*

$$(\exists A > 0) \quad (\forall r > 1) \quad (\forall k \in \mathbb{Z}) \quad : \quad |a_k(r, F)| + |a_k(\frac{1}{r}, F)| \leqslant \frac{A\lambda_1(r)}{|k| + 1}. \tag{27}$$

Proof. In view of (23), for $k \neq 0$

$$\frac{a_k(2r, F)}{2^k} - a_k(r, F) = \frac{1}{2^{k+1}i} \left(\alpha_k (2r)^k - \overline{\alpha_{-k}} (2r)^{-k}\right) - \frac{1}{2i} \left(\alpha_k r^k - \overline{\alpha_{-k}} r^{-k}\right) -$$
$$- \frac{n_k^1(2r, \frac{1}{f}) - n_k^1(2r, f)}{2^k ik} + \frac{n_k^1(r, \frac{1}{f}) - n_k^1(r, f)}{ik} + \frac{1}{2ik} \left(\sum_{r < |a_j| \leqslant 2r} \frac{r^k}{a_j^k} - \sum_{r < |b_j| \leqslant 2r} \frac{r^k}{b_j^k} \right) +$$
$$+ \frac{1}{2^{2k+1}ik} \left(\sum_{1 < |a_j| \leqslant 2r} \frac{\overline{a_j}^k}{r^k} - \sum_{1 < |b_j| \leqslant 2r} \frac{\overline{b_j}^k}{r^k} \right) - \frac{1}{2ik} \left(\sum_{1 < |a_j| \leqslant r} \frac{\overline{a_j}^k}{r^k} - \sum_{1 < |b_j| \leqslant r} \frac{\overline{b_j}^k}{r^k} \right).$$

Hence, in view of Remark 3

$$\begin{aligned} |a_k(r, F)| \leqslant & \frac{1}{2^k} |a_k(2r, F)| + \frac{1}{2} \left(1 - \frac{1}{2^{2k}}\right) \frac{|\alpha_{-k}|}{r^k} + \frac{n_0^1\left(2r, \frac{1}{f}\right) + n_0^1(2r, f)}{2^k k} + \\ & + \frac{n_0^1\left(r, \frac{1}{f}\right) + n_0^1(r, f)}{k} + \frac{n_0^1\left(2r, \frac{1}{f}\right) - n_0^1\left(r, \frac{1}{f}\right)}{2k} + \frac{n_0^1(2r, f) - n_0^1(r, f)}{2k} + \\ & + \frac{n_0^1\left(2r, \frac{1}{f}\right) + n_0^1(2r, f)}{2^{k+1}k} + \frac{n_0^1\left(r, \frac{1}{f}\right) + n_0^1(r, f)}{2k}, \quad k \neq 0, \quad r > 1. \end{aligned} \tag{28}$$

The fact that f is of finite λ-type implies [9] that

$$|c_k(r, f)| + \left| c_k\left(\frac{1}{r}, f\right) \right| \leqslant \frac{B_1 \lambda(r)}{|k| + 1}, \quad k \in \mathbb{Z}, \tag{29}$$

for some $B_1 > 0$ and for all $r > 1$. Furthermore

$$(\exists B_2 > 0) \ (\forall r > 1) \ : \quad n_0^1(r, f) + n_0^2(r, f) \leqslant n_0(r, f) \leqslant B_2 \lambda(r). \tag{30}$$

First consider a positive integer k. In this case (17), (29), and (30) together with Remark 4 yield

$$
\begin{aligned}
|a_k(2r,F)| \leqslant & k\int\limits_1^{2r}\frac{|c_k(t,f)|}{t}dt+\left|\frac{\alpha_k-\overline{\alpha_{-k}}}{2i}-\frac{1}{2ki}((2r)^{-k}-1)n_k(\mathbb{T})\right|\leqslant \\
\leqslant & \frac{k}{k+1}B_1\lambda_1(2r)+|a_k(1,F)|+\frac{B_2\lambda(r)}{2k}\leqslant C_1\lambda_1(r)
\end{aligned}
\tag{31}
$$

for some $C_1>0$ and for all $r>1$. Now using (30), (31) in (28), we obtain that there exists $C_2>0$ such that

$$
|a_k(r,F)|\leqslant\frac{C_2\lambda_1(r)}{|k|+1} \tag{32}
$$

for all $r>1$ and $k>0$. As we have noticed in the proof of Lemma 6, $a_0(r,F)$ is constant. Using this fact and the property $a_{-k}(r,F)=\overline{a_k(r,F)}$, we obtain that (32) holds for all integer k, possibly with a constant different from C_2.

Similarly, in view of (24),

$$
\begin{aligned}
\frac{a_k(\frac{1}{2r},F)}{2^k}-a_k\left(\frac{1}{r},F\right)= & \frac{1}{2^{k+1}i}\left(\alpha_k(2r)^{-k}-\overline{\alpha_{-k}}(2r)^k\right)-\frac{1}{2i}\left(\alpha_k r^{-k}-\overline{\alpha_{-k}}r^k\right)+ \\
& +\frac{n_k^2\left(2r,\frac{1}{f}\right)-n_k^2(2r,f)}{2^k ik}-\frac{n_k^2\left(r,\frac{1}{f}\right)-n_k^2(r,f)}{ik}+ \\
& +\frac{1}{2ik}\left(\sum_{\frac{1}{2r}\leqslant|a_j|<\frac{1}{r}}(\overline{a_j}r)^k-\sum_{\frac{1}{2r}\leqslant|b_j|<\frac{1}{r}}(\overline{b_j}r)^k\right)+ \\
& +\frac{1}{2^{2k+1}ik}\left(\sum_{\frac{1}{2r}\leqslant|a_j|<1}\frac{1}{(a_jr)^k}-\sum_{\frac{1}{2r}\leqslant|b_j|<1}\frac{1}{(b_jr)^k}\right)- \\
& -\frac{1}{2ik}\left(\sum_{\frac{1}{r}\leqslant|a_j|<1}\frac{1}{(a_jr)^k}-\sum_{\frac{1}{r}\leqslant|b_j|<1}\frac{1}{(b_jr)^k}\right),\quad k\neq0,\quad r>1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|a_k(\frac{1}{r},F)|\leqslant & \frac{1}{2^k}\left|a_k\left(\frac{1}{2r},F\right)\right|+\frac{1}{2}\left(1-\frac{1}{2^{2k}}\right)\frac{|\alpha_k|}{r^k}+\frac{n_0^2\left(2r,\frac{1}{f}\right)+n_0^2(2r,f)}{2^k k} \\
& +\frac{n_0^2\left(r,\frac{1}{f}\right)-n_0^2(r,f)}{k}+\frac{n_0^2\left(2r,\frac{1}{f}\right)-n_0^2\left(r,\frac{1}{f}\right)}{2k}+\frac{n_0^2(2r,f)-n_0^2(r,f)}{2k}+ \\
& +\frac{n_0^2\left(2r,\frac{1}{f}\right)+n_0^2(2r,f)}{2^{k+1}k}+\frac{n_0^2\left(r,\frac{1}{f}\right)+n_0^2(r,f)}{2k},\quad k\neq0,\quad r>1.
\end{aligned}
\tag{33}
$$

If k is a positive integer then from (18), using (29), (30), and Remark 4 we obtain

$$
\begin{aligned}
\left|a_k\left(\frac{1}{2r},F\right)\right|\leqslant & k\int\limits_1^{2r}\frac{\left|c_k\left(\frac{1}{t},f\right)\right|}{t}dt+\left|\frac{\alpha_k-\overline{\alpha_{-k}}}{2i}+\frac{1}{2ki}((2r)^{-k}-1)n_k(\mathbb{T})\right|\leqslant \\
\leqslant & \frac{k}{k+1}B_1\lambda_1(2r)+|a_k(1,F)|+\frac{B_2\lambda(r)}{2k}\leqslant C_3\lambda_1(r)
\end{aligned}
$$

for some $C_3 > 0$ and for all $r > 1$. Again, using (30), the previous inequality and the property $a_{-k}(\frac{1}{r}, F) = \overline{a_k(\frac{1}{r}, F)}$ in (33), we get

$$\left| a_k \left(\frac{1}{r}, F \right) \right| \leqslant \frac{C_4 \lambda_1(r)}{|k| + 1}$$

for all $r > 1$ and $k \in \mathbb{Z}$. □

4. Connection between the indicators of completely regularly growing meromorphic function in $\mathbb{C}^*$

Let f be a meromorphic in $\mathbb{C}^*$ function of c.r.g. with respect to λ. It does not matter what type of c.r.g. the function f actually is of, it is assumed to be of finite λ-type in $\mathbb{C}^*$ anyway. This implies that the growth indicators, h_1 and h_2 in the case of c.r.g.1 or h in the case of c.r.g.2, belong to $L_2[0, 2\pi]$. For holomorphic functions f this was proved in [10]. The assumption that f is meromorphic in $\mathbb{C}^*$ give rise to no significant changes in the proof of that result.

We note that by the definition, the indicators h_1, h_2, and h are the growth indicators of $\log|f|$ with respect to λ. However, Lemmata 6 and 8 allow us to introduce the notion of the growth indicators of $\arg F$ with respect to λ_1.

We denote

$$a_k^1 = \lim_{r\to+\infty} \frac{a_k(r, F)}{\lambda_1(r)}, \quad a_k^2 = \lim_{r\to+\infty} \frac{a_k(\frac{1}{r}, F)}{\lambda_1(r)} \quad \text{if} \quad f \in \Lambda_H^{\circ,1},$$

$$a_k^* = \lim_{r\to+\infty} \frac{a_k(r, F) - a_k(\frac{1}{r}, F)}{\lambda_1(r)} \quad \text{if} \quad f \in \Lambda_H^{\circ,2}.$$

Using Lemma 8, we obtain

$$|a_k^1| \leqslant \frac{A}{|k| + 1}, \quad |a_k^2| \leqslant \frac{A}{|k| + 1}, \quad |a_k^*| \leqslant \frac{A}{|k| + 1}, \quad k \in \mathbb{Z}. \tag{34}$$

Thus, by the Riesz-Fischer Theorem [14, p. 79] there exist unique functions $g_1(\theta, f) = \sum_{k\in\mathbb{Z}} a_k^1 e^{ik\theta}$, $g_2(\theta, f) = \sum_{k\in\mathbb{Z}} a_k^2 e^{ik\theta}$, or $g(\theta, f) = \sum_{k\in\mathbb{Z}} a_k^* e^{ik\theta}$, which belong to $L_2[0, 2\pi]$. We call these functions *the growth indicators of* $\arg F$ with respect to λ_1.

By Lemma 6 we have

$$a_k^1 = -ikc_k^1, \quad a_k^2 = ikc_k^2, \quad a_k^* = -ikc_k^*, \quad k \in \mathbb{Z}. \tag{35}$$

Let

$$\lambda_2(r) = \int_1^r \frac{dt}{t} \int_1^t \frac{\lambda(\tau)}{\tau} d\tau, \quad r \geqslant 1$$

and $\Lambda_H^{\circ,1}$ be the class of holomorphic functions of c.r.g.1 in $\mathbb{C}^*$. Using the inverse formulas for the Fourier coefficients $c_k(r, f) + c_k\left(\frac{1}{r}, f\right)$ from [15] it was proved in [10, Theorem 3] that if $f \in \Lambda_H^{\circ,1}$ then the sum of the growth indicators $h_1 + h_2$ is ω-trigonometrically convex [1] for $\omega \in [\varkappa, \rho]$, where

$$\varkappa^2 = \liminf_{r\to+\infty} \frac{\lambda(r)}{\lambda_2(r)}, \qquad \rho^2 = \limsup_{r\to+\infty} \frac{\lambda(r)}{\lambda_2(r)}.$$

The presence of the poles of f does not complicate the proof essentially.

It turns out that for our purpose we need both h_1 and h_2 to possess the property of ω-trigonometrical convexity separately in the case $f \in \Lambda^{\circ,1}$. And if $f \in \Lambda^{\circ,2}$ we need this property for h as well. Analyzing the proof of Theorem 3 from [10], we conclude that the key role in showing ω-trigonometrical convexity is played by the inverse formulae for the Fourier coefficients. While such formulae from [15] for the sum $c_k(r, f) + c_k\left(\frac{1}{r}, f\right)$ can be obtained in the case $f \in \Lambda^{\circ,2}$, we need formulae (21) and (22) for the case $f \in \Lambda^{\circ,1}$. So, once we have them, in the similar way as in [10] by taking a meromorphic in $\mathbb{C}^*$ function f either of c.r.g.1 or c.r.g.2, and using the inverse formulae for the Fourier coefficients $c_k(r,f)$, $c_k(\frac{1}{r},f)$ given by Lemma 5, one can prove that the indicators h_1, h_2, or h are ω-trigonometrically convex for $\omega\in[\varkappa,\rho]$ in the appropriate case.

Each trigonometrically convex function is differentiable almost everywhere [1]. It follows from (35) that the Fourier coefficients of the functions g_1 and $-h_1'$, g_2 and h_2', g and $-h'$ coincide and

$$g_1(\theta,f)=-h_1'(\theta,f),\quad g_2(\theta,f)=h_2'(\theta,f),\quad g(\theta,f)=-h(\theta,f) \tag{36}$$

almost everywhere.

5. Proof of the main results

Proof of Theorem 1. Let g_1, g_2 be the functions defined in the previous section. The Fourier coefficients of $\frac{\arg F(re^{i\theta})}{\lambda_1(r)}-g_1(\theta,f)$, $\frac{\arg F(\frac{1}{r}e^{i\theta})}{\lambda_1(r)}-g_2(\theta,f)$ are $\frac{a_k(r,F)}{\lambda_1(r)}-a_k^1$ and $\frac{a_k(\frac{1}{r},F)}{\lambda_1(r)}-a_k^2$ respectively, $k\in\mathbb{Z}$.

By the Parseval's identity [14] we have

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\frac{\arg F(re^{i\theta})}{\lambda_1(r)}-g_1(\theta,f)\right|^2 d\theta\right\}^{\frac12}=\left\{\sum_{k\in\mathbb{Z}}\left|\frac{a_k(r,F)}{\lambda_1(r)}-a_k^1\right|^2\right\}^{\frac12},\ r>1, \tag{37}$$

and

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\frac{\arg F(\frac{1}{r}e^{i\theta})}{\lambda_1(r)}-g_2(\theta,f)\right|^2 d\theta\right\}^{\frac12}=\left\{\sum_{k\in\mathbb{Z}}\left|\frac{a_k(\frac{1}{r},F)}{\lambda_1(r)}-a_k^2\right|^2\right\}^{\frac12},\ r>1. \tag{38}$$

It follows from the convergence of the series $\sum\limits_{k=1}^{+\infty}\frac{1}{k^2}$ that for any $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that

$$\sum_{k=k_0+1}^{+\infty}\frac{1}{(k+1)^2}<\frac{\varepsilon^2}{32A^2}, \tag{39}$$

where A is the constant from (27). Applying Minkowski inequality [16], and using (27), (39), we obtain

$$\begin{aligned}\left\{\sum_{|k|>k_0}\left|\frac{a_k(r,F)}{\lambda_1(r)}-a_k^1\right|^2\right\}^{\frac12}&\leqslant\left\{\sum_{|k|>k_0}\left|\frac{a_k(r,F)}{\lambda_1(r)}\right|^2\right\}^{\frac12}+\left\{\sum_{|k|>k_0}|a_k^1|^2\right\}^{\frac12}\leqslant\\&\leqslant 2\left\{\sum_{|k|>k_0}\frac{A^2}{(|k|+1)^2}\right\}^{\frac12}=2\sqrt{2}A\left\{\sum_{k=k_0+1}^{+\infty}\frac{1}{(|k|+1)^2}\right\}^{\frac12}<\frac{\varepsilon}{2}.\end{aligned} \tag{40}$$

Similarly,

$$\left\{\sum_{|k|>k_0}\left|\frac{a_k(\frac{1}{r},F)}{\lambda_1(r)}-a_k^2\right|^2\right\}^{\frac12}<\frac{\varepsilon}{2}. \tag{41}$$

By Lemma 6 there exist $r_0>1$ such that

$$\left|\frac{a_k(r,F)}{\lambda_1(r)}-a_k^1\right|<\frac{\varepsilon}{4k_0},\quad\left|\frac{a_k(\frac{1}{r},F)}{\lambda_1(r)}-a_k^2\right|<\frac{\varepsilon}{4k_0} \tag{42}$$

for all $r>r_0$ and $|k|\leqslant k_0$.

Thus, for any $\varepsilon > 0$ and for $r > r_0$, in view of (37), (40), and (42), we get

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\frac{\arg F(re^{i\theta})}{\lambda_1(r)} - g_1(\theta, f)\right|^2 d\theta\right\}^{\frac{1}{2}} = \left\{\sum_{|k|\leqslant k_0}\left|\frac{a_k(r,F)}{\lambda_1(r)} - a_k^1\right|^2\right\}^{\frac{1}{2}} +$$

$$+\left\{\sum_{|k|>k_0}\left|\frac{a_k(r,F)}{\lambda_1(r)} - a_k^1\right|^2\right\}^{\frac{1}{2}} < \sqrt{2k_0+1}\frac{\varepsilon}{4k_0} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, from (38), (41), and (42) it follows that for any $\varepsilon > 0$ and for $r > r_0$

$$\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}\left|\frac{\arg F(\frac{1}{r}e^{i\theta})}{\lambda_1(r)} - g_2(\theta, f)\right|^2 d\theta\right\}^{\frac{1}{2}} < \varepsilon.$$

Together with (36) this proves (5), (6) for $p = 2$. Furthermore, in view of (27), (34), and (35), by applying Hausdorff-Young Theorem [14] we obtain (5), (6) for $p > 2$. Taking into account the monotonicity of the p^{th} integral means, we establish that relations (5), (6) hold for all $p \in [1, +\infty)$. □

Proof of Theorem 2. One can prove this theorem by considering the Fourier coefficients of $\frac{\arg F(re^{i\theta}) - \arg F(\frac{1}{r}e^{i\theta})}{\lambda_1(r)} - g(\theta, f)$, using appropriate part of Lemma 6, and arguing just as in the proof of Theorem 1. □

Proof of Theorems 3, 4. Relations (8), (9), (10) are implied immediately by Theorem 2 in [10] and Theorems 1, 2 by Minkowski inequality [16] and Remark 4. □

REFERENCES

1. B.Ya. Levin. *Distribution of zeros of entire functions. Revised edition.* Amer. Math. Soc, Providence, RI (1980).
2. L.A. Rubel, B.A. Taylor. *A Fourier series method for meromorphic and entire functions* // Bull. Amer. Math. Soc. **72**:5, 858-860 (1966).
3. A.A. Kondratyuk. *The Fourier series method for entire and meromorphic functions of completely regular growth* // Matem. Sborn. **106**:3, 386-408. [Math. USSR-Sbornik. **35**:1, 63-84 (1979).]
4. A.A. Kondratyuk. *The Fourier series method for entire and meromorphic functions of completely regular growth. II* // Matem. Sborn. **113**:1, 118-132 (1980). [Math. USSR-Sbornik. **41**:1, 101-113 (1982).]
5. A.A. Kondratyuk. *The Fourier series method for entire and meromorphic functions of completely regular growth. III* // Matem. Sborn. **120**:3, 331-343 (1983). [Math. USSR-Sbornik. **48**:2, 327-338 (1984)]
6. A.A. Kondratyuk. *Fourier series and meromorphic functions.* Izd. "Vishcha Shkola", L'vov (1988). (in Russian)
7. A.Ya. Khrystiyanyn, A.A. Kondratyuk. *On the Nevanlinna theory for meromorphic functions on annuli. I* // Mat. Stud. **23**:1, 19-30 (2005).
8. A. Ya. Khrystiyanyn, A. A. Kondratyuk, *On the Nevanlinna theory for meromorphic functions on annuli. II* // Mat. Stud. **24**:2, 57-68 (2005).
9. A. Kondratyuk, I. Laine. *Meromorphic functions in multiply connected domains* // Proc. workshop "Fourier series method in complex analysis", Merkrijärvi, 2005. Dep. Math. Univ. Joensuu. **10**, 9-111 (2006).
10. M. Goldak, A. Khrystiyanyn. *Holomorphic functions of completely regular growth in the punctured plane* // Visnyk Lviv. Univ. Ser. Mekh. Mat. 75, 91-96 (2011). (in Ukrainian).

11. O. Vyshyns'kyi, A. Khrystiyanyn. *On the simultaneous regular growth of the logarithm of modulus and argument of a holomorphic in the punctured plane function* // Visnyk Lviv. Univ. Ser. Mekh. Mat. 79, 33-47 (2014). (in Ukrainian).
12. W.K. Hayman. *Meromorphic functions*. Clarendon Press, Oxford (1964).
13. I.P. Natanson. *Theory of functions of a real variable*. Ungar Publishing Company, New York (1964).
14. A. Zygmund. *Trigonometric series I, II*. Cambridge Univ. Press, Cambridge (1959).
15. M. Goldak, A. Khrystiyanyn. *Inverse formulas for the Fourier coefficients of meromorphic functions on annuli* // Visnyk Lviv. Univ. Ser. Mech. Math. 71, 71-77 (2009). (in Ukrainian).
16. A.N. Kolmogorov, S.V. Fomin. *Elements of the theory of functions and functional analysis*, Dover Puplications Inc., Mineola, New York (1999).

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