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# ON COERCIVE PROPERTIES AND SEPARABILITY OF BIHARMONIC OPERATOR WITH MATRIX POTENTIAL

**O.Kh. KARIMOV**

**Abstract.** In the work we consider the coercive properties of a nonlinear biharmonic operator with a matrix operator in the space $L_2(\mathbb{R}^n)^l$ and we prove its separability in this space. The considered nonlinear operators are not small perturbation of linear operators. The case of the linear biharmonic operator is considered separately.

**Keywords:** biharmonic differential operator, matrix potential, coercive inequalities, nonlinearity, separability.

**Mathematics Subject Classification:** 35Q40, 35J10

## 1. Introduction

In the paper we study the separability of the nonlinear biharmonic operator

$$L[u] = \Delta^2 u(x) + V(x, u(x))u(x)$$

with a matrix potential and this operator is not a weak perturbation of a linear operator. We obtain sufficient condition for the separability of this operator in the space $L_2(\mathbb{R}^n)^l$ and we establish the appropriate coercive inequalities.

Fundamental results on the separability theory of the differential operators are due to W.N. Everitt and M. Gierz. In works [1]–[4] they obtained a series of important results on the separability of the Sturm-Liouville operator and of its powers. They also considered a multi-dimensional Schrödinger operator. An essential contribution in further developing of this theory was made by K.Kh. Boimatov, M. Otelbaev and by their pupils (see [5]–[8] and the references therein). The coercive properties of nonlinear Schrödinger and Dirac operators were considered in [6]. The separability of a nonlinear Schrödinger operator was studied in [8].

The separability of partial differential operators was studied first by K.Kh. Boimatov in work [5]. The separability of the linear biharmonic operator $L[u] = \Delta^2 u(x) + V(x)u(x)$ was studied before in works [9], [10]. The separability of nonlinear second order differential operators with varying matrix coefficients in an $n$-dimensional Euclidean space was studied before in work [11]. The present work generalizes work [9] for a nonlinear case.

It should be noted that the separability of nonlinear differential operators was studied mostly in the case when the considered operator is a weak perturbation of a linear operator. In distinction to this case, the nonlinear differential operators we consider here are not necessarily a weak perturbation of a linear operator.

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## 2. Formulation of main result

In the space $L_2(\mathbb{R}^n)^l$, where $l$ is a natural number, we consider the differential equation

$$\Delta^2 u(x) + V(x, u(x))u(x) = f(x), \qquad u(x) \in W^4_{2,loc}(\mathbb{R}^n)^l, \tag{2.1}$$

where $V(x,\omega)$, $x \in \mathbb{R}^n$, $\omega \in \mathbb{C}^l$, are square positive definite Hermitian matrices in $\operatorname{End}\mathbb{C}^l$ . Hereinafter $B^l$, $B$ is a linear space, stands for the space of the elements $(y_1, y_2, \dots, y_l)$ with the entries $y_j$ in $B$.

**Definition 2.1.** *Equation (2.1) and the associated differential operator are called separable in $L_2(\mathbb{R}^n)^l$, if $\Delta^2 u(x)$, $V(x, u(x))u(x) \in L_2(\mathbb{R}^n)^l$ for all $u(x) \in L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$ such that $f(x) \in L_2(\mathbb{R}^n)^l$.*

Given $z^{(i)} = (z_1^{(i)}, \cdots, z_l^{(i)})$ $(i = 1, 2)$, we let

$$\langle z^{(1)}, z^{(2)}\rangle = \sum_{j=1}^{l} z_j^{(1)} \overline{z_j^{(2)}}.$$

We denote

$$(u, v) = \int_{\mathbb{R}^n} \langle u(x), v(x)\rangle dx,$$

if the integral in the right hand side converges absolutely.

In what follows we assume that $V(x,\omega) \in C^2(\mathbb{R}^n \times \mathbb{C}^l; \operatorname{End}\mathbb{C}^l)$.

We introduce new matrix functions

$$F(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_l, \eta_1, \eta_2, \dots, \eta_l) = V^{1/2}(x,\omega),\ (x_i \in R,\ \xi_j,\ \eta_j \in R),$$
$$Q(x_1, x_2, \dots, x_n, \xi_1, \xi_2, \dots, \xi_l, \eta_1, \eta_2, \dots, \eta_l) = F^2(x,\omega),\ (x_i \in R,\ \xi_j,\ \eta_j \in R),$$

where $\omega$ is defined by the identity $\omega = (\xi_1 + i\eta_1, \dots, \xi_l + i\eta_l)$. Here $V^{\frac{1}{2}}(x,\omega)$ is introduced as the square root of a positive definite Hermitian matrix.

We assume that for all $x \in \mathbb{R}^n$, $\omega = (\xi_1 + i\eta_1, \dots, \xi_l + i\eta_l)$, $\Omega = (\mu_1 + i\nu_1, \dots, \mu_l + i\nu_l)$, $(\xi_j,\ \eta_j,\ \mu_j,\ \nu_j \in R)$ and $u \in W_2^1(\mathbb{R}^n)$, the matrix function $F(x,\omega)$ satisfies the conditions

$$\sum_{i=1}^{n} \left\| F^{-\frac{1}{2}} \frac{\partial^2 F}{\partial x_i^2} F^{-\frac{3}{2}}; \mathbb{C}^l \right\|^2 \leqslant \sigma_1, \tag{2.2}$$

$$\sum_{i=1}^{n} \left\| F^{-\frac{1}{2}} \frac{\partial F}{\partial x_i} \frac{\partial u}{\partial x_i}; L_2(\mathbb{R}^n)^l \right\|^2 \leqslant \sigma_2 \left\| F^{\frac{3}{2}} u; L_2(\mathbb{R}^n)^l \right\|^2, \tag{2.3}$$

$$\left\| \sum_{j=1}^{l} \mu_j F^{-\frac{1}{2}} \frac{\partial^2 F}{\partial x_i \partial \xi_j} \omega + \nu_j F^{-\frac{1}{2}} \frac{\partial^2 F}{\partial x_i \partial \eta_j} \omega; \mathbb{C}^l \right\| \leqslant \delta_1 \left\| F^{\frac{1}{2}} \Omega; \mathbb{C}^l \right\|, \tag{2.4}$$

$$\left\| \sum_{j=1}^{l} \mu_j F^{-\frac{1}{2}} \frac{\partial F}{\partial \xi_j} \omega + \nu_j F^{-\frac{1}{2}} \frac{\partial F}{\partial \eta_j} \omega; \mathbb{C}^l \right\| \leqslant \delta_2 \left\| F^{\frac{1}{2}} \Omega; \mathbb{C}^l \right\|. \tag{2.5}$$

We also assume that for all $x \in \mathbb{R}^n$, $\omega = (\xi_1 + i\eta_1, \dots, \xi_l + i\eta_l)$, $\Omega = (\mu_1 + i\nu_1, \dots, \mu_l + i\nu_l)$, $(\xi_j, \eta_j, \mu_j, \nu_j \in R)$ and $u \in W_2^1(\mathbb{R}^n)$ the inequalities

$$\sum_{i=1}^{n} \left\| Q^{-\frac{1}{2}} \frac{\partial^2 Q}{\partial x_i^2} Q^{-1}; \mathbb{C}^l \right\|^2 \leqslant \sigma_3, \tag{2.6}$$

$$\sum_{i=1}^{n} \left\| Q^{-\frac{1}{2}} \frac{\partial Q}{\partial x_i} \frac{\partial u}{\partial x_i}; L_2(\mathbb{R}^n)^l \right\|^2 \leqslant \sigma_4 \|Vu; L_2(\mathbb{R}^n)^l\|^2, \tag{2.7}$$

$$\left\| \sum_{j=1}^{l} \mu_j F^{-1} \frac{\partial^2 Q}{\partial x_i \partial \xi_j} \omega + \nu_j F^{-1} \frac{\partial^2 Q}{\partial x_i \partial \eta_j} \omega; \mathbb{C}^l \right\| \leqslant \delta_3 \left\| F\Omega; \mathbb{C}^l \right\|, \tag{2.8}$$

$$\left\| \sum_{j=1}^{l} \mu_j F^{-1} \frac{\partial Q}{\partial \xi_j} \omega + \nu_j F^{-1} \frac{\partial Q}{\partial \eta_j} \omega; \mathbb{C}^l \right\| \leqslant \delta_4 \left\| F\Omega; \mathbb{C}^l \right\|. \tag{2.9}$$

hold true.

Now we are in position to formulate the main result.

**Theorem 2.1.** *Assume that conditions (2.2)–(2.9) hold and let the numbers* $\sigma_j$, $\delta_j$, $(j = \overline{1,4})$, *are such that*

$$\sigma_1 + 2\sigma_2 < 4, \qquad \delta_1 + 2\delta_2 < 1, \qquad \sigma_3 + 2\sigma_4 < 4, \qquad \delta_3 + 2\delta_4 < 1. \tag{2.10}$$

*Then equation (2.1) is separable in* $L_2(\mathbb{R}^n)^l$ *and for all vector functions* $u(x) \in L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$ *such that* $f(x) \in L_2(\mathbb{R}^n)^l$ *the belongings*

$$\Delta^2 u,\ V(x,u)u,\ V^{\frac{1}{2}}(x,u)\frac{\partial^2 u}{\partial x_i^2} \in L_2(\mathbb{R}^n)^l, \quad i = 1, 2, \ldots, n.$$

*hold. At that, the coercive inequality*

$$\begin{aligned} &\|\Delta^2 u(x); L_2(\mathbb{R}^n)^l\| + \|V(x,u(x))u(x); L_2(\mathbb{R}^n)^l\| \\ &\qquad + \sum_{i=1}^{n} \left\| V^{\frac{1}{2}} \frac{\partial^2 u(x)}{\partial x_i^2}; L_2(\mathbb{R}^n)^l \right\| \leqslant M \|f(x); L_2(\mathbb{R}^n)^l\| \end{aligned} \tag{2.11}$$

*is valid, where a positive number* $M$ *is independent of* $u(x)$, $f(x)$.

**Example.** *The assumption of the theorem are satisfied for equation (2.1) as* $V(x, u(x)) = (1 + |u(x)|^2)^\rho$, $n = 1$, *that is,* $Q(x, \xi, \eta) = (1 + \xi^2 + \eta^2)^\rho$ *as* $\rho \leqslant min\{\frac{\delta_2}{2}; \frac{\delta_4}{4}\}$.

## 3. Auxiliary lemmata

**Lemma 3.1.** *Assume that in equation (2.1) the vector function* $f(x)$ *belongs to the space* $L_2(\mathbb{R}^n)^l$ *and the vector function* $u(x)$ *belongs to the class* $L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$. *Then the vector functions* $V^{1/2}(x, u(x))u(x)$, $\dfrac{\partial^2 u}{\partial x_i^2}$ $(i = 1, 2, \ldots, n)$ *belong to the space* $L_2(\mathbb{R}^n)^l$.

*Proof.* Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be a fixed nonnegative function equalling one as $|x| < 1$. For each positive number $\varepsilon$ we let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$.

Employing the identities

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \quad \text{and} \quad (f, \varphi_\varepsilon u) = (\Delta^2 u, \varphi_\varepsilon u) + (V(x,u)u, \varphi_\varepsilon u),$$

we have

$$\begin{aligned} (f, \varphi_\varepsilon u) = &\sum_{k=1}^{n} \left( \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} \varphi_\varepsilon \frac{\partial^2 u}{\partial x_i^2} \right) + 2 \sum_{k=1}^{n} \left( \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} \frac{\partial \varphi_\varepsilon}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \\ &+ \sum_{k=1}^{n} \left( \frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n} \frac{\partial^2 \varphi_\varepsilon}{\partial x_i^2} u \right) + (Vu, \varphi_\varepsilon u), \end{aligned} \tag{3.1}$$

where $(\cdot, \cdot)$ denotes the scalar product in the space $L_2(\mathbb{R}^n)^l$.

Since the function $\varphi_\varepsilon$ is real-valued and

$$\left| \frac{\partial \varphi_\varepsilon}{\partial x_i} \right| \leqslant M_1 \varepsilon, \quad \left| \frac{\partial^2 \varphi_\varepsilon}{\partial x_k \partial x_i} \right| \leqslant M_0 \varepsilon^2, \quad \forall x \in \mathbb{R}^n,$$

where
$$M_1 = \sup|\nabla\varphi_\varepsilon(x)|, \qquad M_0 = \sup|\Delta\varphi_\varepsilon(x)|,$$
then passing to the limit as $\varepsilon \to 0$, by identity (3.1) we find
$$\operatorname{Re}(f,u) \geqslant \sum_{k,i=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \frac{\partial^2 u}{\partial x_i^2}\right) + (Vu, u),$$
which completes the proof. □

**Lemma 3.2.** *Assume that conditions (2.2)–(2.5) hold and let a vector function $u(x)$ in the class $L_2(\mathbb{R}^n)^l \cap W^4_{2,loc}(\mathbb{R}^n)^l$ solves equation (2.1) with the right hand side $f(x) \in L_2(\mathbb{R}^n)^l$. Then the vector functions $F^{\frac{3}{2}}(x,u(x))u(x)$, $F^{\frac{1}{2}}(x,u(x))\dfrac{\partial^2 u}{\partial x_k^2}$, $k = 1,\dots,n$, belong the space $L_2(\mathbb{R}^n)^l$.*

*Proof.* Let the function $\varphi_\varepsilon(x)$ be the same as in proof of the Lemma 3.1. It is obvious that
$$(f, \varphi_\varepsilon F u) = (\Delta^2 u, \varphi_\varepsilon F u) + (V(x,u)u, \varphi_\varepsilon F u).$$
In view the identity
$$\frac{\partial(\varphi_\varepsilon F u)}{\partial x_i} = \frac{\partial \varphi_\varepsilon}{\partial x_i}Fu + \varphi_\varepsilon \frac{\partial F}{\partial x_i}u + \sum_{j=1}^{l}\varphi_\varepsilon \operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \xi_j}u + \sum_{j=1}^{l}\varphi_\varepsilon \operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \eta_j}u + \varphi_\varepsilon F\frac{\partial u}{\partial x_i},$$
after some simple transformations we obtain
$$\begin{aligned}(f, \varphi_\varepsilon F u) =& A_1^\varepsilon(u) + 2A_2^\varepsilon(u) + 2A_3^\varepsilon(u) + A_4^\varepsilon(u) + 2A_5^\varepsilon(u) \\ &+ A_6^\varepsilon(u) + 2A_7^\varepsilon(u) + 2A_8^\varepsilon(u) + 2A_9^\varepsilon(u) + 2A_{10}^\varepsilon(u) + (Vu, \varphi_\varepsilon F u),\end{aligned} \tag{3.2}$$
where
$$A_1^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\frac{\partial^2\varphi_\varepsilon}{\partial x_i^2}Fu\right), \quad A_2^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\frac{\partial\varphi_\varepsilon}{\partial x_i}\frac{\partial F}{\partial x_i}u\right),$$
$$A_3^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\frac{\partial\varphi_\varepsilon}{\partial x_i}F\frac{\partial u}{\partial x_i}\right), \quad A_4^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon\frac{\partial^2 F}{\partial x_i^2}u\right),$$
$$A_5^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon\frac{\partial F}{\partial x_i}\frac{\partial u}{\partial x_i}\right), \quad A_6^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon F\frac{\partial^2 u}{\partial x_i^2}\right),$$
$$A_7^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\frac{\partial\varphi_\varepsilon}{\partial x_i}\sum_{j=1}^{l}\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \xi_j} + \operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \eta_j}u\right)\right),$$
$$A_8^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon\sum_{j=1}^{l}\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial^2 F}{\partial x_i\partial \xi_j} + \operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial^2 F}{\partial x_i\partial \eta_j}\right)u\right),$$
$$A_9^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon\sum_{j=1}^{l}\left(\operatorname{Re}\frac{\partial^2 u_j}{\partial x_i^2}\frac{\partial F}{\partial \xi_j} + \operatorname{Im}\frac{\partial^2 u_j}{\partial x_i^2}\frac{\partial F}{\partial \eta_j}\right)u\right),$$
$$A_{10}^\varepsilon(u) = \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}, \sum_{i=1}^{n}\varphi_\varepsilon\sum_{j=1}^{l}\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \xi_j} + \operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial F}{\partial \eta_j}\right)\frac{\partial u}{\partial x_i}\right).$$
Hereinafter the values of $F$, $\dfrac{\partial F}{\partial x_i}$, $\dfrac{\partial F}{\partial \xi_j}$, $\dfrac{\partial F}{\partial \eta_j}$, $\dfrac{\partial^2 F}{\partial x_i\partial \xi_j}$, $\dfrac{\partial^2 F}{\partial x_i\partial \eta_j}$ are taken at the point $(x_1, \dots, x_n, \operatorname{Re}u_1(x), \dots, \operatorname{Re}u_l(x), \operatorname{Im}u_1(x), \dots, \operatorname{Im}u_l(x))$.

Estimating the functionals, we find that by Lemma 3.1 the functionals $A_1^\varepsilon(u)$, $A_2^\varepsilon(u)$, $A_3^\varepsilon(u)$, $A_7^\varepsilon(u)$ tend to zero as $\varepsilon \to 0$.

For the functionals $A_m^\varepsilon(u)$, $m = 4,5,6,8,9,10$, we obtain the following estimates

$$|A_4^\varepsilon(u)| \leqslant \frac{\beta_1}{2}\sum_{k=1}^n \left(F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}, \varphi_\varepsilon F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}\right) + \frac{\sigma_1}{2\beta_1}(Vu, \varphi_\varepsilon Fu),$$

$$|A_5^\varepsilon(u)| \leqslant \frac{\beta_2}{2}\sum_{k=1}^n \left(\frac{\partial^2 u}{\partial x_k^2}, \varphi_\varepsilon F\frac{\partial^2 u}{\partial x_k^2}\right) + \frac{\sigma_2}{2\beta_2}(Vu, \varphi_\varepsilon Fu),$$

$$|A_6^\varepsilon(u)| \leqslant \sum_{k=1}^n \left\|\varphi_\varepsilon^{\frac12} F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}; L_2(\mathbb{R}^n)^l\right\|^2, \qquad |A_8^\varepsilon(u)| \leqslant \delta_1 \sum_{k=1}^n \left\|\varphi_\varepsilon^{\frac12} F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}; L_2(\mathbb{R}^n)^l\right\|^2,$$

$$|A_9^\varepsilon(u)| \leqslant \delta_2 \sum_{k=1}^n \left\|\varphi_\varepsilon^{\frac12} F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}; L_2(\mathbb{R}^n)^l\right\|^2, \quad |A_{10}^\varepsilon(u)| \leqslant \delta_2 \sum_{k=1}^n \|\varphi_\varepsilon^{\frac12} F^{\frac12}\frac{\partial^2 u}{\partial x_k^2}; L_2(\mathbb{R}^n)^l\|^2.$$

Here $\beta_1$, $\beta_2$ are arbitrary positive numbers and $\sigma_1$, $\sigma_2$, $\delta_1$ and $\delta_2$ are the constants in conditions (2.2)–(2.5). While estimating the functionals $A_9^\varepsilon(u)$ and $A_{10}^\varepsilon(u)$, we have employed inequality (2.5) twice: in the case

$$\omega = (\omega_1, \omega_2, \dots, \omega_l) = u(x) = (u_1(x), u_2(x), \dots, u_l(x)),$$

and

$$\omega = (\omega_1, \omega_2, \dots, \omega_l) = \frac{\partial u}{\partial x_i} = \left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}, \dots, \frac{\partial u_l}{\partial x_i}\right).$$

On the basis of the obtained estimates, by identity (3.2) we have

$$|(f, \varphi_\varepsilon Fu)| \geqslant \left(1 - \frac{\sigma_1}{2\beta_1} - \frac{\sigma_2}{\beta_2}\right)\cdot (Vu, \varphi_\varepsilon Fu) - |A_1^\varepsilon(u)| - |A_2^\varepsilon(u)| - |A_3^\varepsilon(u)| - |A_7^\varepsilon(u)|$$
$$+ \left(1 - \frac{\beta_1}{2} - \beta_2 - 2\delta_1 - 2\delta_2 - 2\delta_2\right)\sum_{k=1}^n \left(\frac{\partial^2 u}{\partial x_k^2}, \varphi_\varepsilon F\frac{\partial^2 u}{\partial x_k^2}\right).$$

Then we apply Cauchy-Schwarz inequality and passing to the limit as $\varepsilon \to 0$, we obtain the inequality

$$\|f; L_2(\mathbb{R}^n)^l\|\|Fu; L_2(\mathbb{R}^n)^l\| \geqslant |(f, Fu)| \geqslant \left(1 - \frac{\sigma_1}{2\beta_1} - \frac{\sigma_2}{\beta_2}\right)(Vu, Fu)$$
$$+ \left(1 - \frac{\beta_1}{2} - \beta_2 - 2\delta_1 - 4\delta_2\right)\sum_{k=1}^n \left(\frac{\partial^2 u}{\partial x_k^2}, F\frac{\partial^2 u}{\partial x_k^2}\right). \tag{3.3}$$

Now we choose positive numbers $\beta_1$, $\beta_2$ so that to satisfy the conditions

$$\frac{\sigma_1}{2\beta_1} + \frac{\sigma_2}{\beta_2} < 1, \quad \frac{\beta_1}{2} + \beta_2 + 2\delta_1 + 4\delta_2 < 1.$$

Since by Lemma 3.1 $Fu \in L_2(\mathbb{R}^n)^l$, it follows from inequality (3.3) that the vector functions $F^{\frac12}\dfrac{\partial^2 u}{\partial x_k^2}$, $(k = 1, 2, \dots, n)$, $F^{\frac32}u$ belong to the space $L_2(\mathbb{R}^n)^l$. The proof is complete. □

## 4. Proof of Theorem 2.1

We proceed to the proof of Theorem 2.1. Proceeding as above and employing the identity

$$(f, \varphi_\varepsilon Vu) = (\Delta^2 u, \varphi_\varepsilon Vu) + (V(x,u)u, \varphi_\varepsilon Vu),$$

by simple transformations we obtain

$$\begin{aligned}(f,\varphi_\varepsilon Vu) =&B_1^\varepsilon(u)+2B_2^\varepsilon(u)+2B_3^\varepsilon(u)+B_4^\varepsilon(u)+2B_5^\varepsilon(u)\\ &+B_6^\varepsilon(u)+2B_7^\varepsilon(u)+2B_8^\varepsilon(u)+2B_9^\varepsilon(u)+2B_{10}^\varepsilon(u)+(Vu,\varphi_\varepsilon Vu),\end{aligned}\tag{4.1}$$

where

$$B_1^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\frac{\partial^2\varphi_\varepsilon}{\partial x_i^2}Qu\right),\quad B_2^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\frac{\partial\varphi_\varepsilon}{\partial x_i}\frac{\partial Q}{\partial x_i}u\right),$$

$$B_3^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\frac{\partial\varphi_\varepsilon}{\partial x_i}Q\frac{\partial u}{\partial x_i}\right),\quad B_4^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon\frac{\partial^2 Q}{\partial x_i^2}u\right),$$

$$B_5^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon\frac{\partial Q}{\partial x_i}\frac{\partial u}{\partial x_i}\right),\quad B_6^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon Q\frac{\partial^2 u}{\partial x_i^2}\right),$$

$$B_7^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\frac{\partial\varphi_\varepsilon}{\partial x_i}\sum_{j=1}^l\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial Q}{\partial\xi_j}+\operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial Q}{\partial\eta_j}\right)u\right),$$

$$B_8^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon\sum_{j=1}^l\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial^2 Q}{\partial x_i\partial\xi_j}+\operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial^2 Q}{\partial x_i\partial\eta_j}\right)u\right),$$

$$B_9^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon\sum_{j=1}^l\left(\operatorname{Re}\frac{\partial^2 u_j}{\partial x_i^2}\frac{\partial Q}{\partial\xi_j}+\operatorname{Im}\frac{\partial^2 u_j}{\partial x_i^2}\frac{\partial Q}{\partial\eta_j}\right)u\right),$$

$$B_{10}^\varepsilon(u)=\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\sum_{i=1}^n\varphi_\varepsilon\sum_{j=1}^l\left(\operatorname{Re}\frac{\partial u_j}{\partial x_i}\frac{\partial Q}{\partial\xi_j}+\operatorname{Im}\frac{\partial u_j}{\partial x_i}\frac{\partial Q}{\partial\eta_j}\right)\frac{\partial u}{\partial x_i}\right).$$

Hereinafter the values of $Q$, $\dfrac{\partial Q}{\partial x_i}$, $\dfrac{\partial Q}{\partial \xi_j}$, $\dfrac{\partial Q}{\partial \eta_j}$, $\dfrac{\partial^2 Q}{\partial x_i\partial \xi_j}$, $\dfrac{\partial^2 Q}{\partial x_i\partial \eta_j}$ are taken at the point $(x_1,\,\dots,\,x_n,\,\operatorname{Re}u_1(x),\,\dots,\,\operatorname{Re}u_l(x),\,\operatorname{Im}u_1(x),\,\dots,\,\operatorname{Im}u_l(x))$.

Estimating the functionals $B_j^\varepsilon(u)$, $j=\overline{1,\,10}$, we find that the functionals $B_1^\varepsilon(u)$, $B_2^\varepsilon(u)$, $B_3^\varepsilon(u)$, $B_7^\varepsilon(u)$ tend to zero as $\varepsilon\to0$.

For the functionals $B_m^\varepsilon(u)$, $m=4,5,6,8,9,10$, we obtain the following estimates:

$$|B_4^\varepsilon(u)|\leqslant\frac{\beta_3}{2}\sum_{k=1}^n\left(F\frac{\partial^2 u}{\partial x_k^2},\varphi_\varepsilon F\frac{\partial^2 u}{\partial x_k^2}\right)+\frac{\sigma_3}{2\beta_3}(Vu,\varphi_\varepsilon Vu),$$

$$|B_5^\varepsilon(u)|\leqslant\frac{\beta_4}{2}\sum_{k=1}^n\left(\frac{\partial^2 u}{\partial x_k^2},\varphi_\varepsilon Q\frac{\partial^2 u}{\partial x_k^2}\right)+\frac{\sigma_4}{2\beta_4}(Vu,\varphi_\varepsilon Vu),$$

$$|B_6^\varepsilon(u)|\leqslant\sum_{k=1}^n\left\|\varphi_\varepsilon^{\frac12}Q^{\frac12}\frac{\partial^2 u}{\partial x_k^2};L_2(\mathbb{R}^n)^l\right\|^2,\qquad |B_8^\varepsilon(u)|\leqslant\delta_3\sum_{k=1}^n\left\|\varphi_\varepsilon^{\frac12}F\frac{\partial^2 u}{\partial x_k^2};L_2(\mathbb{R}^n)^l\right\|^2,$$

$$|B_9^\varepsilon(u)|\leqslant\delta_4\sum_{k=1}^n\left\|\varphi_\varepsilon^{\frac12}F\frac{\partial^2 u}{\partial x_k^2};L_2(\mathbb{R}^n)^l\right\|^2,\qquad |B_{10}^\varepsilon(u)|\leqslant\delta_4\sum_{k=1}^n\|\varphi_\varepsilon^{\frac12}F\frac{\partial^2 u}{\partial x_k^2};L_2(\mathbb{R}^n)^l\|^2.$$

Here $\beta_3$, $\beta_4$ are positive numbers and $\sigma_3$, $\sigma_4$, $\delta_3$ and $\delta_4$ are the constants in conditions (2.6)–(2.10). While estimating the functionals $B_9^\varepsilon(u)$ and $B_{10}^\varepsilon(u)$, inequality (2.9) is used twice: in the case

$$\omega=(\omega_1,\omega_2,\dots,\omega_l)=u(x)=(u_1(x),u_2(x),\dots,u_l(x)),$$

and
$$\omega = (\omega_1, \omega_2, \ldots, \omega_l) = \frac{\partial u}{\partial x_i} = \left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}, \ldots, \frac{\partial u_l}{\partial x_i}\right).$$
By the obtained estimates and identity (4.1) we have
$$|(f, \varphi_\varepsilon V u)| \geqslant \left(1 - \frac{\sigma_3}{2\beta_3} - \frac{\sigma_4}{\beta_4}\right)(Vu, \varphi_\varepsilon V u) - |B_1^\varepsilon(u)| - |B_2^\varepsilon(u)| - |B_3^\varepsilon(u)| - |B_7^\varepsilon(u)|$$
$$+ \left(1 - \frac{\beta_3}{2} - \beta_4 - 2\delta_3 - 2\delta_4 - 2\delta_4\right) \sum_{k=1}^n \left(\frac{\partial^2 u}{\partial x_k^2}, \varphi_\varepsilon V \frac{\partial^2 u}{\partial x_k^2}\right).$$
Applying Cauchy-Schwarz inequality and passing to the limit as $\varepsilon \to 0$, we obtain the inequality
$$\|f; L_2(\mathbb{R}^n)^l\| \|Vu; L_2(\mathbb{R}^n)^l\| \geqslant |(f, Vu)| \geqslant \left(1 - \frac{\sigma_3}{2\beta_3} - \frac{\sigma_4}{\beta_4}\right)(Vu, Vu)$$
$$+ \left(1 - \frac{\beta_3}{2} - \beta_4 - 2\delta_3 - 4\delta_4\right) \sum_{k=1}^n \left(\frac{\partial^2 u}{\partial x_k^2}, V \frac{\partial^2 u}{\partial x_k^2}\right).$$
Then we choose positive numbers $\beta_3$, $\beta_4$ to satisfy the conditions
$$\frac{\sigma_3}{2\beta_3} + \frac{\sigma_4}{\beta_4} < 1, \quad \frac{\beta_3}{2} + \beta_4 + 2\delta_3 + 4\delta_4 < 1.$$
By the obtained inequality via simple transformations we arrive at coercive inequality (2.11).

The separability of nonlinear operator (2.1) in the space $L_2(\mathbb{R}^n)^l$ is implied by coercive inequality (2.11). The proof of Theorem 2.1 is complete.

## 5. Linear case

For the sake of a pictorial presentation, we formulate the statement of Theorem 2.1 in the case of a linear biharmonic operator. We assume that $V(x, \omega)$ is independent of $\omega$ and is of the form $V(x, \omega) = V(x)$, where $V(x) = V^*(x) \in C^2(\mathbb{R}^n, \operatorname{End}\mathbb{C}^l)$. We also suppose that for all $x \in \mathbb{R}^n$ and $u \in W_2^1(\mathbb{R}^n)$ the conditions
$$\sum_{i=1}^n \left\| F^{-\frac{1}{2}} \frac{\partial^2 F}{\partial x_i^2} F^{-\frac{3}{2}}; \mathbb{C}^l \right\|^2 \leqslant \sigma_1,$$
$$\sum_{i=1}^n \left\| F^{-\frac{1}{2}} \frac{\partial F}{\partial x_i} \frac{\partial u}{\partial x_i}; L_2(\mathbb{R}^n)^l \right\|^2 \leqslant \sigma_2 \left\| F^{\frac{3}{2}} u; \; ; L_2(\mathbb{R}^n)^l \right\|^2,$$
$$\sum_{i=1}^n \left\| V^{-\frac{1}{2}} \frac{\partial^2 V}{\partial x_i^2} V^{-1}; \mathbb{C}^l \right\|^2 \leqslant \sigma_3,$$
$$\sum_{i=1}^n \left\| V^{-\frac{1}{2}} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_i}; \; L_2(\mathbb{R}^n)^l \right\|^2 \leqslant \sigma_4 \|Vu; \; ; L_2(\mathbb{R}^n)^l\|^2$$
hold, where $\sigma_j$, $j = \overline{1, 4}$, are some numbers.

**Theorem 5.1.** *Assume that the conditions formulated in this section hold and let the numbers $\sigma_j$, $j = \overline{1, 4}$, be such that $0 < \sigma_1 + 2\sigma_2 < 4$, $0 < \sigma_3 + 2\sigma_4 < 4$. Then equation (2.1) is separable in $L_2(\mathbb{R}^n)^l$ and for all vector functions $u(x) \in L_2(\mathbb{R}^n)^l \cap W_{2,loc}^4(\mathbb{R}^n)^l$ such that $f(x) \in L_2(\mathbb{R}^n)^l$ the belongings*
$$\Delta^2 u, \; Vu, \; V^{\frac{1}{2}} \frac{\partial^2 u}{\partial x_i^2} \in L_2(\mathbb{R}^n)^l, \quad i = 1, 2, \ldots, n,$$

*hold true. At that, the coercive inequality*

$$\left\|\Delta^2 u(x); L_2(\mathbb{R}^n)^l\right\| + \left\|V(x)u(x); L_2(\mathbb{R}^n)^l\right\| + \sum_{i=1}^{n} \left\|V^{\frac{1}{2}} \frac{\partial^2 u(x)}{\partial x_i^2}; L_2(\mathbb{R}^n)^l\right\| \leqslant M \left\|f(x); L_2(\mathbb{R}^n)^l\right\|$$

*hold, where a positive number $M$ is independent of $u(x)$, $f(x)$.*

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Olimzhon Khudoiberdievich Karimov,
Institute of Mathematics AS RT,
Ainy st., 299/4,
734063, Dushanbe, Republic of Tadjikistan
E-mail: karimov_olim@mail.ru