ISSN 2304-0122 Ufa Mathematical Journal. Vol. 8. No 4 (2016). P. 131-143.

doi:10.13108/2016-8-4-131 UDC 517.956

ON SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS IN CYLINDRICAL DOMAINS

A.V. NEKLUDOV

Abstract. In a semi-infinite cylinder, we consider a second order elliptic equation with a lower order term. On the lateral boundary of the cylinder we impose the homogeneous Neumann condition. We show that each bounded solution tends to a constant at infinity and once the lower order term does not decay too fast, this constant vanishes. We establish that for a sufficiently fast decay of the lower order term, we have a trichotomy of the solutions as for the equation without the lower order term: the solution tends to a general non-zero constant or grows linearly or grows exponentially. The decay conditions for the lower order term are formulated in an integral form.

Keywords: Elliptic equation, Neumann boundary value condition, unbounded domain, low order term, asymptotic behavior of solutions, trichotomy of solutions.

Mathematics Subject Classification: 35J15, 35J25

1. Introduction

The behavior of solutions to elliptic equations in cylindrical or close domains with the Dirichlet, Neumann or periodic w.r.t. all variables except one boundary conditions on the lateral surface was studied rather well for the divergent type equations with no lower order terms [1]-[4]. For the equations with lower order terms, the most studied case is for the coefficients periodic w.r.t. the variable directed along the axis of the cylinder [5], [6].

In the present work we study the behavior of the generalized solutions to the elliptic second order equations with a lower order term subject to the Neumann condition on the lateral surface by means of energy estimates of Saint-Venant's principle kind [2]–[4]. The main attention is paid to the dependence of the properties of the solution on the behavior of the coefficient $q(x)$ at the lower order term of the equation. We show that under rather fast decay of the lower coefficient, the behavior of the solutions is similar to the behavior of the solutions to the divergent type equations with no lower terms and subject to Neumann condition: the tending of bounded solution to a constant, trichotomy of solutions. In the case of a slowly decaying lower order term, the behavior of bounded solutions is similar to the behavior of solutions to equations with no lower terms and subject to Dirichlet condition (each bounded solution tends to zero).

2. Main notations and definitions

In the n-dimensional cylinder $\Omega = (0, +\infty) \times \widehat{\Omega}$ we consider the elliptic equation

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) - q(x)u = 0, \tag{1}$$

A.V. Nekludov, On solutions of second order elliptic equations in cylindrical domains.

Submitted October 28, 2015.

where $x = (x_1, x_2, \dots, x_n) = (x_1, \widehat{x}) \in \mathbb{R}^n_x$, $\widehat{\Omega} \subset \mathbb{R}^{n-1}_{\widehat{x}}$ be a bounded domain with a Lipschitz boundary, $a_{ij}(x)$ are measurable functions in Ω, $a_{ij} = a_{ji}$, $\lambda_1|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqslant \lambda_2|\xi|^2$, $\xi \in \mathbb{R}^n$, $\lambda_1, \lambda_2 = \mathrm{const} > 0$, $q(x) \geqslant 0$ is a locally bounded measurable function.

On the lateral boundary of the cylinder $\Gamma = (0,\infty) \times \partial\widehat{\Omega}$ we impose Neumann boundary condition

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} = 0, \tag{2}$$

where $\partial u/\partial\nu \equiv \sum_{i,j=1}^n a_{ij}(x)\partial u/\partial x_i \cos(\vec{n}, x_j)$, $\vec{n}$ is the unit normal to Γ.

We introduce the following notations: $\Omega(a,b) = \Omega \cap \{x : a < x_1 < b\}$, $\Omega_t = \Omega(t, t+1)$, $\Gamma(a,b) = \Gamma \cap \{x : a < x_1 < b\}$, $\Gamma_t = \Gamma(t,t+1)$, $S_t = \{x : x_1 = t,\ \widehat{x} \in \widehat{\Omega}\}$, $\nabla u = \operatorname{grad} u$, $m_0 = \mathrm{mes}_{n-1}\widehat{\Omega}$, $\overline{u}(t) = m_0^{-1}\int_{\Omega_t} u\,dx$.

By solutions to (1)-(2) in Ω we mean generalized solutions, that is, the functions belonging to Sobolev space $W_2^1(\Omega(0,t))$ for each $t > 0$ and satisfying the integral identity

$$\int_{\Omega(0,t)} \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx + \int_{\Omega(0,t)} quv\,dx = 0 \tag{3}$$

for all functions $v \in W_2^1(\Omega(0,t))$ such that $v|_{S_0\cup S_t} = 0$.

3. Auxiliary statements

Lemma 1. *Let $u(x)$ be a solution to equation* (1) *in Ω_t satisfying condition* (2) *on Γ_t. Then the estimates*

$$\sup\nolimits_{S_{t+1/2}} |u| \leqslant c_0\left(\int_{\Omega_t} u^2\,dx\right)^{1/2}, \qquad \sup\nolimits_{S_{t+1/2}}(u - C) \leqslant c_1\left(\int_{\Omega_t}(u-C)^2\,dx\right)^{1/2}$$

hold true, where c_0 is independent of u, t; c_1 is independent of u, t, $C > 0$.

Proof. It is known, see, for instance, [7], that the solution to a second order elliptic equation satisfying the homogeneous Neumann condition on Γ_t, for each point Γ_t, by means of local flattening and symmetry principle, can be continued to a domain ω containing a neighbourhood of this boundary point and the structure of the equation is preserved.

Let $C \geqslant 0$, $k > 0$, $x^0 \in \omega$, $\rho, \sigma \in (0,1)$, $\varphi(x) \in C^1(\mathbb{R}^n)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ as $|x - x^0| \leqslant \rho(1-\sigma)$, $\varphi(x) = 0$ as $|x - x^0| \geqslant \rho$, $|\nabla\varphi| \leqslant \mathrm{const}/(\rho\sigma)$. We choose ρ such that $\operatorname{supp}\varphi \subset \omega$. Letting $v = \max\{u - C - k, 0\}\varphi$ in integral identity (3) and taking into consideration that $\int_{\{x:\,u-C-k>0\}} qu(u-C-k)\,\varphi dx \geqslant 0$, we obtain the estimate

$$\int_{A_{k,\rho(1-\sigma)}} |\nabla w|^2\,dx \leqslant c_2(\rho\sigma)^{-2}\int_{A_{k,\rho}}(w-k)^2\,dx,$$

where $w = u - C$, $A_{k,\varkappa} = \{x : w(x) > k\} \cap \{x : |x - x^0| < \varkappa\}$, c_2 is independent of w, k, ρ, σ, x^0.

This follows [8, Ch. II, Sect. 5.3] that for each domain $\omega' \subset\subset \omega$ the estimate

$$\sup\nolimits_{\omega'} w \leqslant c\left(\int_{\omega} w^2\,dx\right)^{1/2} \leqslant c_1\left(\int_{\Omega_t} w^2\,dx\right)^{1/2}$$

holds true. Covering $\Gamma(t+1/4, t+3/4)$ by finitely many constructed neighbourhoods, we obtain that this estimate is true for $\sup_{\Omega(t+1/4,t+3/4)} w$ and therefore, for $\sup_{S_{t+1/2}} w$. Thus, the second of the needed estimates is proved. Moreover, as $C = 0$, similar to the obtained estimate for $\sup u$, we obtain the estimate for $\sup(-u)$. The proof is complete. □

For a solution $u(x)$ to equation (1) satisfying (2), in the standard way we introduce the notion of the "heat flow" through the section S_t of cylinder Ω:

$$P(t,u)=\lim_{h\to 0+}\left(h^{-1}\int_{\Omega(t,t+h)}\sum_{i=1}^{n}a_{i1}\frac{\partial u}{\partial x_i}\,dx\right)=\int_{S_t}\sum_{i=1}^{n}a_{i1}\frac{\partial u}{\partial x_i}\,d\widehat{x};$$

the latter identity is true for almost each $t\geqslant 0$. Let $0\leqslant t<T$, $h_1>0$, $h_2>0$. We choose $v=\Phi$ (3), where $\Phi=\Phi(x_1)$ is continuous function, $\Phi=1$ as $t+h_1\leqslant x_1\leqslant T$, $\Phi(t)=\Phi(T+h_2)=0$, Φ is linear for $t\leqslant x_1\leqslant t+h_1$ and for $T\leqslant x_1\leqslant T+h_2$:

$$h_1^{-1}\int_{\Omega(t,t+h_1)}\sum_{i=1}^{n}a_{i1}\frac{\partial u}{\partial x_i}\,dx-h_2^{-1}\int_{\Omega(T,T+h_2)}\sum_{i=1}^{n}a_{i1}\frac{\partial u}{\partial x_i}\,dx+\int_{\Omega(t,T+h_2)}qu\Phi\,dx=0. \tag{4}$$

Making h_1 to tend to zero and then doing the same with h_2, we obtain the relation

$$P(T,u)-P(t,u)=\int_{\Omega(t,T)}qu\,dx. \tag{5}$$

It is easy to see that as $t>0$, in the definition of the flow, the domain of the integration $\Omega(t,t+h)$ can be replaced by $\Omega(t-h,t)$.

We consider the equation with no lower order term corresponding to equation (1):

$$L_0V\equiv\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial V}{\partial x_i}\right)=0. \tag{6}$$

It is well-known, see, for instance [9, Thm. 2], that there exists a positive solution $V(x)$ to equation (6) in Ω satisfying the homogeneous Neumann condition $\big(\partial V/\partial\nu\big)\big|_\Gamma=0$ on Γ and the estimate

$$C_1x_1\leqslant V(x)\leqslant C_2x_1,\qquad C_1,C_2=\text{const}>0,$$

as $x_1>1$. The function $V(x)$ also satisfies [10, Form. (12)] the conditions

$$\int_{\Omega_t}|\nabla V|^2\,dx\leqslant c_1=\text{const},\qquad P(t,V)=1,\qquad t\geqslant 0,$$

the second condition is satisfied by multiplying V by a constant. As $t>0$, the function V satisfies the integral identity

$$\int_{\Omega(0,t)}\sum_{i,j=1}^{n}a_{ij}\frac{\partial V}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx=0 \tag{7}$$

for all functions $v\in W_2^1(\Omega(0,t))$ such that $v|_{S_0\cup S_t}=0$.

Lemma 2. *Let $u(x)$ be a bounded in Ω solution to* (1)-(2), $M_0=\sup_{S_0}u$. *Then the estimate*

$$u(x)\leqslant\max\{M_0,0\}$$

holds true in Ω.

Proof. Let $V(x)$ be the solution to equation (6) defined above. We fix $\varepsilon>0$. It is obvious that for the function $w=u-\varepsilon V$ we have $w\leqslant M_0$ on S_0 and on $S_{T(\varepsilon)}$ for sufficiently large $T(\varepsilon)$. Since $Lw=\varepsilon qV\geqslant 0$ and $(\partial w/\partial\nu)|_\Gamma=0$, the function w can not have a positive maximum in $\Omega(0,T(\varepsilon))\cup\Gamma(0,T(\varepsilon))$, that is $w\leqslant\max\{M_0,0\}$. Making ε to tend to 0, we arrive at the statement of the lemma. □

Lemma 3. *Let $u(x)$ be a bounded in Ω solution to* (1)-(2). *Then*

$$\int_\Omega\left(|\nabla u|^2+qu^2\right)dx<\infty.$$

Proof. Choosing $v = u\Phi$ in (3), where $\Phi = \Phi(x_1) \in C^2(\mathbb{R})$, $0 \leqslant \Phi \leqslant 1$, $\Phi = 1$ as $1 \leqslant x_1 \leqslant N$, $\Phi = 0$ as $x_1 \leqslant 0$ and as $x_1 \geqslant N+1$, $(\Phi')^2 \leqslant c\Phi$, $c = \text{const}$, employing the ellipticity of the equation and the estimate of form $ab \leqslant \varepsilon a^2/2 + b^2/(2\varepsilon)$, we obtain the estimate

$$\int_{\Omega(0,N+1)} \big(|\nabla u|^2 + qu^2\big)\Phi\, dx \leqslant c_0 + c_1 \int_{\Omega_N} |u||\nabla u||\Phi'|\, dx \leqslant\leqslant c_0 + \int_{\Omega_N} \big(c_2 u^2 + |\nabla u|^2 \Phi\big)\, dx,$$

$c_i = \text{const} > 0$. Then

$$\int_{\Omega(1,N)} \big(|\nabla u|^2 + qu^2\big)\, dx \leqslant c_0 + c_2 \int_{\Omega_N} u^2\, dx \tag{8}$$

that implies the statement of the lemma. □

Lemma 4. *Let $u(x)$ be a solution to (1)–(2) in Ω, $V(x)$ is the solution to equation (6) defined above. Then*

$$\overline{u}(N) = \overline{V}(N) \int_N^{N+1} P(t,u)dt - \int_{\Omega(0,N+1)} quV\Phi\, dx + I_N,$$

where

$$|I_N| \leqslant c_0 \bigg(\int_{\Omega_N} |\nabla u|^2\, dx \bigg)^{1/2} + c_1, \quad c_0, c_1 = \text{const} > 0,$$

$\Phi = \Phi(x_1)$ is a continuous function, $\Phi(x_1) = 1$ as $1 \leqslant x_1 \leqslant N$, $\Phi(0) = \Phi(N+1) = 0$, Φ is linear as $0 \leqslant x_1 \leqslant 1$ and $N \leqslant x_1 \leqslant N+1$.

Letting $v = u\Phi$ in integral identity (7), we obtain

$$\int_{\Omega(0,N+1)} \sum_{i,j=1}^n a_{ij} \frac{\partial V}{\partial x_i} \frac{\partial u}{\partial x_j} \Phi\, dx = \int_{\Omega_N} \sum_{i=1}^n a_{i1} \frac{\partial V}{\partial x_i} u\, dx - \int_{\Omega_0} \sum_{i=1}^n a_{i1} \frac{\partial V}{\partial x_i} u\, dx.$$

Choosing the test function $v = V\Phi$ in integral identity (3) for u, we obtain

$$\int_{\Omega(0,N+1)} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial V}{\partial x_j} \Phi\, dx = -\int_{\Omega(0,N+1)} quV\Phi\, dx$$
$$+ \int_{\Omega_N} \sum_{i=1}^n a_{i1} \frac{\partial u}{\partial x_i} V\, dx - \int_{\Omega_0} \sum_{i=1}^n a_{i1} \frac{\partial u}{\partial x_i} V\, dx.$$

By two latter identities and the symmetricity of the matrix a_{ij} we get that

$$\int_{\Omega_N} \sum_{i=1}^n a_{i1} \frac{\partial V}{\partial x_i} u\, dx = \int_{\Omega_N} \sum_{i=1}^n a_{i1} \frac{\partial u}{\partial x_i} V\, dx - \int_{\Omega(0,N+1)} quV\Phi\, dx + I_0,$$

where $I_0 = \text{const}$ is independent of N. This yields

$$\begin{aligned} \overline{u}(N) =& \overline{V}(N) \int_N^{N+1} P(t,u)dt - \int_{\Omega(0,N+1)} quV\Phi\, dx \\ &+ \int_{\Omega_N} \sum_{i=1}^n a_{i1} \Big((V - \overline{V}(N)) \frac{\partial u}{\partial x_i} - (u - \overline{u}(N)) \frac{\partial V}{\partial x_i} \Big)\, dx + I_0. \end{aligned} \tag{9}$$

Employing Cauchy-Schwarz and Poincaré inequalities and the estimate for the Dirichlet integral for V, we obtain

$$\begin{aligned}
&\left|\int_{\Omega_N}\sum_{i=1}^{n} a_{i1}\Big((V-\overline{V}(N))\frac{\partial u}{\partial x_i}-(u-\overline{u}(N))\frac{\partial V}{\partial x_i}\Big)\,dx\right|\\
&\leqslant c_2\Big[\Big(\int_{\Omega_N}\big(V-\overline{V}(N)\big)^2\,dx\Big)^{1/2}\Big(\int_{\Omega_N}|\nabla u|^2\,dx\Big)^{1/2}\\
&\quad+\Big(\int_{\Omega_N}\big(u-\overline{u}(N)\big)^2\,dx\Big)^{1/2}\Big(\int_{\Omega_N}|\nabla V|^2\,dx\Big)^{1/2}\Big]\\
&\leqslant c_3\Big(\int_{\Omega_N}|\nabla V|^2\,dx\Big)^{1/2}\Big(\int_{\Omega_N}|\nabla u|^2\,dx\Big)^{1/2}\leqslant c_4\Big(\int_{\Omega_N}|\nabla u|^2\,dx\Big)^{1/2},
\end{aligned}$$

$c_i > 0$ are independent of N. Then by (9) we obtain the statement of the lemma.

4. Behavior of bounded solutions

Theorem 1. *Let $u(x)$ be a bounded in Ω solution* (1)-(2), *$q(x) \geqslant 0$ in Ω. Then for some* $C = \text{const}$

$$\int_{\Omega_t}(u-C)^2\,dx \to 0, \quad t\to\infty.$$

If the condition $\|q\|_{L_p(\Omega_t)} \to 0$, $t\to\infty$, $p > n/2$ is satisfied as well or $C = 0$, then

$$\sup\nolimits_{S_t}|u-C|\to 0, \quad t\to\infty.$$

Proof. The boundedness of the solution implies the boundedness of $\overline{u}(t)$. Hence, for some sequence $t_k\to\infty$, $k\to\infty$, we have $\overline{u}(t_k)\to C = \text{const}$. Then, employing Poincaré inequality and a finiteness of the Dirichlet integral for $u(x)$ by Lemma 3, we obtain that

$$\int_{\Omega_{t_k}}(u-C)^2\,dx \leqslant 2\int_{\Omega_{t_k}}\big(u-\overline{u}(t_k)\big)^2\,dx + 2m_0\big(\overline{u}(t_k)-C\big)^2 \to 0, \quad k\to\infty.$$

Let us show that $\|u-C\|_{L_2(\Omega_t)}\to 0$, $t\to\infty$. We assume the opposite, then $\|u-C'\|_{L_2(\Omega_{t'_k})}\to 0$ as $k\to\infty$ for some sequence $t'_k\to\infty$ and a constant $C'\neq C$. In view of the continuity of the function $\overline{u}(t)$, without loss of generality we can assume that C and C' are of the same sign, for instance, $0\leqslant C < C'$. In accordance with Lemma 1 we have

$$\sup\nolimits_{S_{t_k+1/2}}(u-C)\leqslant \alpha_k \equiv c\|u-C\|_{L_2(\Omega_{t_k})}\to 0, \quad k\to\infty, \quad c=\text{const}.$$

By Lemma 2 we obtain that $u\leqslant C+\alpha_k$ as $x_1 > t_k+1/2$, which contradicts the condition $C < C'$.

The statement of the theorem on the uniform convergence of u to a constant as $C\neq 0$ is implied by the fact that $L_0(u-C) = qu$ and De Georgi estimate [2] $\sup_{S_{t+1/2}}|u-C|\leqslant c\big(\|u-C\|_{L_2(\Omega_t)}+\|qu\|_{L_p(\Omega_t)}\big)$. As $C=0$, this is implies by Lemma 1. The proof is complete. □

Theorem 2. *Assume that the function $q(x)\geqslant 0$ satisfies one of the following conditions:*
1) $q(x)\geqslant q_0 = \text{const} > 0$ in Ω,
2) $\int_\Omega x_1 q(x)\,dx = \infty$, $\|q\|_{L_p(\Omega_t)}\to 0$, $t\to\infty$, $p > n/2$.
Then for each bounded in Ω solution (1)-(2)

$$\sup\nolimits_{S_t}|u(x)|\to 0, \quad t\to\infty.$$

Proof. Suppose that Condition 1) holds. Then by Lemmata 1 and 3 we obtain

$$\sup\nolimits_{S_t}u^2\leqslant c_0\int_{\Omega_{t-1/2}}u^2\,dx\to 0,$$

$t \to \infty$, $c_0 > 0$ is independent of t.

Suppose that Condition 2) holds. Assume that $u \to C \neq 0$ as $x_1 \to \infty$. We can assume that $C > 0$. By Lemma 4 we have

$$\overline{V}(N) \int_N^{N+1} P(t,u)dt = \int_{\Omega(0,N+1)} quV\Phi\, dx + I_N, \tag{10}$$

where $|I_N| \leqslant c_1 = \text{const}$, $\Phi = \Phi(x_1) = 1$ as $0 \leqslant x_1 \leqslant N$, $\Phi = N + 1 - x_1$ as $N \leqslant x_1 \leqslant N + 1$. Since by the assumption$u \to C > 0$, $x_1 \to \infty$, and in accordance with Theorem 1, this convergence is uniform in $\widehat{x} \in \widehat{\Omega}$, then $u(x) > 0$ in $\Omega(t_0, \infty)$ for sufficiently large t_0. Then it follows from (5) that $P(t,u)$ is a non-decreasing function of t as $t > t_0$. Since by Lemma 3 $\int_\Omega |\nabla u|^2\, dx < \infty$, then $P(t,u) \to 0$, $t \to \infty$, and therefore, $P(t,u) < 0$ for sufficiently large t. Since $u \to C > 0$, it follows from Condition 2) that $\int_\Omega quV\, dx = +\infty$. Then the left hand side and the right hand side in (10) have opposite signs if N is large enough. The obtained contradiction implies that $C = 0$. The proof is complete. □

5. Fast decaying lower coefficient: the existence of solution with a linear growth, trichotomy of solutions

It is known [11, Ch. VI, Thm. 5] that for each solution to ordinary differential equation

$$u'' - q(t)u = 0, \qquad \int_{t_0}^{\infty} tq(t)\, dt < \infty,$$

on the half-line $t > t_0$, one of the asymptotics $u(t) \sim ct$, $c = \text{const} \neq 0$ and $u(t) \to \text{const}$ holds true as $t \to \infty$. In what follows we show that under an appropriate integral condition for $q(x)$, for the solutions to (1)-(2) in Ω a similar result is true with an additional third option: exponential growth (trichotomy of solutions).

Theorem 3. *Let $q(x) \geqslant 0$ in Ω, $\int_\Omega x_1 q(x)\, dx < \infty$, $\|q\|_{L_p(\Omega_t)} \leqslant c$ as $t \geqslant t_0 = \text{const} > 0$, $p > n/2$, $c > 0$ is some constant depending on $\widehat{\Omega}$, λ_1, λ_2. Then there exists a positive in Ω solution $U(x)$ to problem (1)–(2) satisfying the conditions*

$$U\big|_{S_0} = 0, \quad A_1 x_1 \leqslant U(x) \leqslant A_2 x_1 \quad (x_1 \geqslant 1), \quad A_1, A_2 = \text{const} > 0,$$

$$P(t,U) \to p_0 = \text{const} > 0, \quad t \to \infty.$$

Proof. Let $V(x) > 0$ be the above introduced positive linearly growing solution to equation (6) in Ω satisfying homogeneous Neumann condition on Γ. For an arbitrary $N \in \mathbb{N}$, in the domain $\Omega(0,N)$ we consider solution $U_N(x)$ to the problem

$$LU_N = 0, \quad U_N\big|_{S_0} = 0, \quad U_N\big|_{S_N} = C_1 N, \quad \frac{\partial U_N}{\partial \nu}\bigg|_{\Gamma(0,N)} = 0.$$

In accordance with the maximum principle, U_N can not has a negative minimum in $\Omega(0,N)$ and on $\Gamma(0,N)$. Therefore, $U_N > 0$ in $\Omega(0,N)$. Choosing the test function $v = U_N\Phi$ in integral identity (3) for $u = U_N$, where $\Phi = \Phi(x_1)$ is a continuous function, $\Phi = 1$ as $0 \leqslant x_1 \leqslant N - h$, $\Phi(N) = 0$, Φ is linear as $N - h \leqslant x_1 \leqslant N$, we obtain

$$\int_{\Omega(0,N)} \sum_{i,j=1}^n a_{ij} \frac{\partial U_N}{\partial x_i}\frac{\partial U_N}{\partial x_j}\Phi\, dx + \int_{\Omega(0,N)} qU_N^2\Phi\, dx = h^{-1}\int_{\Omega(N-h,N)} U_N \sum_{i=1}^n a_{i1}\frac{\partial U_N}{\partial x_i}\, dx$$
$$= h^{-1}\int_{\Omega(N-h,N)} (U_N - C_1 N)\sum_{i=1}^n a_{i1}\frac{\partial U_N}{\partial x_i}\, dx + h^{-1}C_1 N\int_{\Omega(N-h,N)} \sum_{i=1}^n a_{i1}\frac{\partial U_N}{\partial x_i}\, dx.$$

Since $(U_N - C_1N)\big|_{S_N} = 0$, by the Fridrichs type inequality

$$\int_{\Omega(N-h,N)} \varphi^2\,dx \leqslant c_0h^2 \int_{\Omega(N-h,N)} |\nabla\varphi|^2\,dx, \quad \varphi\big|_{S_N} = 0, \quad c_0 = \text{const},$$

we obtain

$$h^{-1}\bigg|\int_{\Omega(N-h,N)} (U_N - C_1N)\sum_{i=1}^n a_{i1}\frac{\partial U_N}{\partial x_i}\,dx\bigg| \leqslant c_1\int_{\Omega(N-h,N)} |\nabla U_N|^2\,dx \to 0, \quad h\to 0.$$

Hereinafter in the proof, $c_i = \text{const} > 0$ depend only on $\widehat{\Omega}$, λ_1, λ_2. Then by the above identity we obtain

$$\int_{\Omega(0,N)} \sum_{i,j=1}^n a_{ij}\frac{\partial U_N}{\partial x_i}\frac{\partial U_N}{\partial x_j}\,dx + \int_{\Omega(0,N)} qU_N^2\,dx = C_1NP(N,U_N).$$

Hence, taking into consideration that $U_N\big|_{S_0} = 0$ and therefore, by [2, Form. (46)], the inequality

$$m_0C_1^2N^2 = \int_{S_N} U_N^2\,d\widehat{x} \leqslant c_2N\int_{\Omega(0,N)} |\nabla U_N|^2\,dx$$

holds, we obtain

$$P(N,U_N) \geqslant c_3N^{-1}\int_{\Omega(0,N)} |\nabla U_N|^2\,dx \geqslant c_4 > 0. \tag{11}$$

For the function $w = U_N - V$ we have $Lw = qV \geqslant 0$ in$\Omega(0,N)$, $(\partial w/\partial\nu)|_{\Gamma(0,N)} = 0$, $w|_{S_0\cup S_N} \leqslant 0$. Then w can not have a positive maximum in $\Omega(0,N)\cup\Gamma(0,N)$. Hence, $w < 0$ in $\Omega(0,N)$. Thus, the inequality

$$0 < U_N < V \tag{12}$$

holds true in $\Omega(0,N)$. Since in accordance (5) for $t < N$

$$P(t,U_N) = P(N,U_N) - \int_{\Omega(t,N)} qU_N\,dx$$

we have, by (11) and (12) we obtain that there exists a $t_0 > 0$ such that for all $t \geqslant t_0$ and $N \geqslant t$

$$P(t,U_N) \geqslant c_4/2 > 0. \tag{13}$$

It follows from estimates (12) and (8) that the sequence U_N $(N \geqslant t)$ is bounded in $W_2^1(\Omega(0,t))$ for each $t > 0$. Hence, applying diagonal process, we obtain a sequence U_{N_k} weakly convergent in $W_2^1(\Omega(0,t))$ and strongly convergent in $L_2(\Omega(0,t))$ to some function U for each $t > 0$. It is obvious that U satisfies (1)–(2) and the estimate $0 \leqslant U(x) \leqslant V(x) \leqslant C_2x_1$ almost everywhere in $\Omega(1,\infty)$ and by the Hölder continuity of generalized solutions to second order elliptic equations [8, Ch. III, Thm. 14.1], $0 \leqslant U(x) \leqslant V(x) \leqslant C_2x_1$ everywhere in $\Omega(1,\infty)$. By (5) we obtain that $P(t,U) \to p_0 = \text{const}$, $t\to\infty$. Since it follows from (4) that $P(t,U_N) = \int_0^1 P(\tau,U_N)\,d\tau + \int_{\Omega(0,t)} qU_N\Psi(x_1)\,dx$, $\Psi = x_1$ as $0 \leqslant x_1 \leqslant 1$, $\Psi = 1$ as $1 \leqslant x_1 \leqslant t$, then $P(t,U) = \lim_{k\to\infty} P(t,U_{N_k})$. Thanks to (13), we obtain that $P(t,U) \geqslant c_4/2$ as $t \geqslant t_0$ and $p_0 \geqslant c_4/2 > 0$.

Let us estimate Dirichlet integral for U. Choosing the test function $v = U\Phi$ in the integral identity of type (3) for $U(x)$, where $\Phi = \Phi(x_1)$ is continuous function, $\Phi = 1$ as $0 \leqslant x_1 \leqslant t$, $\Phi(t+h) = 0$; Φ is linear as $t \leqslant x_1 \leqslant t+h$; $h > 0$, we obtain

$$\int_{\Omega(0,t+h)} \sum_{i,j=1}^n a_{ij}\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}\Phi\,dx + \int_{\Omega(0,t+h)} qU^2\Phi\,dx = h^{-1}\int_{\Omega(t,t+h)} U\sum_{i=1}^n a_{i1}\frac{\partial U}{\partial x_i}\,dx.$$

Making h to tend to zero, we obtain that for almost each $t > 0$

$$\int_{\Omega(0,t)} \sum_{i,j=1}^{n} a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \, dx + \int_{\Omega(0,t)} qU^2 \, dx = \int_{S_t} U \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, d\widehat{x}. \tag{14}$$

Hence, for almost each $t > 0$ we obtain

$$I(t) \equiv \int_{\Omega(0,t)} |\nabla U|^2 \, dx \leqslant c_5 \int_{S_t} U \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, d\widehat{x} \leqslant c_6 t \sqrt{I'(t)}.$$

Then, integrating inequality $I'I^{-2} \geqslant c_6^{-2} t^{-2}$ from t to T and making T to tend to ∞, we obtain $I(t) \leqslant c_6^2 t$.

Let $N_0 \in \mathbb{N}$ be such that $\int_{\Gamma(N_0,\infty)} qU \, dx < c_4 C_1/(3C_2)$ and $P(t,U) \geqslant c_4/2$ as $t \geqslant N_0$. Employing Poincaré inequality and the estimate for Dirichlet integral of U, by Lemma 4 for $u = U$ in the domain $\Omega(N_0, \infty)$ we obtain

$$\begin{aligned} \overline{U}(N) \geqslant & \overline{V}(N) \int_N^{N+1} P(t,U) dt - \int_{\Omega(N_0,N+1)} qUV \, dx - c_7 N^{1/2} \\ \geqslant & c_4 C_1 N/2 - C_2(N+1) c_4 C_1/(3C_2) - c_7 N^{1/2} \geqslant c_8 N. \end{aligned}$$

for sufficiently large $N \geqslant N_0$.

Let us estimate the deviation of U from $\overline{U}(N)$ in the domain Ω_N. Since the function $U - \overline{U}(N)$ satisfies the equation $L_0(U - \overline{U}(N)) = qU$ in Ω and homogeneous Neumann condition on Γ, then for $p > n/2$, in view of De Georgi estimate [2], Poincaré inequality and the estimates for the function U and its Dirichlet integral we obtain that

$$\begin{aligned} \sup_{S_{N+1/2}} \left(U - \overline{U}(N)\right)^2 \leqslant & c_9 \left(\int_{\Omega_N} \left(U - \overline{U}(N)\right)^2 dx + ||qU||^2_{L_p(\Omega_N)} \right) \\ \leqslant & c_{10}(N + c^2 N^2) \leqslant c_8^2 N^2/4, \quad N \geqslant N_0' = \text{const} \end{aligned}$$

if $c_{10} c^2 \leqslant c_8^2/5$. In view of the linear lower bound for $\overline{U}(N)$, we obtain the required lower bound for $U(x)$. The proof is complete. □

Lemma 5. *Let $q(x) \geqslant 0$ in Ω, $\|q\|_{L_p(\Omega_t} \leqslant c'$ as $t \geqslant t_1 = \text{const}$ for some $p > n/2$, c' is some constant independent of $\widehat{\Omega}$, λ_1, λ_2; $u(x)$ is the solution to* (1)–(2) *and for some sequence $t_k \to \infty$ the condition $\sup_{\Omega_{t_k}} |u| = o(\exp(At_k))$ holds, $k \to \infty$, where $A > 0$ is some constant depending on $\widehat{\Omega}$, λ_1, λ_2. Then there exists a sequence $t_k' \to \infty$, $k \to \infty$, such that the estimate*

$$\overline{u}(t_k') - \frac{1}{2}|\overline{u}(t_k')| - I_1 \leqslant u(x) \leqslant \overline{u}(t_k') + \frac{1}{2}|\overline{u}(t_k')| + I_1, \qquad x \in S_{t_k'+1/2},$$

holds true and $I_1 \geqslant 0$ is independent of k.

Proof. Employing estimate (8), we obtain

$$\int_{\Omega(0,t_k)} |\nabla u|^2 \, dx \leqslant I_0 + c_1 \int_{\Omega_{t_k}} u^2 \, dx = o(\exp(2At_k)), \quad k \to \infty, \tag{15}$$

$c_i = c_i(\widehat{\Omega}, \lambda_1, \lambda_2) > 0$, $I_0 \geqslant 0$ is independent of $k \in \mathbb{N}$. Let us show that for some sequence $t_k' \to \infty$

$$\int_{\Omega_{t_k'}} |\nabla u|^2 \, dx \leqslant \delta \int_{\Omega(0,t_k')} |\nabla u|^2 \, dx, \quad \delta = \exp\{2A\} - 1 > 0. \tag{16}$$

Indeed, otherwise for an arbitrary $t \geqslant t_0 = \text{const}$

$$\int_{\Omega_t} |\nabla u|^2 \, dx = \int_{\Omega(0,t+1)} |\nabla u|^2 \, dx - \int_{\Omega(0,t)} |\nabla u|^2 \, dx > \delta \int_{\Omega(0,t)} |\nabla u|^2 \, dx,$$

that in view of (15) we obtain that

$$\int_{\Omega(0,t)} |\nabla u|^2\,dx < (1+\delta)^{-1}\int_{\Omega(0,t+1)} |\nabla u|^2\,dx < \dots$$
$$\dots < (1+\delta)^{-N_k}\int_{\Omega(0,t+N_k)} |\nabla u|^2\,dx = (1+\delta)^{-N_k} o(\exp\{2A(t+N_k)\}) \to 0, \quad k\to\infty,$$

if we take $N_k \in \mathbb{N}$ such that $t_k - 1 \leqslant t + N_k \leqslant t_k$. Thus, $\nabla u \equiv 0$ and estimate (16) holds true. Then by (15) and Poincaré inequality we obtain

$$\int_{\Omega_{t'_k}} |\nabla u|^2\,dx \leqslant \delta\Big(I_0 + c_1\int_{\Omega_{t'_k}} u^2\,dx\Big) \leqslant c_2\delta\Big(\int_{\Omega_{t'_k}} |\nabla u|^2\,dx + \overline{u}^2(t'_k) + I_0\Big).$$

If $\delta \leqslant c_2^{-1}/2$, then

$$\int_{\Omega_{t'_k}} |\nabla u|^2\,dx \leqslant 2c_2\delta(\overline{u}^2(t'_k) + I_0). \tag{17}$$

Let us estimate the deviation of $u(x)$ from $\overline{u}(t'_k)$. Employing Poincaré inequality and estimate (17), by Lemma 1 we obtain

$$\sup_{S_{t'_k+1/2}} u^2 \leqslant c_3\Big(\int_{\Omega_{t'_k}} |\nabla u|^2\,dx + \overline{u}^2(t'_k)\Big) \leqslant c_4\big((\delta+1)\overline{u}^2(t'_k) + \delta I_0\big).$$

Hence, since $L_0(u - \overline{u}(t'_k)) = qu$, we employ De Georgi estimate [2] and inequality (17) once again, for $k \geqslant k_0 = \text{const}$ we obtain

$$\begin{aligned}\sup_{S_{t'_k+1/2}} \big(u-\overline{u}(t'_k)\big)^2 &\leqslant c_5\Big(\int_{\Omega_{t'_k}} \big(u-\overline{u}(t'_k)\big)^2\,dx + ||qu||^2_{L_p(\Omega_{t'_k})}\Big)\\ &\leqslant c_6\Big(\int_{\Omega_{t'_k}} |\nabla u|^2\,dx + (c')^2\big((\delta+1)\overline{u}^2(t'_k) + \delta I_0\big)\Big)\\ &\leqslant c_7\Big(\delta(\overline{u}^2(t'_k) + I_0) + (c')^2\big((\delta+1)\overline{u}^2(t'_k) + \delta I_0\big)\Big) \leqslant \frac{1}{4}\big(\overline{u}^2(t'_k) + I_0\big)\end{aligned}$$

if $c_7(c')^2 \leqslant 1/8$ and $c_7\delta(1+(c')^2) \leqslant 1/8$. Thus, the statement of the lemma is true for the sequence t'_k, $k \geqslant k_0$, $c' = (8c_7)^{-1/2}$, $\delta = \min\{c_2^{-1}/2, \big(8c_7(1+(c')^2)\big)^{-1}\}$, $A = 2^{-1}\ln(1+\delta)$. □

Lemma 6. *Suppose that $u(x)$ satisfies the assumptions of Lemma 5 and moreover, $\int_\Omega x_1 q(x)\,dx < \infty$ and $\|q\|_{L_p(\Omega_t)} \leqslant c$ as $t \geqslant t_0 = \text{const}$, where $c > 0$ is a constant in Theorem 3. Then*

$$|u(x)| \leqslant Cx_1, \quad C = \text{const} > 0$$

for all $x_1 \geqslant 1$.

Proof. We suppose the opposite, then for some sequence $\widetilde{t}_k \to \infty$

$$\sup_{S_{\widetilde{t}_k}} |u|/\widetilde{t}_k \to \infty, \quad k\to\infty. \tag{18}$$

Let U be a linearly growing solution to (1)-(2) in Ω. The existence of such solution was proved in Theorem 3. Applying the maximum principle to the functions $u \pm c_0 U$ for sufficiently large $c_0 > 0$, by (18) we obtain that $\sup_{S_t} |u|/t \to \infty$, $t \to \infty$. Let t'_k be a sequence, for which Lemma 5 holds true. Without loss of generality we can assume that $\sup_{S_{t'_k+1/2}} u > 0$. Then by Lemma 5 we obtain that $\inf_{S_{t'_k+1/2}} u/t'_k \to +\infty$, $k\to\infty$. Applying the maximum principle to the function $U - c_1 - \varepsilon u$ for sufficiently large $c_1 > 0$ and making ε to tend to 0, we obtain that $U \leqslant c_1$ in $\Omega(t'_1 + 1/2, \infty)$, which contradicts the linear growth of U. The obtained contradiction means that relation (18) is wrong that proves the lemma. □

Lemma 7. *Suppose that the assumptions of Lemma 6 holds true and moreover, the condition $P(t,u) \to 0$, $t \to \infty$ is satisfied. Then the solution $u(x)$ to* (1)-(2) *is bounded in* Ω.

Proof. According to Lemma 6, $|u(x)| \leqslant Cx_1$, $x_1 \geqslant 1$. Then $\int_{\Omega(0,t)} x_1 qu\,dx = o(t)$, $t \to \infty$. This follows from Lemma 4 that

$$|\overline{u}(t)| \leqslant o(t) + c_1 \biggl(\int_{\Omega_t} |\nabla u|^2\,dx \biggr)^{1/2}, \quad t \to \infty,$$

$c_1 > 0$ is independent of t. Estimating Dirichlet integral for u in the same way as this was done for the function U in the proof of Theorem 3, we obtain that $\int_{\Omega(0,t)} |\nabla u|^2\,dx \leqslant c_2 t$, $c_2 > 0$ is independent of t. Then $\overline{u}(t) = o(t)$. Employing Lemma 5, we obtain that $\sup_{S_{t_k}} |u| = o(t_k)$ for some sequence $t_k \to \infty$, that is, $u(x) \leqslant c_0 + \varepsilon U$ on $S_{t_1} \cup S_{t_k}$ as $k > k_0(\varepsilon)$. Applying the maximum principle and making ε to tend to 0, we obtain that $u(x) \leqslant c_0$ for sufficiently large x_1. In the same way we obtain the estimates from below. The proof is complete. □

The main result on the trichotomy of solutions in the case of a fast decaying is a follows.

Theorem 4. *Let $q(x) \geqslant 0$ in Ω, $\int_\Omega x_1 q(x)\,dx < \infty$, $\|q\|_{L_p(\Omega_t)} \leqslant \min\{c, c'\}$ as $t \geqslant t_0 = \mathrm{const}$, c, c' are constants in Theorem 3 and Lemma 5, respectively. Then each solution to* (1)-(2) *behaves in one of the following three ways:*

1) $u(x)$ is bounded in Ω;

2) $\sup_{\Omega_t} |u| \geqslant C_0 \exp(At)$, where the constant $A > 0$ is independent of $\widehat{\Omega}$, λ_1, λ_2; $C_0 = \mathrm{const} > 0$;

3) $C_1 x_1 \leqslant u(x) \leqslant C_2 x_1$ as $x_1 \geqslant x_1^{(0)} = \mathrm{const} > 0$, $C_1, C_2 = \mathrm{const}, C_1 C_2 > 0$.

Proof. According to Lemma 6, there exists $A > 0$ such that each solution to (1)-(2) not obeying 2) satisfies the inequality $|u(x)| \leqslant c_0 x_1$ as $x_1 \geqslant 1$, $c_0 = \mathrm{const}$. It follows from (5) that for such solution there exists the finite limit $\lim\limits_{t\to\infty} P(t,u)$. Then for the solution $w \equiv u - p_1 U$ to (1)-(2), where U is a linearly growing solution (1)-(2) in Theorem 3, $p_1 = \mathrm{const}$, we obtain $\lim\limits_{t\to\infty} P(t,w) = 0$. According to Lemma 7, the function w is bounded in Ω. Thus, we obtain that $u = w + p_1 U$ satisfies either Condition 1) as $p_1 = 0$ or Condition 3) as $p_1 \neq 0$. The proof is complete. □

In conclusion let us show that in the case of a fast decaying lower order term, the limiting constant C of the bounded solution can be written explicitly in terms of the values of the solutions on the base S_0 of the cylinder.

Theorem 5. *Suppose that the function $q(x)$ satisfies the assumptions of Theorem 3. Then the limiting constant C of the bounded in Ω solution to* (1)-(2) *$u(x)$ satisfies the representation*

$$C = \lim_{h\to 0} h^{-1} \int_{\Omega(0,h)} u \sum_{i=1}^n a_{i1} \frac{\partial U}{\partial x_i}\,dx,$$

where $U(x)$ is the linearly growing solution to (1)-(2) *in Theorem 3 satisfying the condition $P(t,U) \to p_0 = 1$, $t \to \infty$.*

Proof. Let $\Phi_{h,N} = \Phi_{h,N}(x_1)$ be a continuous function, $\Phi_{h,N}(x_1) = 1$ as $h \leqslant x_1 \leqslant N$, $\Phi_{h,N}(0) = \Phi_{h,N}(N+1) = 0$, $\Phi_{h,N}$ is linear as $0 \leqslant x_1 \leqslant h$ and $N \leqslant x_1 \leqslant N+1$. Letting $v = u\Phi_{h,N}$ in integral identity (3) for $U(x)$, we obtain

$$\int_{\Omega(0,N+1)} \sum_{i,j=1}^n a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial u}{\partial x_j} \Phi_{h,N}\,dx = -\int_{\Omega(0,N+1)} quU\Phi_{h,N}\,dx$$
$$+ \int_{\Omega_N} u \sum_{i=1}^n a_{i1} \frac{\partial U}{\partial x_i}\,dx - h^{-1} \int_{\Omega(0,h)} u \sum_{i=1}^n a_{i1} \frac{\partial U}{\partial x_i}\,dx.$$

Let $\Phi_N(x_1) = 1$ as $0 \leqslant x_1 \leqslant N$, $\Phi_N(x_1) = N + 1 - x_1$ as $N \leqslant x_1 \leqslant N + 1$. Choosing the test function $v = U\Phi_N$ in integral identity (3) for u, we obtain

$$\int_{\Omega(0,N+1)} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial U}{\partial x_j} \Phi_N \, dx = -\int_{\Omega(0,N+1)} quU\Phi_N \, dx + \int_{\Omega_N} U \sum_{i=1}^{n} a_{i1} \frac{\partial u}{\partial x_i} \, dx.$$

By the last two identities, in view of symmetricity of the matrix a_{ij}, we get

$$\int_{\Omega_N} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx = \int_{\Omega_N} U \sum_{i=1}^{n} a_{i1} \frac{\partial u}{\partial x_i} \, dx + h^{-1} \int_{\Omega(0,h)} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx$$
$$+ \int_{\Omega(0,h)} \Big(\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial U}{\partial x_j} + quU \Big) (\Phi_{h,N} - 1) \, dx.$$

Making h to tend to zero, we obtain

$$\int_{\Omega_N} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx = \int_{\Omega_N} U \sum_{i=1}^{n} a_{i1} \frac{\partial u}{\partial x_i} \, dx + \lim_{h\to 0} h^{-1} \int_{\Omega(0,h)} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx.$$

Hence,

$$\begin{aligned} \overline{u}(N) \int_N^{N+1} P(t, U) \, dt =& \overline{U}(N) \int_N^{N+1} P(t, u) dt + \lim_{h\to 0} h^{-1} \int_{\Omega(0,h)} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx \\ &+ \int_{\Omega_N} \sum_{i=1}^{n} a_{i1} \Big((U - \overline{U}(N)) \frac{\partial u}{\partial x_i} - (u - \overline{u}(N)) \frac{\partial U}{\partial x_i} \Big) \, dx. \end{aligned} \tag{19}$$

The left hand side in (19) tends to C as $N \to \infty$. Since for the bounded solution $u(x)$ we have $\int_\Omega |\nabla u|^2 \, dx < \infty$, then by (5) we obtain that $P(t, u) \to 0$, $t \to \infty$ and $P(t, u) = -\int_{\Omega(t,\infty)} qu \, dx$. Then

$$|P(t, u)| \leqslant c_0 \int_{\Omega(t,\infty)} q \, dx \leqslant c_0 t^{-1} \int_{\Omega(t,\infty)} x_1 q \, dx = o(t^{-1}), \quad t \to \infty.$$

Hereinafter $c_i = \text{const} > 0$. Then the first term in the right hand side in (19) tends to zero as $N \to \infty$.

Since $\int_{\Omega(0,N)} |\nabla U|^2 \, dx \leqslant c_1 N$, there exists a sequence $N_k \to \infty$, $k \to \infty$, for which $\int_{\Omega_{N_k}} |\nabla U|^2 \, dx \leqslant c_2$. Applying Cauchy-Schwarz and Poincaré inequalities, in view of Lemma 3 we obtain

$$\Big| \int_{\Omega_{N_k}} \sum_{i=1}^{n} a_{i1} \Big((U - \overline{U}(N_k)) \frac{\partial u}{\partial x_i} - (u - \overline{u}(N_k)) \frac{\partial U}{\partial x_i} \Big) \, dx \Big|$$
$$\leqslant c_3 \Big(\int_{\Omega_{N_k}} |\nabla u|^2 \, dx \Big)^{1/2} \Big(\int_{\Omega_{N_k}} |\nabla U|^2 \, dx \Big)^{1/2} \to 0, \quad k \to \infty.$$

Thus, by (19) we obtain the statement of the theorem. □

We observe that the obtained expression for the limiting constant C depends only on the values of the function $u(x)$ on S_0. Indeed, for the functions u_1 and u_2 such that $(u_1 - u_2)|_{S_0} = 0$ we have

$$h^{-1} \Big| \int_{\Omega(0,h)} (u_1 - u_2) \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, dx \Big|$$
$$\leqslant c \Big(\int_{\Omega(0,h)} |\nabla (u_1 - u_2)|^2 \, dx \Big)^{1/2} \Big(\int_{\Omega(0,h)} |\nabla U|^2 \, dx \Big)^{1/2} \to 0, \qquad h \to 0.$$

It is obvious that for the classical solution the limiting constant C is expressed explicitly in terms of the integral over S_0:

$$C = \int_{S_0} u \sum_{i=1}^{n} a_{i1} \frac{\partial U}{\partial x_i} \, d\widehat{x}.$$

In the simplest case of the Laplace operator $L = \Delta$ we have $U = m_0^{-1} x_1$, $C = m_0^{-1} \int_{S_0} u \, d\widehat{x}$. It is obviously implied by the identity $\int_{S_t} \frac{\partial u}{\partial x_1} \, d\widehat{x} = \text{const}$ and for the bounded solutio this constant vanishes.

BIBLIOGRAPHY

1. E.M. Landis, G.P. Panasenko. *A variant of a Phragmén-Lindelöf theorem for elliptic equations with coefficients that are periodic functions of all variables except one* // Trudy Semin. im I.G. Petrovskogo, **5**, 105-136 (1979). [Topics Modern Math. Petrovskii Semin. **5**, 133-172 (1985).]
2. O.A. Oleinik, G.A. Iosif'yan. *On the behavior at infinity of solutions of second order elliptic equations in domains with noncompact boundary* // Matem. Sborn. **112**:4, 588-610 (1980). [Math. USSR. Sb. **40**:4, 527-548 (1981).]
3. O.A. Oleinik, G.A. Yosifian. *On the asymptotic behavior at infinity of solutions in linear elasticity* // Arch. Rat. Mech. Anal. **78**:1, 29-53 (1982).
4. A.V. Neklyudov. *On Neumann problem for high order divergent elliptic equations in unbounded domains close to a cylinder* // Trudy Semin. im I.G. Petrovskogo. **16**, 191-217 (1991). (in Russian).
5. A.L. Piatnitski. *On the behaviour at infinity of the solution of a second-order elliptic equation given on a cylinder* // Uspekhi Matem. Nauk. **37**:2, 231-232 (1982). [Russ. Math. Surv. **37**:2, 249-250 (1982).]
6. A.V. Neklyudov. *Solutions of nondivergent second-order elliptic equations in a nonbounded domain* // Vestn. Mosk. Univ. Ser. I. 1, 93-95 (1989). [Mosc. Univ. Math. Bull. **44**:1, 80-83 (1989).]
7. V. A. Kondrat'ev. *Positive solutions to weakly nonlinear elliptic equations of second order on cylindrical domains* // Trudy Mat. Inst. Steklova. **250**, 183-191 (2005). [Proc. Steklov Inst. Math. **250**, 169-177 (2005).]
8. O.A. Ladyzhenskaya and N.N. Ural'tseva. *Linear and quasilinear elliptic equations.* Nauka, Moscow (1973). [Academic Press, New York (1968).]
9. S.S. Lakhturov. *The asymptotic behaviour of solutions of the second boundary-value problem in unbounded domains* // Uspekhi Matem. Nauk. **35**:4, 195-196 (1980). [Russ. Math. Surveys. **35**:4, 175-176 (1980).]
10. A.V. Neklyudov. *The behavior of solutions of semilinear elliptic equations of second order of the form $Lu = e^u$ in the infinite cylinder* // Matem. Zametki. **85**:3, 408-420 (2009). [Math. Notes. **85**:3-4, 397-408 (2009).]
11. R. Bellman. *Stability theory of differential equations.* McGraw-Hill Book Company, New York (1953).

Alexey Vladimirovich Neklyudov,
Bauman Moscow State Technical University,
Rubstovskaya quay 2/18,
Moscow, 105005, Russia
E-mail: nekl5@yandex.ru