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ON SIMULTANEOUS SOLUTION OF THE KDV EQUATION AND A FIFTH-ORDER DIFFERENTIAL EQUATION

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Abstract. In the paper we consider an universal solution to the KdV equation. This solution also satisfies a fifth order ordinary differential equation. We pose the problem on studying the behavior of this solution as $t \to \infty$. For large time, the asymptotic solution has different structure depending on the slow variable $s = x^2/t$. We construct the asymptotic solution in the domains $s < -3/4$, $-3/4 < s < 5/24$ and in the vicinity of the point $s = -3/4$. It is shown that a slow modulation of solution's parameters in the vicinity of the point $s = -3/4$ is described by a solution to Painlevé IV equation.

Keywords: asymptotics, matching of asymptotic expansions, Korteweg-de Vries equation, non-dissipative shock waves.

Mathematics Subject Classification: 35Q53, 35N10

1. Introduction

In works by A.M. Il'in and S.V. Zakharov [1–3] there was initiated the study on influence of a small dissipation on the processes of transforming weak discontinuities into the strong ones. It was shown in these works that in the leading term, this process is described by a special solution to the Burgers equation. It was shown in work [4] that in the problems with a small dispersion, a similar role is played by two special solutions to the Korteweg-de Vries equation (KdV)

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

In the present work we study one of these solutions with the prescribed asymptotics:

$$u\big|_{x\to\infty} = 0, \qquad u\big|_{x\to-\infty} = (t + \sqrt{t^2 - 4x})/2. \tag{1.2}$$

The solutions $u(x,t)$ plays an universal role [4] in problem on appearance of non-dissipative shock waves [4,5]. In work [4], for the solution to problem (1.1), (1.2), the asymptotic solutions was constructed in some directions as $x^2 + t^2 \to \infty$; in the domain of non-damped oscillations this asymptotic solution was determined by quasi-simple solutions to the Whitham equations. In the present work the asymptotics to this solution as $t \to \infty$ is studied in more details. Namely, we propose an ansätz for the zone, in which fast oscillations arise, we determine the equation for the phase shift in the zone of the Whitham oscillations, we construct the asymptotics for the solution before the zone of these oscillations. We show that Painlevé IV equation determines the leading term of the asymptotics in the vicinity of the zone of Whitham oscillations appearance.

It was shown in [4] that the solution $u(x,t)$ satisfies the fifth order ordinary differential equation in the variable x:

$$\left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18}\right)'_x + \frac{2u + xu_x - 3t(u_{xxx} + uu_x)}{6} = 0. \tag{1.3}$$

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Equation (1.3) is a combination of stationary parts of the symmetries for KdV equation. One of them is the highest (generalized) fifth order symmetry:

$$u_{\tau_5} = \left(u_{xxxx} + \frac{5u_{xx}u}{3} + \frac{5u_x^2}{6} + \frac{5u^3}{18} \right)'_x, \tag{1.4}$$

the second one is the classical dilation symmetry:

$$u_{\tau_r} = 2u + xu_x - 3t(u_{xxx} + uu_x). \tag{1.5}$$

Equation (1.3) can be called as the first highest analogue of Painlevé I equation, see [4, Eq. (6.2)].

The asymptotic solution to problem (1.1, 1.2, 1.3) as $t \to \infty$ has various structure depending on the direction [4]. These directions are determined by the values of the variable

$$s = \frac{x}{t^2}. \tag{1.6}$$

The Whitham oscillations zone associated with $-\frac{3}{4} < s < \frac{5}{24}$, some neighbourhood of the point $s = -\frac{3}{4}$ corresponds to the zone of the appearance of Whitham oscillations, $s < -\frac{3}{4}$ is the zone before the Whitham oscillations.

The work is devoted to studying the asymptotic solutions in these domains and to matching these asymptotics. It should be mentioned that to solve this problem, together with usual averaging methods [6], we employ the condition that the sought solution satisfy simultaneously two equations. This condition allows us to obtain the equation in a slow variable, see [7], [8].

2. Asymptotics as $s < -3/4$

We make the change of variables

$$u = t\,U(t, s), \quad s = \frac{x}{t^2}.$$

Under such change, equations (1.1,1.3) cast into the form:

$$t^{-5}U_{sss} + tU_t - 2sU_s + UU_s + U = 0, \tag{2.1}$$

$$t^{-10}U_{sssss} + \frac{1}{6}t^{-5}(20U_sU_{ss} + (10U - 3)U_{sss}) + \frac{1}{6}(5U^2 + s - 3U)U_s + \frac{1}{3}U = 0. \tag{2.2}$$

In equation (2.2), all the derivatives of the third and higher order in the variable x can be replaced by the equation (2.1):

$$\begin{gathered} \frac{1}{3}t^{-5}(U_s + 9)U_{ss} - t^{-4}U_{sst} + \frac{1}{6}(U^2 - 4sU + 24s^2 - 5s)U_s - \\ -\frac{1}{6}(4U - 3 + 12s)tU_t - \frac{1}{6}(4U + 12s - 5)U = 0. \end{gathered} \tag{2.3}$$

The leading term of the asymptotics depends on the slow variable only s:

$$U = V_0(s) + \cdots, \quad t \to \infty, \quad s < -\frac{3}{4}.$$

Substituting this formula into equations (2.1) and(2.3) lead us to two equations for $V_0(s)$:

$$\frac{1}{6}(V_0^2 - 4sV0 - 5s + 24s^2)V_0' - \frac{1}{6}V_0(4V_0 - 5 + 12s) = 0, \quad (V_0 - 2s)V_0' + V_0 = 0. \tag{2.4}$$

This system implies the algebraic equation:

$$V_0^2 - V_0 + s = 0,$$

whose solutions satisfy system (2.4). It follows from (1.2) that we should choose on the roots to this equation:

$$V_0 = (1 + \sqrt{1 - 4s})/2. \tag{2.5}$$

It follows from the results of paper [4] that fast oscillations arise in the terms of the asymptotics, and this is the asymptotics for the solution is constructed as a partial sum of the series:

$$U = V_0+V_1(p,V_0)t^{-5/2}+V_2(p,V_0)t^{-5}+V_3(p,V_0)t^{-15/2}+V_4(p,V_0)t^{-10}+V_5(p,V_0)t^{-25/2}+\dots \quad (2.6)$$

For the fast variable p we construct its series:

$$p = t^{5/2}p_{-1}(V_0)+p_{\ln}\ln t+p_0(V_0)+p_1(V_0)t^{-5/2}+p_2(V_0)t^{-5}+p_3(V_0)t^{-15/2}+p_4(V_0)t^{-10}+\dots \quad (2.7)$$

The coefficients $V_k(p,s)$ we impose the condition of 2π-periodicity w.r.t. the variable p. In the coefficients of the asymptotic expansion, the slow variable s is replaced by the dependence on V_0 for simplicity of calculations.

Substituting series (2.6) and (2.7) into (2.1) and (2.3), for V_1 we obtain two equations:

$$-\frac{(p'_{-1})^3}{(2V_0-1)^3}\frac{\partial^3 V_1}{\partial p^3}+\frac{1}{2}(5p_{-1}-2p'_{-1}V_0)\frac{\partial V_1}{\partial p}=0,$$
$$-\frac{5}{2}\frac{p_{-1}(p'_{-1})^2}{(2V0-1)^2}\frac{\partial^3 V_1}{\partial p^3}+\frac{1}{12}(p_{-1}(12V_0^2-16V_0+3)-2V_0(2V_0-1)(6V_0-5)p'_{-1})\frac{\partial V_1}{\partial p}=0.$$

Excluding $\frac{\partial^3 V_1}{\partial p^3}$ from this system, we obtain the relation:

$$\Big(2V_0(6V_0-5)p'_{-1}+15(2V_0-1)p_{-1}\Big)\Big((2V_0-1)p'_{-1}-5p_{-1}\Big)\frac{\partial V_1}{\partial p}=0.$$

Since V_1 should depend on p, by the last relation we can find p_{-1}. The bounded terms of asymptotics exist under the only choice of p_{-1}, namely, as

$$p_{-1}(V_0)=-\frac{2\sqrt{2}}{15\sqrt{3}}V_0^{3/2}(6V_0-5). \quad (2.8)$$

Here the constatns of the integrations are determined by condition of 2π-periodicity of V_1. The equation for V_1 becomes:

$$\frac{\partial^3 V_1}{\partial p^3}+\frac{\partial V_1}{\partial p}=0;$$

we write its solution as

$$V_1 = D_1(V_0)+A_1(V_0)\cos p, \quad (2.9)$$

where $D_1(V_0)$, $A_1(V_0)$ are functions in slow variable to be determined by the existence and boundedness conditions for the next terms of the asymptotics. Hereinafter, the third integration constant corresponds to the phase shift $p_j(V_0)$.

For the function V_2 we obtain two inhomogeneous equations in the variable p:

$$\begin{aligned}\frac{\partial^3 V_2}{\partial p^3}+\frac{\partial V_2}{\partial p}&=F_{21}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0),\\ \frac{\partial^3 V_2}{\partial p^3}+\frac{\partial V_2}{\partial p}&=F_{22}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0).\end{aligned} \quad (2.10)$$

The existence and boundedness condition for the solutions is that the right hand sides should coincide and they should be orthogonal to the solutions of the homogeneous equation, that is,

$$\begin{aligned}&F_{21}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0)=F_{22}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0),\\ &\int_0^{2\pi}F_{21}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0)dp=0,\\ &\int_0^{2\pi}F_{21}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0)\cos p\,dp=0,\\ &\int_0^{2\pi}F_{21}(p,V_0,A_1,D_1,A'_1,D'_1,p'_0)\sin p\,dp=0.\end{aligned} \quad (2.11)$$

The solutions to system (2.11) are of the form:

$$\begin{aligned} A_1 &= \frac{C_1}{\sqrt{V_0}(2V_0-3)},\\ D_1 &= -\frac{2p_{ln}\sqrt{6}}{5\sqrt{V_0}(2V_0-1)},\\ p_0 &= \frac{p_{ln}}{5}\ln((2V_0-3)^2V_0^3)+p^0, \end{aligned} \tag{2.12}$$

where C_1, p^0 are arbitrary constants. The equation for V_2 is of the form:

$$\frac{\partial^3 V_2}{\partial p^3}+\frac{\partial V_2}{\partial p}=\frac{3C_1^2}{4(2V_0-3)^2V_0^2}\sin 2p, \tag{2.13}$$

and its solution is

$$V_2=D_2(V_0)+A_2(V_0)\cos p+\frac{C_1^2}{8(2V_0-3)^2V_0^2}\cos 2p. \tag{2.14}$$

For the next terms in the asymptotics we obtain systems of form (2.10) with solvability condition of form (2.11). We failed to prove the solvability of these systems for all the terms. By straightforward calculations we checked that up to V_5 all terms are constructed uniquely and no new constants arise. We provide explcit formulae for D_2, A_2, p_1:

$$\begin{aligned} D_2 &= -\frac{C_1^2}{24V_0^2(2V_0-1)(2V_0-3)}-\frac{12(4V_0-1)p_{\ln}^2}{25V_0^2(2V_0-1)^3}+\frac{4}{(2V_0-1)^4},\\ A_2 &= -\frac{3\sqrt{6}(14V_0+15)C_1p_{\ln}}{20(2V_0-3)^3V_0^2}\\ p_1 &= -\frac{\sqrt{6}}{V_0^{3/2}(2V_0-3)^2}\left(\frac{(4V_0^2-96V_0+63)p_{\ln}^2}{50(2V_0-1)}-\frac{(2V_0+9)C_1^2}{288}\right.\\ &\qquad\left.+\frac{(2V_0+1)(212V_0^2-204V_0+45)}{24(2V_0-1)^2}\right). \end{aligned} \tag{2.15}$$

These formulae determine V_2 completely.

The coefficients $p_{\ln}$, p^0, C_1 in series (2.6) and (2.7) are still arbitrary. They can be determined by comparing the series with formula (5.3) and the next formula in [4]. We find:

$$p_{\ln}=\frac{5\ln 2}{2\pi},\quad C_1=-\frac{3\sqrt{6}\ln 2}{\pi},\quad p_0=\frac{\ln 2\ln 24}{2\pi}-\frac{\pi}{2}+2\arg\Gamma\left(\frac{i\ln 2}{2\pi}\right). \tag{2.16}$$

Employing expressions (2.9), (2.12), (2.14), (2.15) for V_1 and V_2, we find the applicability domain for asymptotic expansion by the condition $V_1\gg t^{-5/2}V_2$. We find $|V_0-3/2|\gg t^{-5/4}$; in terms of the variable s the domain is of the form $|s+3/4|\gg t^{-5/4}$. This is why slow variable for the internal variable is of the form:

$$y=(s+3/4)t^{5/4}. \tag{2.17}$$

Replacing the variable s by formula (2.17) in series (2.6) and (2.7), we obtain:

$$\begin{aligned} U &\approx \frac{3}{2}+\left(-\frac{y}{2}+\left(\frac{6\ln 2}{\pi y}+\frac{108\ln^2 2}{\pi^2y^3}-\frac{432\ln 2(5\pi^2-9\ln^2 2)}{\pi^3y^5}+\dots\right)\cos p\right)t^{-5/4}+\dots\\ p &\approx -\frac{4}{5}t^{5/2}+yt^{5/4}-\frac{5\ln 2\ln t}{4\pi}-\frac{y^2}{12}-\frac{\ln 2\ln(-y)}{\pi}-\frac{3\ln 2\ln(3/2)}{2\pi}+p^0+\dots \end{aligned} \tag{2.18}$$

These formulae determine the asymptotics of the coefficients in the internal expansion as $y\to-\infty$.

3. Asymptotic solution in the vicinity of the point $s = -3/4$

To construct the asymptotic solution in the vicinity of the point $s = -3/4$, we make the change:

$$U = \frac{3}{2} + t^{-5/4}W(y,t), \quad y = (s + 3/4)t^{5/4}. \tag{3.1}$$

Equations (2.1) and (2.3) become

$$\begin{aligned} &t^{-9/4}W_{yyy} + t^{-1}(WW_y - \frac{3}{4}yW_y - \frac{1}{4}W) + t^{1/4}(3W_y + \frac{3}{2}) + W_t = 0 \\ &t^{-5/2}W_{tyy} - t^{-7/2}\frac{W_{yy}}{12}(4W_y - 21) + t^{-1}\frac{1}{24}(W_y(68y - 12W) + 22W + 27y) - \\ &\qquad - t^{-9/4}\frac{1}{48}(W_y(8W^2 - 12yW - 27y^2) + W(8W + 9y)) + \\ &\qquad + t^{-5/4}\frac{W_y}{12}(8W + 9y) - W_t + t^{1/4}2(2W_y + 1) = 0. \end{aligned} \tag{3.2}$$

An asymptotic solution W is constructed as:

$$W = W_0(\psi, y) + t^{-5/4}W_1(\psi, y) + t^{-5/2}W_2(\psi, y) + t^{-15/4}W_3(\psi, y) + \dots, \quad t \to \infty, \tag{3.3}$$

with the fast variable

$$\psi = -\frac{4}{5}t^{5/2} + yt^{5/4} - \frac{5\ln 2}{4\pi}\ln t + \psi_0(y) + t^{-5/4}\psi_1(y) + t^{-5/2}\psi_2(y) + \dots, \ t \to \infty. \tag{3.4}$$

By substituting (3.3) and (3.4) into (3.2) we obtain the equations for the coefficients of series (3.3). The equations for the leading term W_0 coincide and are of the form:

$$\partial_\psi^3 W_0 + \partial_\psi W_0 = 0. \tag{3.5}$$

We write its soltion as

$$W_0 = H_0(y) + R_0(y)\cos\psi, \tag{3.6}$$

where $H_0(y)$, $R_0(y)$ are unknown function of slow variable, the term with $\sin\psi$ is taken in the phase shift $\psi_0(y)$. The equation for W_1 is

$$\begin{aligned} \partial_\psi^3 W_1 + \partial_\psi W_1 &= \frac{R_0}{2}(y + 2H_0)\sin\psi - \frac{3}{2}(1 + 2H_0') + \frac{R_0^2}{2}\sin 2\psi, \\ \partial_\psi^3 W_1 + \partial_\psi W_1 &= \frac{5R_0}{6}(y + 2H_0)\sin\psi + \frac{1}{2}(1 + 2H_0') + \frac{R_0^2}{2}\sin 2\psi. \end{aligned} \tag{3.7}$$

The existence and boundedness conditions for the solutions of this system gives:

$$H_0(y) = -\frac{y}{2}.$$

Under such choice of H_0, the solution to system (3.7) exists and bounded as $\psi \to \infty$. Function W_1 is of the form:

$$W_1 = H_1(y) + R_1(y)\cos 2\psi + \frac{R_0^2}{12}\cos 2\psi. \tag{3.8}$$

For the terms of the asymptotics we obtain equations of form (2.10), the solvability and boundedness conditions for the solution are of form (2.11). By the existence and boundedness condition for W_2 we obtain:

$$H_1 = -\frac{R_0^2}{12} - \frac{y^2}{8} - \frac{1}{\pi}, \tag{3.9}$$

$$R_0'' = R_0(\psi_0')^2 + \frac{5}{12} y R_0 \psi_0' + \frac{1}{72} R_0^3 + \frac{y^2}{24} R_0^2 - \frac{1}{12\pi} R_0, \tag{3.10a}$$

$$R_0 \psi_0'' = -2R_0'\psi_0' - \frac{5}{12} y R_0' - \frac{1}{4} R_0. \tag{3.10b}$$

The solution $W_2(p, y)$ becomes

$$W_2 = R_2 \cos\psi + H_2(y) + \frac{R_0^3}{192} \cos 2\psi - \frac{R_0^2}{6} \psi_0' \cos 2\psi + \frac{R_0}{6} R_1 \cos 2\psi - \frac{R_0}{6} R_0' \sin 2\psi. \tag{3.11}$$

Then we checked that all the terms up to V_5 are in the class of bounded periodic in p functions. The functions R_1, R_2, R_3, ψ_1, ψ_2, ψ_3, H_3, H_4 are determined uniquely, with no additional arbitrary coefficients.

Let us show that system (3.10) is equivalent to Painlevé IV equation. We observe that this system possesses the first integral quadratic in the derivatives:

$$I = (R_0')^2 + R_0^2(\psi_0')^2 + \frac{y}{2} R_0^2 \psi_0' + \frac{3\pi y^2 + 4}{48\pi} R_0^2 - \frac{1}{144} R_0^4. \tag{3.12}$$

System (3.10a) and (3.12) can be considered as a one second differential equation for $R_0(y)$ with the parameter ψ_0'. We make a change in this system:

$$\psi_0' = iR_0'/R_0 + P(y) - y/4, \tag{3.13}$$

where i is the imaginary unit. Excluding the function $R_0(y)$ from (3.10a) and (3.12), for a new unknown function $P(y)$ we obtain the Painlevé IV equation:

$$2P(y)P''(y) = (P'(y))^2 - 3P^4(y) + \frac{1}{3} y P^3(y) + \left(-\frac{y^2}{144} + \frac{1}{6}(i - \ln 2/\pi)\right) P^2(y). \tag{3.14}$$

The function $Q(z) = (2 + 2i)\sqrt{3}P((-2 + 2i)\sqrt{3}z)$ satisfies the usual Painlevé IV equation. In (3.14) we have substituted the value I found by means of formulae (2.18), which give the asymptotics of the functions ψ_0, R_0 as $y \to -\infty$, and therefore, they allow to find the value I and the asymptotics $P(y)$ as $y \to -\infty$:

$$I = \frac{\ln^2 2}{4\pi^2},$$

$$P(y) = \frac{y}{12} + \frac{1}{y}(i - \ln 2/\pi) + \frac{6}{y^3}\left(\frac{6i \ln 2}{\pi} - \frac{3\ln^2 2}{\pi^2} + 4\right) + \dots. \tag{3.15}$$

By straightforward calculations we can check that equation (3.14) has a solution with asymptotics (3.15). However, at present, we do not know the asymptotics of this solution as $y \to \infty$; such problem can be solved by the approaches from work [9].

4. Asymptotic solution in the zone of Whitham oscillations

In the zone of Whitham oscillations, the asymptotic solution U to system (2.1), (2.3) is constructed as a series in inverse powers of t:

$$U = U_0(\varphi, s) + t^{-5/4} U_1(\varphi, s) + t^{-5/2} U_2(\varphi, s) + \dots, \quad t \to \infty. \tag{4.1}$$

Here U_0, U_1 and U_2 are 2π-periodic function in the fast variable φ. This variable is of the form

$$\varphi = t^{5/2} f(s) + n(s),$$

where $f(s)$, $n(s)$ are unknown functions.

For the function U_0 we obtain the following nonlinear system of equation in the fast variable φ:

$$\begin{aligned} &(f')^2\frac{\partial^3 U_0}{\partial\varphi^3} + (U_0 - a(s))\,\frac{\partial U_0}{\partial\varphi} = 0,\\ &a(s)(f')^2\frac{\partial^3 U_0}{\partial\varphi^3} - \frac{1}{3}(f')^2\frac{\partial U_0}{\partial\varphi}\frac{\partial^2 U_0}{\partial\varphi^2} + \frac{1}{6}\left(U_0^2 + s + 4a(s)U_0 - 3a(s)\right)\frac{\partial U_0}{\partial\varphi} = 0. \end{aligned} \tag{4.2}$$

Here we denote

$$a(s) = 2s - \frac{5f}{2f'}.$$

Excluding the expression $\partial^3_\varphi U_0$ from (4.2), we obtain the second order equation for the function U_0:

$$(f')^2\frac{\partial^2 U_0}{\partial\varphi^2} + \frac{1}{2}U_0^2 - a(s)U_0 + 3a(s)^2 + \frac{s - 3a(s)}{2} = 0. \tag{4.3}$$

Equation (4.3) can be integrated once:

$$\left(f'\frac{\partial U_0}{\partial\varphi}\right)^2 + \frac{1}{3}U_0^3 - a(s)U_0^2 + (6a^2 - 3a + s)U_0 + b(s) = 0. \tag{4.4}$$

Here $b(s)$ is an arbitrary function arising an integration constant.

We shall an explicit formula for U_0 later, while now we assume that this is some 2π-periodic function satisfying equation (4.4). By this equation all the derivatives of $\partial_\varphi U_0$ can be written as fractional-rational expression in terms of

$$U_0,\ \partial_\varphi U_0,\ \partial_s U_0,\ \partial_s^2 U_0, \ldots.$$

Employing this conditions, we can find the boundedness condition for the next terms of the asymptotics, which are the equations for slowly varying functions $f(s)$, $n(s)$, $b(s)$.

The equations for U_1 have the form:

$$\begin{aligned} &(f')^2\partial^3_\varphi U_1 + (U_0 - a)\,\partial_\varphi U_1 + \partial_\varphi U_0 U_1 = \frac{F_1(U_0, \partial_\varphi U_0, \partial_s U_0, a, a', n', s)}{f},\\ &a(s)(f')^2\partial^3_\varphi U_1 - \frac{1}{3}(f')^2(\partial^2_\varphi U_1\partial_\varphi U_0 + \partial^2_\varphi U_0\partial_\varphi U_1) + \frac{1}{6}\partial_\varphi U_1(U_0^2 + s + 4aU_0 - 3a)\\ &\qquad + \frac{1}{3}\partial_\varphi U_0(U_0 + 2a)U_1 = \frac{F_2(U_0, \partial_\varphi U_0, \partial_s U_0, a, b, a', b', n', s)}{f\partial_\varphi U_0}. \end{aligned} \tag{4.5}$$

Here F_1, F_2 are polynomial functions of their arguments. Excluding the higher derivatives of U_1 w.r.t. the variable φ from system (4.5), we arrive at a relation not involving function U_1, which is a compatibility condition for this system:

$$\begin{aligned} &\Big(f(360a - 30s - 45 - 540a^2)a' - 10fb'\\ &+ 2f'(108a^3 - 108a^2 + 6as + 6b + 27a - 2s)\Big)U_0\\ &+ 15f(54a^2 - 72a^3 - 12as + 4b - 9a + 3s)a' + 15f(4a - 1)b'\\ &+ 6f'(72a^4 - 66a^3 + 12a^2 s - 16ab + 15a^2 + 5as - 5b) = 0. \end{aligned} \tag{4.6}$$

Since equation (4.6) should be satisfied identically, the coefficients at various powers of U_0 should vanish. Therefore, we obtain the closed system of equations for $a(s)$, $b(s)$:

$$\begin{aligned} &a' = \frac{(2a - 1)(288a^3 - 192a^2 + 24sa + 27a - 4s - 4b)}{(a - 2s)(-576a^3 + 504a^2 - 126a - 48sa + 8b + 12s + 9)},\\ &b' = (36a - 3s - 54a^2 - 9/2)a' - \frac{108a^3 - 108a^2 + 6as + 6b + 27a - 4s}{2a - 4s}. \end{aligned} \tag{4.7}$$

System (4.5) is compatible if and only if $a(s)$ and $b(s)$ are determined by equations (4.7). If this condition is satisfied, all the derivatives of U_1 w.r.t. φ of order higher than two can be expressed via lower derivatives, for instance,

$$(f')^2\partial_\varphi^2 U_1 = (a-U_0)U_1 + (n' + \partial_s U_0/\partial_\varphi U_0)G_1(U_0,a,s)/s + G_2(U_0,a,b,s)/f/\partial_\varphi U_0,$$

where G_1, G_2 are some functions.

The equation for U_2 are of the form:

$$\begin{aligned}&(f')^2\partial_\varphi^3 U_1 + (U_0-a)\,\partial_\varphi U_1 + \partial_\varphi U_0 U_1 = \frac{F_3}{f},\\ &a(s)(f')^2\partial_\varphi^3 U_1 - \frac{1}{3}(f')^2(\partial_\varphi^2 U_1\partial_\varphi U_0 + \partial_\varphi^2 U_0\partial_\varphi U_1) + \frac{1}{6}\partial_\varphi U_1(U_0^2 + s + 4aU_0 - 3a)\\ &\qquad + \frac{1}{3}\partial_\varphi U_0(U_0+2a)U_1 = \frac{F_4}{f\partial_\varphi U_0}.\end{aligned} \tag{4.8}$$

Here F_3, F_4 are functions depending on previous terms.

Excluding the derivatives of the function U_2 from these equations, we obtain the relation:

$$\begin{aligned}&\partial_{\varphi s}U_1 - \frac{\partial_\varphi^2 U_0}{\partial_\varphi U_0}\partial_s U_1 + \left(\frac{\partial_\varphi^2 U_0\partial_s U_0}{(\partial_\varphi U_0)^2} + \frac{G_3(U_0,a,b)}{(f\partial_\varphi U_0)^2(2U_0+3-12a)}\right)\partial_\varphi U_1\\ &\quad - \left(\frac{\partial_\varphi^3 U_0\partial_s U_0}{(\partial_\varphi U_0)^2} - \frac{G_4(U_0,a,b)}{(f\partial_\varphi U_0)^2(2U_0+3-12a)}\right)U_1 = G_5(U_0,a,b,n',n'').\end{aligned} \tag{4.9}$$

Differentiating this equation in φ, we obtain a relation of the same form and excluding $\partial_{\varphi s}U_1$ from these equations, we obtain

$$\begin{aligned}\partial_\varphi U_1 = &\frac{\partial_\varphi^2 U_0}{\partial_\varphi U_0}U_1 + \frac{n''G_6(s,a,b,f) + n'G_7(U_0,s,a,b,f)}{\partial_\varphi U_0}\\ &+ G_8(\partial_s^3 U_0, \partial_s^2 U_0, \partial_s U_0, U_0, a, b, f, s).\end{aligned} \tag{4.10}$$

Substituting (4.10) into equation (4.9), we obtain the relation of the form:

$$\begin{aligned}&\partial_\varphi U_0(n''' + A_1 n'' + A_2 n') + \partial_s^3 U_0 + B_1\partial_s^2 U_0\partial_s U_0 + B_2\partial_s^2 U_0\\ &\quad + B_3(\partial_s U_0)^3 + B_4(\partial_s U_0)^2 + B_5\partial_s U_0 + B_6 = 0,\end{aligned} \tag{4.11}$$

where

$$A_i = A_i(s,f,a,b), \quad B_i = B_i(U_0,s,f,a,b)$$

are some functions.

Without loss of generality we can assume that the function U_0 is even in φ. Then in (4.11), the first part is odd, while the other is even w.r.t. φ. Therefore, by (4.11) we get immediately two equations:

$$n''' + A_1 n'' + A_2 n' = 0, \tag{4.12}$$

$$\partial_s^3 U_0 + B_1\partial_s^2 U_0\partial_s U_0 + B_2\partial_s^2 U_0 + B_3(\partial_s U_0)^3 + B_4(\partial_s U_0)^2 + B_5\partial_s U_0 + B_6 = 0. \tag{4.13}$$

Let us determined the leading term of asymptotic solution (4.1). We seek a solution to equation (4.4) as

$$U_0 = A(s)\mathrm{dn}^2\left(\frac{B(s)}{f'(s)}p; k(s)\right) + C(s), \tag{4.14}$$

where A, B, k, C are functions in slow variables, dn is the Jacobi elliptic function. They are determined by substituting (4.14) into (4.3), equating to zero the coefficients at various powers of dn and postulating the oscillation period to be 2π. By this system we find:

$$\begin{aligned}
A =&6B^2, \qquad f' = \frac{\pi B}{K(k)}, \qquad a = -4k^2B^2 + 8B^2 + C,\\
f =&\frac{2\pi B(4B^2k^2 - 8B^2 - C + 2s)}{5K(k)},\\
b =&4608(2k^2-3)(4k^4-17k^2+19)B^6 - 384(k^2-2)(7k^2-13)(10C-3)B^4\\
&+8(4C(41k^2-79)(5C-3)+12(2k^2-3)s+63(k^2-2))B^2\\
&-\frac{1}{3}(10C-3)(32C(5C-3)+12s+9).
\end{aligned} \tag{4.15}$$

Hereinafter, $K(k)$, $E(k)$ are complete elliptic integrals. Moreover, we have one more algebraic relation:

$$45(2k^4 - 7k^2 + 7)B^4 - 4(k^2-2)(10C-3)B^2 + 5C^2 - 3C + s = 0. \tag{4.16}$$

At the present step, all functions of slow variables are expressed via B, k, C, we have one algebraic equation (4.16) and three differential equation, (4.7) and the implication of the identity $(f)' = f'$, where f, f' were determined independently in (4.15). Differentiating (4.16), we obtain a differential implication. Excluding the derivatives B', k', C' from four differential implications, we obtain an additional algebraic relations in terms of B, k, C, $q = E(k)/K(k)$. It can be also differentiated w.r.t. s and again substitute the found derivatives to obtain an additional algebraic equation.

Employing these relations, the functions $B(s)$, $k(s)$, $C(s)$ are found implicitly:

$$\begin{aligned}
B^2 =&\frac{5(k^2q + k^2 + q - 1)}{12(3k^3q + k^4 + 2k^2q + 2k^2 + 3q - 3)}, \qquad C = 6B^2(k^2-1)\\
s =&\frac{1}{3k^4+2k^2+3}\left(\frac{3(k^2+1)^2}{4} - \frac{1}{3}\left(k^2+1+\frac{5k^2(k^2-1)^2}{3k^4q+k^4+2k^2q+2k^2+3q-3}\right)^2\right).
\end{aligned}$$

The dependence on the slow variable s in the functions A, B, C, k, f, a is determined.

In this paper we do not give an answer to the question on which solution to equation (4.12) corresponds to the studied solution. However, by matching with the expansion in the vicinity of the point $s = -3/4$ we can find the asymptotics of the function $n(s)$ as $s \to -3/4$.

In order to do it, we find the asymptotics of solution (4.1) as $s \to -3/4$:

$$\begin{aligned}
U &= \frac{3}{2} + t^{-5/4}\left(-\frac{y}{2} + \left(\frac{y}{3} + \dots\right)\cos\varphi\right) + \dots, \qquad y = (s+3/4)t^{5/2},\\
\varphi &= -\frac{4}{5}t^{5/2} + t^{5/4}y + \left(-\frac{y^2}{9} + \dots\right) + \dots
\end{aligned}$$

These formulae allow us to find the leading term in the asymptotics of the function $P(y)$ as $y \to \infty$. By formula (3.13) we find:

$$P(y) = \frac{y}{36} + \frac{-i}{y} + \dots, \qquad y \to \infty.$$

By means of equation (3.14) we can find the next terms in the asymptotics for this solution:

$$P = \frac{y}{36} + \frac{\ln 2/\pi - i}{y} + \frac{6(3\ln^2 2/\pi^2 - 4 - 6i\ln 2/\pi)}{y^3} + \dots, \quad y \to \infty.$$

Returning back to the variable s for the function $n(s)$, we find

$$n(s) = \frac{\ln 2}{\pi}\ln(s+3/4) + \left(p_0 - \frac{3\ln 2\ln(3/2)}{2\pi}\right) + \dots, \quad s \to -3/4.$$

We see that in this case the function $n(s)$ is not a constant in contrast to similar problems [8], [7].

5. Conclusion

In the work we study the solution introduced in paper [4]. The main result is the description of the asymptotics of the leading front. We show that the main term is described by Painlevé IV equation. In the work we also found the equation for the phase shift in the Whitham oscillations zone. We show that in this case the function $n(s)$ is not constant in contrast to similar cases in [7, 8].

In future we plan to show that equation (3.14) has a solution with prescribed asymptotics (3.15) and (4) as $y \to \pm\infty$ and to determine function $n(s)$.

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