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ON SOLUTIONS OF CAUCHY PROBLEM FOR EQUATION $u_{xx} + Q(x)u - P(u) = 0$ WITHOUT SINGULARITIES IN A GIVEN INTERVAL

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Abstract. The paper is devoted to Cauchy problem for equation $u_{xx}+Q(x)u-P(u) = 0$, where $Q(x)$ is a π-periodic function. It is known that for a wide class of the nonlinearities $P(u)$ the "most part" of solutions of Cauchy problem for this equation are singular, i.e., they tend to infinity at some finite point of the real axis. Earlier in the case $P(u) = u^3$ this fact allowed us to propose an approach for a complete description of solutions to this equation bounded on $\mathbb{R}$. One of the ingredients in this approach is the studying of the set $\mathcal{U}_L^+$ introduced as the set of the points (u_*, u'_*) in the initial data plane, for which the solutions to the Cauchy problem $u(0) = u_*$, $u_x(0) = u'_*$ are not singular in the segment $[0; L]$. In the present work we prove a series of statements on the set $\mathcal{U}_L^+$ and on their base, we classify all possible type of the geometry of such sets. The presented results of the numerical calculations are in a good agreement with theoretical statements.

Keywords: ODE with periodic coefficients, singular solutions, nonlinear Schrödinger equation.

Mathematics Subject Classification: 34L30, 34C11, 35Q55, 37B10

1. Introduction

The differential equation

$$u_{xx} + Q(x)u - P(u) = 0 \tag{1}$$

arises in numerous problem of various physical nature. In particular, it describes stationary distributions of a field in an inhomogeneous waveguide [1] and structures arising in Bose-Einstein condensate [2]. Therefore, the issue on possible types of solutions to this equation is of high interest for various physical applications.

Several years ago, the following fact was supposed to be used for studying the structures described by equation (1) [3]. Under some restrictions for the nonlinearity $P(u)$, the "most" part of the solutions to the Cauchy problem for equation (1) turns out to be *singular*, that is, tending to infinity at some point of the real line. Therefore, the set of the solutions bounded on whole real line and the most interesting for physical applications turns out to be "poorer". The general impression on this set can be obtained by numerical solving the Cauchy problem for a large domain in the plane of the initial data (u, u_x). In the case, when $Q(x)$ is a periodic function, a more detailed study of this set is possible by means of the Poincaré map [4] generated by equation (1). At that, to describe the set of non-singular solutions in terms of the Poincaré map, on succeeds to apply the methods of symbolic dynamics.

In work [3] this approach was realized for the case, when $P(u) = u^3$ and $Q(x) = \omega - A\cos 2x$, when A and ω are real parameters. It was shown that under three sufficient conditions there

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exists a homeomorphism between the set of all bounded solutions to equation (1) and the set of all bi-infinite sequence formed by the symbols of some finite alphabet. By means of numerics, there were found the domains in the plane of the parameters (ω, A), in which these conditions are satisfied. Later this approach was applied to the case, when the nonlinearity is of the form $R(x)u^3$, where $R(x)$ is also a periodic function [5].

One of main notions employed in realizing this approach, is the set of the initial data for the Cauchy problem, $(u(0), u_x(0))$, such that the corresponding solutions to equation (1) remain bounded in some interval S. In work [3] this set was denoted by $\mathcal{U}_L^+$ if $S = [0; L]$ and $\mathcal{U}_L^-$ if $S = [-L; 0]$. In these terms, the initial condition in the Cauchy problem for the solutions defined on the whole line belong to the set $\mathcal{U}_\infty = \mathcal{U}_\infty^+ \cap \mathcal{U}_\infty^-$. If $Q(x)$ is a π-periodic function, the structure of the set $\mathcal{U}_\infty$ can be described by means of the sets $\mathcal{U}_\pi^+$, $\mathcal{U}_\pi^-$ and the actions of the Poincaré map on them.

Thus, in the case when $Q(x)$ is a π-periodic function, the issue on possible geometry of the sets $\mathcal{U}_\pi^+$ and $\mathcal{U}_\pi^-$ is of high interest. The present work is devoted to studying of possible types of these sets under rather typical assumptions for the nonlinearity $P(u)$.

The work has the following structure. In Section 2 we present the main definitions used in the work and formulate the restrictions for the inhomogeneous term $Q(x)$ and the nonlinear term $P(u)$. In Section 3 we prove some technical statements. One of them is the statement on the continuous dependence of the singularity point of the solution on the initial data in the Cauchy problem. This statement specifies the classical statement on semi-continuity of the end-points of the existence interval for a solution [6, Ch. 5] for a given class of equations. In Section 4 the statements of Section 3 are employed for proving the main theorems of the work. The main result of the paper is Theorem 3 presented in Section 4 banning the existence of the “holes” in the sets $\mathcal{U}_\pi^\pm$ and the existence of bounded connected components for these sets. Section 5 is based on the results of Section 4 and is devoted to the describing of possible forms of the set $\mathcal{U}_\pi^+$. In Section 6 we provide the results of numerical studies of the sets $\mathcal{U}_\pi^+$ for some particular nonlinearities $P(u)$ being of interest for the applications. Finally, in Conclusion 7 we formulate the summary and give some directions of developing the obtained results.

2. Some definitions

Throughout the work we assume that the function $Q(x)$ satisfies the following conditions:

(**Q1**) $Q(x) \in C^1(\mathbb{R})$;

(**Q2**) $Q(-x) = Q(x)$;

(**Q3**) $Q(x + \pi) = Q(x)$.

We shall assume the following restrictions for the function $P(u)$:

(**P1**) $P(u) \in C^1(\mathbb{R})$;

(**P2**) there exist $C, \lambda, A > 0$, such that $P(A) > 0$ and as $u > A$, the inequality

$$P_u(u) > Cu^\lambda \tag{2}$$

holds true.

Remark. *It follows from Condition (P2) that the constant C in (2) can be chosen such that together with (2), as $u > A$, the inequality*

$$P(u) > Cu^{\lambda+1} \tag{3}$$

holds true. In what follows we assume that C, λ, A are such that the inequalities (2) and (3) hold simultaneously.

In what follows, in a series of statements we assume that
(**P3**) The function $-P(-u)$ satisfies Condition (P2).

We introduce definitions.

Definition 1. *Singular solutions. A solution $u(x)$ to equation (1) is said to be singular if there exists a value $x = x_1$ such that $\lim_{x\to x_1} u(x) = \pm\infty$. In this situation we say that $u(x)$ collapses to $+\infty$ at the point $x = x_1$ if $\lim_{x\to x_1} u(x) = +\infty$ or $u(x)$ collapses to $-\infty$ at the point $x = x_1$ if $\lim_{x\to x_1} u(x) = -\infty$.*

Definition 2. *Functions $h^{\pm}(x, u, u')$. We introduce a function $h^+ : \mathbb{R}^3 \to \mathbb{R}$ as follows: $h^+(x_0, u_0, u_0') = x_1 - x_0$ if solution $u(x)$ to Cauchy problem (1) with initial data $u(x_0) = u_0$ and $u_x(x_0) = u_0'$ collapses (no matter to $+\infty$ or to $-\infty$) at the point $x = x_1$, $x_1 > x_0$. At that we assume that $h^+(x_0, u_0, u_0') = +\infty$ if for initial data (u_0, u_0') at the point $x = x_0$, the solution to the Cauchy problem does not collapse at each $x > x_0$. In the same way we introduce the function $h^- : \mathbb{R}^3 \to \mathbb{R}$ letting $h^-(x_0, u_0, u_0') = x_0 - x_1$ if the solution $u(x)$ to Cauchy problem for equation (1) with initial data $u(x_0) = u_0$ and $u_x(x_0) = u_0'$ collapses at the point $x = x_1$, $x_1 < x_0$.*

Example. *Let $Q(x) \equiv 0$ and $P(u) = u^3$. It is obvious that such choice of $Q(x)$ and $P(u)$ satisfies Conditions (Q1)-(Q3) and (P1)-(P2). The solution to Cauchy problem (1) with initial data $u(0) = u_x(0) = \sqrt{2}$ is*

$$u(x) = \frac{\sqrt{2}}{1 - x}.$$

Therefore, $u(x)$ collapses to $+\infty$ at the point $x = 1$ and at that, $h^+(0, \sqrt{2}, \sqrt{2}) = 1$. On the other hand, $u(x)$ is defined on the whole semi-axis $x < 0$ and this is why $h^-(0, \sqrt{2}, \sqrt{2}) = +\infty$.

Let us mention some properties of the functions $h^{\pm}(x, u, u')$.

- By (Q3) we have $h^{\pm} : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{R}$, where $\mathbb{S}^1$ is a circle.
- By (Q2) we have $h^+(0, u, u') = h^-(0, u, -u')$.
- Let $u(x)$ be a solution to equation (1), then for $\alpha \in \mathbb{R}$

$$h^+\big(x_0 + \alpha, u(x_0 + \alpha), u_x(x_0 + \alpha)\big) = h^+\big(x_0, u(x_0), u_x(x_0)\big) - \alpha,$$
$$h^-\big(x_0 + \alpha, u(x_0 + \alpha), u_x(x_0 + \alpha)\big) = h^-\big(x_0, u(x_0), u_x(x_0)\big) + \alpha.$$

In further exposition, a special role is played by the functions $h^{\pm}(x, u, u')$ at $x = 0$ considered as functions on u and u'. We introduce the notation

$$h_0^{\pm}(u, u') \equiv h^{\pm}(0, u, u').$$

Definition 3. *Functions $H^{\pm}(u, u', \Lambda)$. We let $H^+(u_0, u_0', \Lambda) = x_1$ if the solution $u(x)$ to the Cauchy problem for the equation*

$$u_{xx} + \Lambda u - P(u) = 0 \tag{4}$$

with the initial data $u(0) = u_0$, $u_x(0) = u_0'$ collapses at the point $x = x_1$, $x_1 > 0$. The function $H^-(u_0, u_0', \Lambda)$ is introduced in the same way, $H^-(u_0, u_0', \Lambda) = -x_1$, if solution $u(x)$ to equation (4) with initial conditions $u(0) = u_0$, $u_x(0) = u_0'$ collapses at the point $x = x_1$, $x_1 < 0$.

For further purposes it turns out to be important that the functions $H^{\pm}(u, u', \Lambda)$ can be expressed by quadratures (see Lemma 2) and are continuous.

Definition 4. *Sets $\mathcal{U}_L^{\pm}$ and $\mathcal{U}_L$. We define sets $\mathcal{U}_L^{\pm}$ and $\mathcal{U}_L$ as follows:*

$$\mathcal{U}_L^+ = \left\{(u_*, u_*') \,\middle|\, h_0^+(u_*, u_*') > L\right\},$$

$$\mathcal{U}_L^- = \left\{(u_*, u_*') \,\middle|\, h_0^-(u_*, u_*') > L\right\},$$

$$\mathcal{U}_L = \mathcal{U}_L^+ \cap \mathcal{U}_L^-.$$

In other words, the set $\mathcal{U}_L^+$ consists of the points in the plane of the initial data (u, u_x), which produce the solutions collapsing "later" that a given value L (or not collapsing).

Let us mention some properties of sets $\mathcal{U}_L^{\pm}$.

- We denote by I the symmetry mapping w.r.t. the axis u in the plane (u, u_x). By the evenness of the function $Q(x)$ (Property (Q2)), the relation $\mathcal{U}_L^+ = I[\mathcal{U}_L^-]$ is true.
- If $P(u)$ is an odd function, then the sets $\mathcal{U}_L^+$ and $\mathcal{U}_L^-$ are mutually symmetric w.r.t. the origin.

Definition 5. *Poincaré map. We define a Poincaré map $T : \mathbb{R}^2 \to \mathbb{R}^2$ generated by equation (1) as follows (see* [4, Ch 1]*):*

$$T\begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \begin{pmatrix} u(\pi) \\ u_x(\pi) \end{pmatrix},$$

where $u(x)$ is the solution to equation (1) with initial conditions $u(0) = u_0$, $u_x(0) = u_0'$.

Let us mention some properties of the mapping T and sets $\mathcal{U}_\pi^{\pm}$.

- T is a diffeomorphism conserving the area.
- T is defined only on the set $\mathcal{U}_\pi^+$. Therefore, the inverse mapping T^{-1} is defined on the set $\mathcal{U}_\pi^-$, at that,
$$T\mathcal{U}_\pi^+ = \mathcal{U}_\pi^-, \quad T^{-1}\mathcal{U}_\pi^- = \mathcal{U}_\pi^+.$$
- The mapping IT, where I is the symmetry mapping w.r.t. the axis u, see above, is an automorphism of the set $\mathcal{U}_\pi^+$.

Throughout the paper, the open ball of radius ε centered at the point (u_0, u_0') is denoted by $V_\varepsilon(u_0, u_0')$

$$V_\varepsilon(u_0, u_0') = \left\{(u_*, u_*') \;\middle|\; (u_* - u_0)^2 + (u_*' - u_0')^2 < \varepsilon^2\right\}.$$

3. Properties of functions $h^+(x, u, u')$ and $h^-(x, u, u')$

3.1. Limiting properties of function $h_0^+(u, u')$ and $h_0^-(u, u')$. We define function $H_0(u, \Lambda)$ by the formula

$$H_0(u, \Lambda) \equiv \int\limits_u^{+\infty} \frac{\mathrm{d}\xi}{\sqrt{F(u, \xi) - \Lambda(\xi^2 - u^2)}}, \tag{5}$$

where

$$F(u, \xi) \equiv 2\int\limits_u^{\xi} P(\eta)\mathrm{d}\eta. \tag{6}$$

Lemma 1. *Let $F(u, \xi)$ be given by formula (6) and $P(u)$ satisfy Conditions (P1)-(P2). Then for each $\Lambda \in \mathbb{R}$ there exists a sufficiently large $A_1 > 0$ such that as $u > A_1$, the function $H_0(u, \Lambda)$ is well-defined and the identity*

$$\lim_{u \to +\infty} H_0(u, \Lambda) = 0$$

holds true.

Proof. We introduce the notation $Z(u, \xi) \equiv F(u, \xi) - \Lambda(\xi^2 - u^2)$. It is obvious that $Z(u, u) = 0$ and integral (5) has a singularity at the point $\xi = u$. At the same time, by Property (P2), for sufficiently large u the inequality $Z_\xi(u, u) = 2P(u) - 2\Lambda u > 0$ holds true. Therefore, for such values of u the zero of function $Z(u, \xi)$ at the point $\xi = u$ is simple and the singularity of integral (5) at the point $\xi = u$ is integrable. Then we note that as $\xi > u$ and $\lambda > 0$ the inequality

$$\xi^2 - u^2 < \frac{2}{u^\lambda(\lambda + 2)} \cdot (\xi^{2+\lambda} - u^{2+\lambda})$$

holds true. Hence, first, for sufficiently large u and as $\xi > u$, the function $Z(u,\xi)$ is positive, and second, the estimate

$$\frac{1}{\sqrt{Z(u,\xi)}} < \frac{\sqrt{\lambda+2}}{\sqrt{C\left(\xi^{2+\lambda}-u^{2+\lambda}\right)-\Lambda(\lambda+2)(\xi^2-u^2)}} < \frac{1}{C_1(u,\Lambda)\sqrt{\xi^{2+\lambda}-u^{2+\lambda}}}$$

holds true, where

$$C_1(u,\Lambda) = \sqrt{\frac{1}{\lambda+2}\cdot\left(C-\frac{2\Lambda}{u^\lambda}\right)}.$$

By the change $\xi = u\zeta$ we get

$$0 < H_0(u,\Lambda) < \frac{1}{C_1(u,\Lambda)u^{\lambda/2}}\int_1^{+\infty}\frac{\mathrm{d}\zeta}{\sqrt{\zeta^{2+\lambda}-1}}.$$

This inequality implies the statement of the lemma. □

We introduce the notations

$$m = \min_{x\in[0;\pi]} Q(x), \quad M = \max_{x\in[0;\pi]} Q(x).$$

Lemma 2. *Let $Q(x)$ and $P(u)$ satisfy Conditions (Q1)-(Q3) and (P1)-(P2), respectively. Then there exists $A_2 > 0$ such that for the initial data in the Cauchy problem for equation (1) located in the sector*

$$R^+(A_2) = \{(u_*, u'_*) \mid u_* \geqslant A_2,\ u'_* \geqslant 0\},$$

the solution to equation (1) collapses at $+\infty$ and $\lim_{u\to+\infty} h_0^+(u,u') = 0$ uniformly in u' and the estimate

$$0 < h_0^+(u,u') \leqslant H_0(u,M) \tag{7}$$

holds true.

Proof. We take a point $(u_0,u'_0) \in R^+(A_2)$. The solution to the Cauchy problem for equation (4), where we let $\Lambda = M$ with initial data $u(0) = u_0$, $u_x(0) = u'_0$ is written in the implicit form

$$x = \pm\int_{u_0}^{u}\frac{\mathrm{d}\xi}{\sqrt{F(u_0,\xi) - M\left(\xi^2-u_0^2\right) + (u'_0)^2}}, \tag{8}$$

where the sign "+" or "-" at the integral corresponds to the sign of u'_0. Let $u_0 > A_1$ in Lemma 1 and $u'_0 \geqslant 0$. In this case the denominator in (8) does not vanish as $\xi > u_0$ and the integral

$$x_1 = \int_{u_0}^{+\infty}\frac{\mathrm{d}\xi}{\sqrt{F(u_0,\xi) - M\left(\xi^2-u_0^2\right) + (u'_0)^2}} \tag{9}$$

converges. At that, the corresponding solution $u(x)$ to the Cauchy problem for equation (4) collapses to $+\infty$, while the singularity point x_1 satisfies the estimate

$$x_1 \leqslant \int_{u_0}^{+\infty}\frac{\mathrm{d}\xi}{\sqrt{F(u_0,\xi) - M\left(\xi^2-u_0^2\right)}} \equiv H_0(u_0,M).$$

Let us estimate function $h_0^+(u,u')$ at point (u_0,u'_0). We rewrite equations (1) and (4) as

$$u_{xx} - f(x,u) = 0, \quad f(x,u) = P(u) - Q(x)u, \tag{10}$$

$$u_{xx} - \breve{f}(u) = 0, \quad \breve{f}(u) = P(u) - Mu. \tag{11}$$

We consider the Cauchy problems $u(0) = u_0$, $u_x(0) = u'_0$ for equations (10) and (11). We denote the solutions to these problems respectively by $u(x)$ and $\breve{u}(x)$. By construction, $f(x,u) \geqslant \breve{f}(u)$

for all x and $u > 0$. By Property (P2), there exists $A_3 > 0$ such that for $u > A_3$, the function $\breve{f}(u)$ is monotonically increasing. As $u_0 \geqslant A_2 > \max\{A_1, A_3\}$ and $u_0' \geqslant 0$, by the Comparison Lemma (see Appendix A) we have $u(x) \geqslant \breve{u}(x)$ as $x \geqslant 0$. It follows that first, $u(x)$ is singular, and second, $h_0^+(u_0, u_0') \leqslant H_0(u_0, M)$. This proves the lemma. □

Corollary. *Assume that $Q(x)$ and $P(u)$ satisfy Conditions (Q1)-(Q3) and (P1)-(P2), respectively. Then there exists $A_2 > 0$ such that for the initial data in the Cauchy problem for equation (1) located in the sector*

$$R^-(A_2) = \{(u_*, u_*') \mid u_* \geqslant A_2,\ u_*' \leqslant 0\}$$

the estimate

$$0 < h_0^-(u, u') \leqslant H_0(u, M) \tag{12}$$

holds true.

Proof. The statement is implied by the invertibility of equation (1); this invertibility holds due to Property (Q2). □

Lemma 3. *Assume that $Q(x)$ and $P(u)$ satisfy Conditions (Q1)-(Q3) and (P1), (P3), respectively. Then there exists $\widetilde{A}_2 < 0$ such that for the initial data in the Cauchy problem for equation (1) located in the sector*

$$L^-(\widetilde{A}_2) = \{(u_*, u_*')| u_* \leqslant \widetilde{A}_2, u_*' \leqslant 0\},$$

the solution to equation (1) collapses to $-\infty$, and at that, $\lim_{u\to-\infty} h_0^+(u, u') = 0$ uniformly in u'.

Proof. The statement is implied by Lemma 2 by changing $u \to -u$ in equation (1). □

Corollary. *Assume that $Q(x)$ and $P(u)$ satisfy Conditions (Q1)-(Q3) and (P1), (P3), respectively. Then there exists $\widetilde{A}_2 < 0$ such that for the initial data in the Cauchy problem for equation (1) located in the sector*

$$L^+(\widetilde{A}_2) = \{(u_*, u_*')| u_* \leqslant \widetilde{A}_2, u_*' \geqslant 0\},$$

the solution to equation (1) collapses to $-\infty$, at that, $\lim_{u\to-\infty} h_0^-(u, u') = 0$ uniformly in u'.

The obtained results characterizing the behavior of the functions $h_0^\pm(u, u')$ in the plane (u, u_x) as $u \to \pm\infty$ are schematically presented in Figure 1.

3.2. Continuity of functions $h_0^+(u, u')$ and $h_0^-(u, u')$.

Lemma 4. *Assume that solution $u(x)$ of equation (1) collapses to $+\infty$ at the point $x = x_1$ while approaching this point from the left. Then*

(a) there exists a semi-neighbourhood $(x_1 - \Delta; x_1)$ of the point $x = x_1$ such that for $\xi \in (x_1 - \Delta; x_1)$ the inequality

$$H^+(u(\xi), u_x(\xi), m(\xi)) < x_1 - \xi = h^+(\xi, u(\xi), u_x(\xi)) < H^+(u(\xi), u_x(\xi), M(\xi)), \tag{13}$$

holds true, where

$$m(\xi) = \min_{x\in[\xi;x_1]} Q(x), \quad M(\xi) = \max_{x\in[\xi;x_1]} Q(x);$$

(b) For each $\xi \in (x_1 - \Delta; x_1)$ there exists a neighbourhood $V_\varepsilon(u(\xi), u_x(\xi))$ such that in each point of this neighbourhood the values

$$H^+(u, u', m(\xi)), \quad h^+(\xi, u, u'), \quad H^+(u, u', M(\xi))$$

are finite and

$$H^+(u, u', m(\xi)) \leqslant h^+(\xi, u, u') \leqslant H^+(u, u', M(\xi)). \tag{14}$$

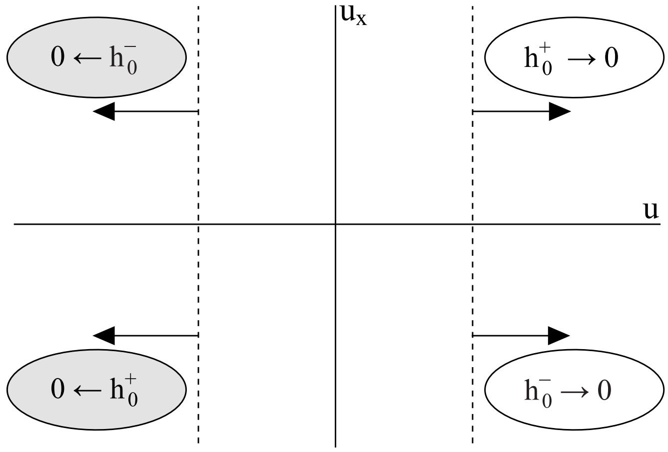

FIGURE 1. Behavior of the functions $h_0^{\pm}(u,u')$ in the plane (u,u_x). The white oval indicates conditions (P1), (P2) for $P(u)$. The grey oval indicates conditions (P1), (P3). Dashed lines indicate the uniform convergence to zero for the functions $h_0^{\pm}(u,u')$ as $u \to \pm\infty$

Proof. (a) Let ξ be some value, $\xi \in (0;x_1)$. Denote $u_0 = u(\xi)$, $u'_0 = u_x(\xi)$. We write equation (1) as

$$u_{xx} - f(x,u) = 0, \quad f(x,u) = P(u) - Q(x)u \tag{15}$$

and consider the following equations

$$u_{xx} - \widehat{f}(\xi,u) = 0, \quad \widehat{f}(\xi,u) = P(u) - m(\xi)u, \tag{16}$$

$$u_{xx} - \check{f}(\xi,u) = 0, \quad \check{f}(\xi,u) = P(u) - M(\xi)u. \tag{17}$$

We denote by $\widehat{u}(x)$, $\check{u}(x)$ respectively the solutions to the Cauchy problems for (16), (17) with the initial conditions $u(\xi) = u_0$, $u_x(\xi) = u'_0$.

By assumption, solution $u(x)$ collapses to $+\infty$ at the point $x = x_1$. It means that $u(x)$ is well-defined on $x \in [\xi;x_1)$ and moving ξ to x_1, we can achieve that the solution $u(x)$ on the segment $x \in [\xi;x_1)$ is greater than A in Condition (P2). Then the inequality

$$\widehat{f}(\xi,u) \geqslant f(x,u)$$

holds true, and the function $f(\xi,u)$ is monotonically increasing in u. By the Comparison Lemma (see Appendix A), for $x \geqslant \xi$ we have $\widehat{u}(x) \geqslant u(x)$. This means that $H^+(u_0,u'_0,m(\xi))$ is finite, the solution $\widehat{u}(x)$ collapses to $+\infty$ and

$$H^+(u_0,u'_0,m(\xi)) \leqslant h^+(\xi,u_0,u'_0). \tag{18}$$

Since $\lim_{\xi\to x_1}(m(\xi) - M(\xi)) = 0$ and the function $H^+(u,u',\Lambda)$ is continuous, we can choose ξ close enough to x_1 so that the solution $\check{u}(x)$ also collapses to $+\infty$. In this case the value $H^+(u_0,u'_0,M(\xi))$ is also finite. We apply the Comparison Lemma to equations (15) and (17). As a result, we conclude that as $x \geqslant \xi$, the inequalities $\check{u}(x) \leqslant u(x)$ and

$$h^+(\xi,u_0,u'_0) \leqslant H^+(u_0,u'_0,M(\xi)) \tag{19}$$

hold true. Combining (18) and (19), we obtain the double inequality (13), which holds true for all ξ belonging to a sufficiently small left semi-neighbourhood x_1.

(b) Since the function $H^+(u,u',\Lambda)$ is continuous, it follows that $H^+(u,u',M(\xi))$ is finite not only at the point $u_0 = u(\xi)$, $u'_0 = u_x(\xi)$, but also in some neighbourhood $V_\varepsilon(u(\xi),u_x(\xi))$. Applying the Comparison Lemma to the solutions of the Cauchy problems with the initial data in the neighbourhood $V_\varepsilon(u(\xi),u_x(\xi))$ for equations (15), (16), (17), we obtain that all three solutions collapse and the singularity points satisfy inequality (14). □

Corollary. *If a solution $u(x)$ to equation (1) collapses to $-\infty$ at the point $x = x_1$ approaching it from the left, the statement of Lemma 4 is still true up to swapping $m(\xi)$ and $M(\xi)$ in formulae (13) and (14).*

The proof of this statement is completely similar to the proof of Lemma 4.

Lemma 5. *Let a point (u_0, u_0') be such that $h_0^+(u_0, u_0') = x_1 < +\infty$. Then $h_0^+(u, u')$ is continuous at the point (u_0, u_0').*

Proof. The solution $u(x)$ to the Cauchy problem $u(0) = u_0$, $u_x(0) = u_0'$ for equation (1) collapses to $+\infty$ at the point $x = x_1$. By Lemma 4, Statement (a), there exists Δ such that for each $\xi \in (x_1 - \Delta; x_1)$ the inequality

$$H^+(u(\xi), u_x(\xi), m(\xi)) \leqslant h^+(\xi, u(\xi), u_x(\xi)) \leqslant H^+(u(\xi), u_x(\xi), M(\xi)) \tag{20}$$

holds true, where

$$m(\xi) = \min_{x \in [\xi; x_1]} Q(x), \quad M(\xi) = \max_{x \in [\xi; x_1]} Q(x).$$

By Statement (b) of Lemma 4, for each $\xi \in (x_1 - \Delta; x_1)$ there exists a neighbourhood $V_{\varepsilon_1}(u(\xi), u_x(\xi))$ such that for each point in this neighbourhood the estimate

$$-H^+(u, u', M(\xi)) \leqslant -h^+(\xi, u, u') \leqslant -H^+(u, u', m(\xi)) \tag{21}$$

holds true. Summing (20) and (21), we obtain

$$\begin{aligned} H^+(u(\xi), u_x(\xi), m(\xi)) - H^+(u, u', M(\xi)) \leqslant & h^+(\xi, u(\xi), u_x(\xi)) - h^+(\xi, u, u') \\ \leqslant & H^+(u(\xi), u_x(\xi), M(\xi)) - H^+(u, u', m(\xi)). \end{aligned}$$

Since $\lim_{\xi \to x_1}(m(\xi) - M(\xi)) = 0$ and by the continuity of the function $H^+(u, u', \Lambda)$, for each $\delta > 0$ there exist ε_2, $0 < \varepsilon_2 < \varepsilon_1$, and $\eta \in (x_1 - \Delta; x_1)$ such that for each point in $V_{\varepsilon_2}(u(\eta), u_x(\eta))$ the estimates

$$\begin{aligned} -\delta < H^+(u(\eta), u_x(\eta), m(\eta)) - H^+(u, u', M(\eta)) < \delta, \\ -\delta < H^+(u(\eta), u_x(\eta), M(\eta)) - H^+(u, u', m(\eta)) < \delta \end{aligned}$$

hold true. This means that for each point in $V_{\varepsilon_2}(u(\eta), u_x(\eta))$ the inequality

$$\left| h^+(\eta, u(\eta), u_x(\eta)) - h^+(\eta, u, u') \right| < \delta$$

holds true, that is, $h^+(\eta, u, u')$ is well-defined in some neighbourhood of the point $(u(\eta), u_x(\eta))$ and is a continuous function of u and u'. We consider the mapping $\mathcal{T}_\eta$ generated by the shift along the trajectories of equation (1) mapping the point $(u(0), u_x(0))$ into the point $(u(\eta), u_x(\eta))$. This mapping is continuous and at that, the inverse mapping $\mathcal{T}_\eta^{-1}$ is also continuous. Therefore, there exists a neighbourhood of the point $(u(0), u_x(0))$,

$$V_\varepsilon(u(0), u_x(0)) \subset \mathcal{T}_\eta^{-1} V_{\varepsilon_2}(u(\eta), u_x(\eta)),$$

in which the function $h_0^+(u, u')$ is well-defined and continuous. □

Remark. *Similar statement is true for the function $h_0^-(u, u')$. Namely, if a point (u_0, u_0') is such that $h_0^-(u_0, u_0') = x_1 < +\infty$, then $h_0^-(u, u')$ is finite in some neighbourhood $V_\varepsilon(u_0, u_0')$ and is continuous at the point (u_0, u_0'). The proof of this statement is completely similar to that of Lemma 5.*

4. Properties of sets $\mathcal{U}_L^{\pm}$ and $\mathcal{U}_L$

Basing on the results of Section 3, we describe some properties of the sets $\mathcal{U}_L^{\pm}$, $0 < L < \infty$.

Theorem 1. *Assume that Conditions (Q1)-(Q3), (P1)-(P2) are satisfied and L is such that the sets $\mathcal{U}_L^{\pm}$ are non-empty. Then*
(a) *$\mathcal{U}_L^{\pm}$ are open sets;*
(b) *the boundary of $\mathcal{U}_L^{\pm}$ consists of the points (u_*, u_*') such that $h_0^{\pm}(u_*, u_*') = L$.*

Proof. Let us prove the theorem for $\mathcal{U}_L^{+}$; the proof for $\mathcal{U}_L^{-}$ is completely similar.
(a) Assume that $p = (u_*, u_*') \in \mathcal{U}_L^{+}$. This means that $h_0^{+}(u_*, u_*') > L$. However, by the continuity of $h_0^{+}(u, u')$, there exists the neighbourhood $V_\varepsilon(u_*, u_*')$ such that $h_0^{+}(u, u') > L$ for all points in this neighbourhood, that is, $V_\varepsilon(u_*, u_*') \subset \mathcal{U}_L^{+}$ and it means that the set $\mathcal{U}_L^{+}$ is open.
(b) If the point $p = (u_*, u_*')$ belongs to the boundary $\mathcal{U}_L^{+}$, then there exists a sequence of the points $p_n = (u_n, u_n')$, $p_n \in \mathcal{U}_L^{+}$, converging to p. For each of the points p_n the inequality $h_0^{+}(u_n, u_n') > L$ holds true. Therefore, $h_0^{+}(u_*, u_*') \geqslant L$. But if $h_0^{+}(u_*, u_*') > L$, then p is an internal point of the domain. Hence, $h_0^{+}(u_*, u_*') = L$ and this is the desired identity. □

Theorem 2. *Let L be fixed, Conditions (Q1)-(Q3), (P1)-(P2) are satisfied and the sets $\mathcal{U}_L^{\pm}$ are non-empty. Then there exists $B > 0$ such that the set $\mathcal{U}_L$ and its closure are located in the half-plane $u \leqslant B$.*

Proof. By Lemma 2 there exists $B_1 > 0$ such that in the sector $R^{+}(B_1)$ the estimate (7) is satisfied for $h_0^{+}(u, u')$, while in the sector $R^{-}(B_1)$ estimate (12) is true for $h_0^{-}(u, u')$. By Lemma 1 there exists $B_2 > 0$ such that for each $u > B_2$ the inequality $H_0(u, M) < L$ holds true. We denote $B = \max\{B_1, B_2\}$. Then it is true that the sector $R^{+}(B)$ contains no points of the set $\mathcal{U}_L^{+}$, while the sector $R^{-}(B)$ has no points of the set $\mathcal{U}_L^{-}$. Hence, the intersection $\mathcal{U}_L = \mathcal{U}_L^{+} \cap \mathcal{U}_L^{-}$ is located in the half-plane $u \leqslant B$ and its closure also located in this half-plane. The proof is complete. □

Now we are in position to formulate the main result of the work.

Theorem 3. *Assume that Conditions (Q1)-(Q3) and (P1)-(P2) hold true and the set $\mathcal{U}_\pi^{+}$ is non-empty. Then*
(a) *the set $\mathcal{U}_\pi^{+}$ consists of finitely many or countable many connected components $V_1, V_2, \ldots$, each of which is a domain, that is, an open connected set;*
(b) *each connected component V_k is simply connected, that is, each closed contour can be continuously contracted to a point without leaving V_k;*
(c) *each connected component V_k is unbounded.*

Proof. Since $\mathcal{U}_\pi^{+}$ is an open set, it can be represented as the union of finitely many or countably many non-intersecting domains V_k, $\mathcal{U}_\pi^{+} = \bigcup_k V_k$ (see [7, Ch. 4, Thm. 17]), which are the connected components of $\mathcal{U}_\pi^{+}$. This proves Statement (a).

Let us prove Statement (b). Consider one of the connected components, V, and assume that it is not simply connected. This means that there exists a closed Jordan contour $\Gamma \subset V$, which can not be continuously contracted into the point without leaving the component V. We denote by Δ_1 the domain enveloped by Γ. We have that Δ_1 contains a point not belonging to V, that is, such that $h_0^{+}(u, u') \leqslant \pi$. We consider the set

$$\Theta = \{(u_*, u_*') \in \Delta_1, h_0^{+}(u_*, u_*') > \pi\}, \quad \Theta \subset \Delta_1, \quad \Theta \subset V,$$

and its closure $\overline{\Theta}$. By the continuity of the function $h_0^{+}(u, u')$, in $\overline{\Theta}$ there is a point $P = (U, U') \in \Delta_1$ such that $h_0^{+}(U, U') = \pi$. At that, the point P can be connected with some point $N \in \Gamma$ by a continuous curve $\alpha \in \Theta$.

Now we apply the Poincaré map T to Γ, Θ and α. The T-image of Γ is a closed Jordan contour $T\Gamma$, which partitions $\mathbb{R}^2$ into the internal part, Δ_2, and the external part, $\mathbb{R}^2 \backslash \Delta_2$. The

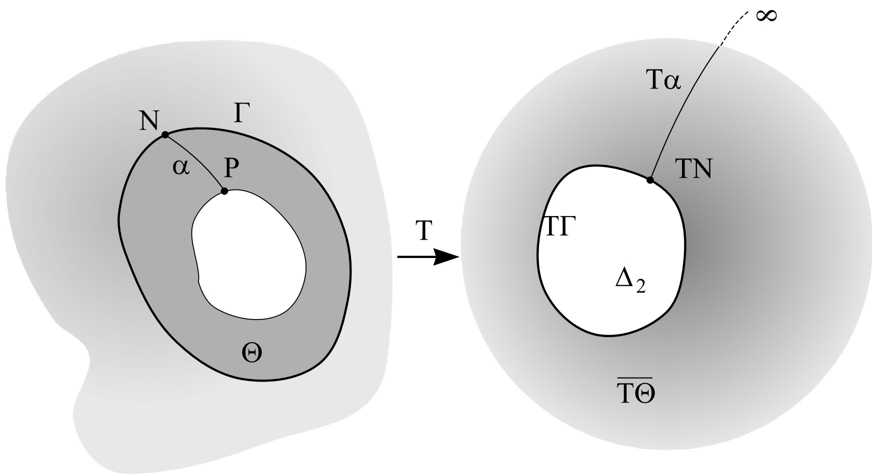

FIGURE 2. To the proof of Statement (b) of Theorem 3

set Δ_2 is bounded, the set $\mathbb{R}^2\backslash\Delta_2$ is unbounded. The T-image of α is the curve $T\alpha$; one of its ends lies in $T\Gamma$, while the other goes to infinity. The T-image of Θ is the set $T\Theta$ located in the exterior of the contour $T\Gamma$, $\mathbb{R}^2\backslash\Delta_2$. It contains the curve $T\alpha$ and therefore, is unbounded.

We consider $\overline{T\Theta}$, the closure of $T\Theta$. Let us show that

$$\overline{T\Theta} = \mathbb{R}^2\backslash\Delta_2. \tag{22}$$

It is obvious that $\overline{T\Theta} \subset \mathbb{R}^2\backslash\Delta_2$. Let us prove the opposite inclusion. Assume that there exists a point q such that $q \in \mathbb{R}^2\backslash\Delta_2$, but $q \notin \overline{T\Theta}$. We connect q with an arbitrary point of the contour $T\Gamma$ by a finite curve β. On the curve β there is a point $\widetilde{q} = (v_*, v'_*)$ being a boundary point of $\overline{T\Theta}$. It means that in each neighbourhood $\widetilde{q}$ there are points not belonging to $\overline{T\Theta}$, and there exists a sequence of the points $q_n = (v_n, v'_n)$, $q_n \in T\Theta$, converging to $\widetilde{q}$. We consider T-preimages of the points q_n, $p_n = T^{-1}q_n = (u_n, u'_n)$. Since T is a diffeomorphism, the sequence $\{p_n\}$ also converges to some point $\widetilde{p} = (u_*, u'_*)$. Since $h_0^+(u_n, u'_n) > \pi$, then $h_0^+(u_*, u'_*) \geqslant \pi$. If $h_0^+(u_*, u'_*) = \pi$, then $q_n \to \infty$, which does not hold. Therefore, $h_0^+(u_*, u'_*) > \pi$. But then $\widetilde{q} = T\widetilde{p}$ and $\widetilde{q}$ is an internal point of $T\Theta$ and not its boundary point. Thus, we have proved (22).

Hence, we have

$$\Theta \subseteq V \subseteq \mathcal{U}_\pi^+,$$

this is why

$$T\Theta \subseteq TV \subseteq T\mathcal{U}_\pi^+ = \mathcal{U}_\pi^-$$

and

$$\overline{T\Theta} \subseteq \overline{\mathcal{U}_\pi^-}. \tag{23}$$

We consider the set $\Theta_0 = I\left[\overline{T\Theta}\right] \cap \overline{T\Theta}$, where I, as above, stands for the symmetry mapping w.r.t. the axis u. By (22), each of the sets $I\left[\overline{T\Theta}\right]$ and $\overline{T\Theta}$ is the plane without a bounded domain, and this is why they include the exterior of some rather large circle. It follows that Θ_0 also includes the exterior of some circle. On the other hand, by (23)

$$\Theta_0 \subseteq I\left[\overline{\mathcal{U}_\pi^-}\right] \cap \overline{\mathcal{U}_\pi^-} = \overline{\mathcal{U}_\pi^+} \cap \overline{\mathcal{U}_\pi^-} = \overline{\mathcal{U}_\pi}.$$

But by Theorem 2, the set $\overline{\mathcal{U}_\pi}$ lies in the left half-plane $u \leqslant B$ for some B. The proved contradiction proves Statement (b).

Let us prove Statement (c). Assume that a connected component, V, is bounded. We consider an arbitrary Jordan contour Γ located inside V. We denote by $\Delta_1 \subset V$ the domain

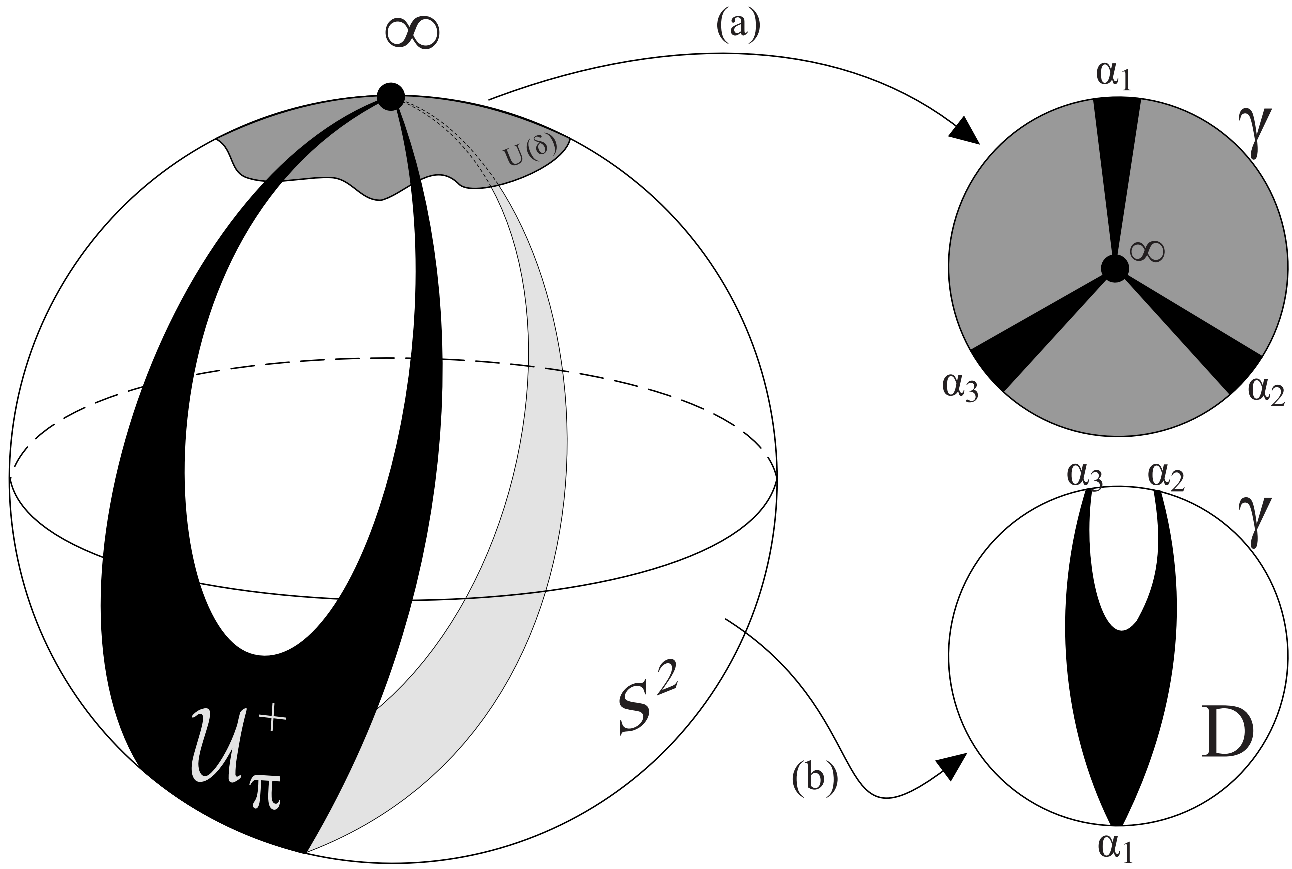

FIGURE 3. Riemann sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. On the sphere, the configuration of the set $\mathcal{U}_\pi^+$ is shown similar to Figure 5(c)

enveloped by Γ. We introduce the notation:

$$\Theta = V \backslash \Delta_1.$$

Let us consider $T\Gamma$ and $T\Theta$. It is obvious that $T\Gamma$ is a Jordan contour in $\mathbb{R}^2$. We denote by Δ_2 the domain enveloped by this contour. Concerning $T\Theta$, Statement (22) turns out to be true and the proof of (22) is completely similar to the corresponding proof in Statement (b). As in Statement (b), we consider the set $\Theta_0 = I\left[\overline{T\Theta}\right] \cap \overline{T\Theta}$ which, on one hand, includes the exterior of some circle, and on the other hand, is contained in $\overline{\mathcal{U}_\pi}$ located in the left half-plane $u \leqslant B$ for some B. Thus, the proof of Statement (c) is complete. $\square$

Remark. *Since the sets $\mathcal{U}_\pi^-$ and $\mathcal{U}_\pi^+$ are located symmetrically w.r.t. the axis u, Theorem 3 remains true if we replace $\mathcal{U}_\pi^+$ by $\mathcal{U}_\pi^-$ in its formulation.*

Remark. *Theorem 3 is also true for the sets $\mathcal{U}_{k\pi}^\pm$, where $k > 1$ is an integer number.*

Remark. *The above identity can be essentially simplified, if the boundaries of the set $\mathcal{U}_\pi^+$ are sufficiently "good" curves. It seems that the boundaries of $\mathcal{U}_\pi^+$ are indeed C^1-smooth curves in a rather general case, but at present, the authors do not know the proof of this fact.*

5. Possible types of set $\mathcal{U}_\pi^+$

Theorems 1, 2 and 3 allow us to describe possible topological types of the sets $\mathcal{U}_\pi^+$. In order to do it, we make the following non-rigorous arguments. We consider the Riemann sphere obtained by completing the plane (u, u_x) by the infinity $\{\infty\}$, $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. We observe that by Theorem 3:

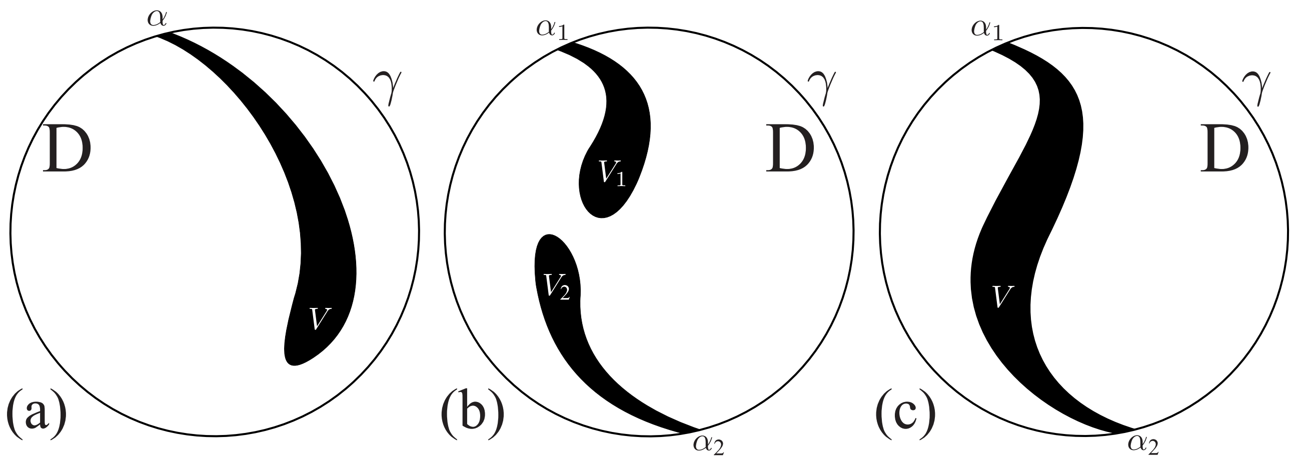

FIGURE 4. All possible configurations of the set $\mathcal{U}_\pi^+$ for equation (1) as $n = 1, 2$. (a): $n = 1, k = 1$. (b): $n = 2, k = 2$. (c): $n = 2, k = 1$.

1. The set $\mathcal{U}_\pi^+$ can not cover the point $\{\infty\}$.
2. Each of the connected component of $\mathcal{U}_\pi^+$ touches the point $\{\infty\}$.

In order to classify all possible locations of the connected components of $\mathcal{U}_\pi^+$, we make use of the following visualization. We consider a neighbourhood W of the point $\{\infty\}$ on $\mathbb{S}^2$ with the boundary on the parallel "sufficiently close" to the point $\{\infty\}$. We assume that the intersection of $\mathcal{U}_\pi^+$ with W is represented by several disjoint components, each of which touches $\{\infty\}$. On the boundary of W these components cut the arcs α_1, $\alpha_2, \ldots$. The amount of arcs n can be finite or infinite. The issue on possible configurations of $\mathcal{U}_\pi^+$ is reduced to possible types of the closures of the connected components of $\mathcal{U}_\pi^+$ between the arcs α_1, $\alpha_2, \ldots$ in $\mathbb{S}^2 \backslash W$. The amount of connected components k is not necessarily equal to n, see below.

Interpreting now $\mathbb{S}^2 \backslash W$ as a circle of rather large radius, we locate n arcs α_1, $\alpha_2, \ldots$ on its boundary, the circumference γ. In the case $n = 1$, on γ there is the only arc α_1 and there is the only possible configuration for $\mathcal{U}_\pi^+$ shown schematically in Figure 4(a). If $n = 2$, there are two arcs α_1 and α_2 on γ and for $\mathcal{U}_\pi^+$, there are two possible topologically different configuration shown in Figure 4(b,c).

On Figures 5 and 6, there are shown all possible topological configurations corresponding to the cases $n = 3$ and $n = 4$. We note that on Figures 6(b) and 6(c), topologically different sets $\mathcal{U}_\pi^+$ are shown, but their values $(n, k) = (4, 3)$ are same. In the same way, Figures 6(d) and 6(e) present different configurations of $\mathcal{U}_\pi^+$ with the same values $(n, k) = (4, 2)$. The issue on the amount of possible configurations for an arbitrary n seems to be nontrivial. It is easy to show that the lower bound for the amount of possible topological configurations for a given amount of arcs n is the the number of partitions of n into the terms $p(n)$. At the same time, each such partition can generate several topologically different configurations.

6. NUMERICAL STUDY OF THE SETS $\mathcal{U}_\pi^\pm$: EXAMPLES

If $Q(x)$ and $P(u)$ are given, we can use numerics for studying the sets $\mathcal{U}_\pi^\pm$. It is natural to base it on scanning the plane of the initial data (u, u_x) in order to construct the 3D plot of the function $h_0^+(u, u')$ and to find the points in this plane, for which $h_0^+(u, u') > \pi$. In the examples we provide below the studying was made as follows. In the plane (u, u_x) we introduce a grid of size $M \times N$ covering the rectangle $[A; B] \times [C; D]$ with a sufficiently small step $\tau_1 = (B - A)/M$ in u and $\tau_2 = (D - C)/N$ in u_x, respectively. For each node of the grid (u_i, u'_j), $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N$, we solve the Cauchy problem for equation (1) with the initial data $u(0) = u_i$, $u_x(0) = u'_j$. If in the segment $[0; \pi]$ the solution to the Cauchy problem takes values greater than some prescribed value $u_{\max}$, we assume that the point (u_i, u'_j) does not belong to the

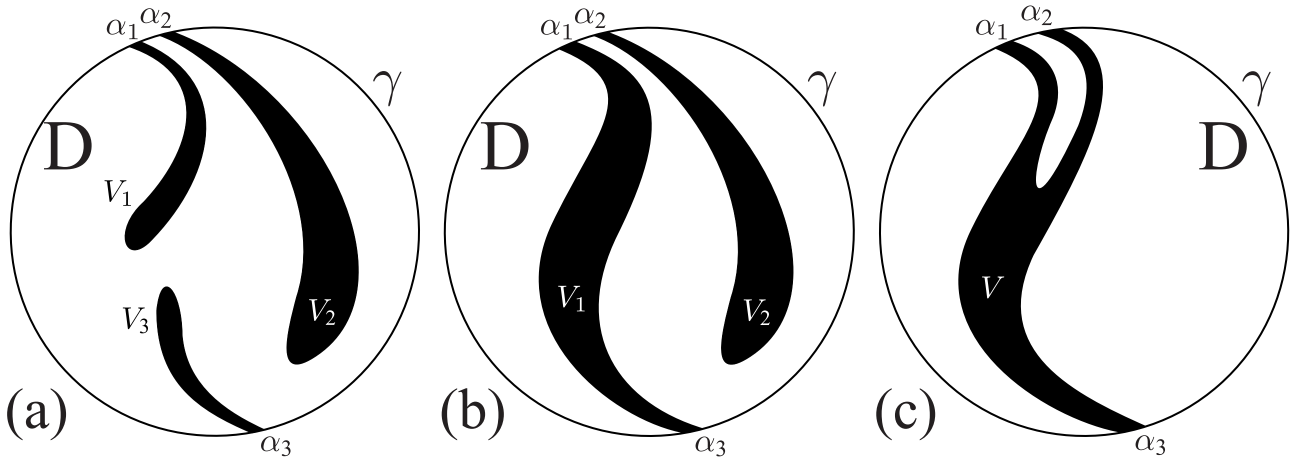

FIGURE 5. All possible configuations of the set $\mathcal{U}_\pi^+$ for equation (1) as $n = 3$. (a): $k = 3$. (b): $k = 2$. (c): $k = 1$.

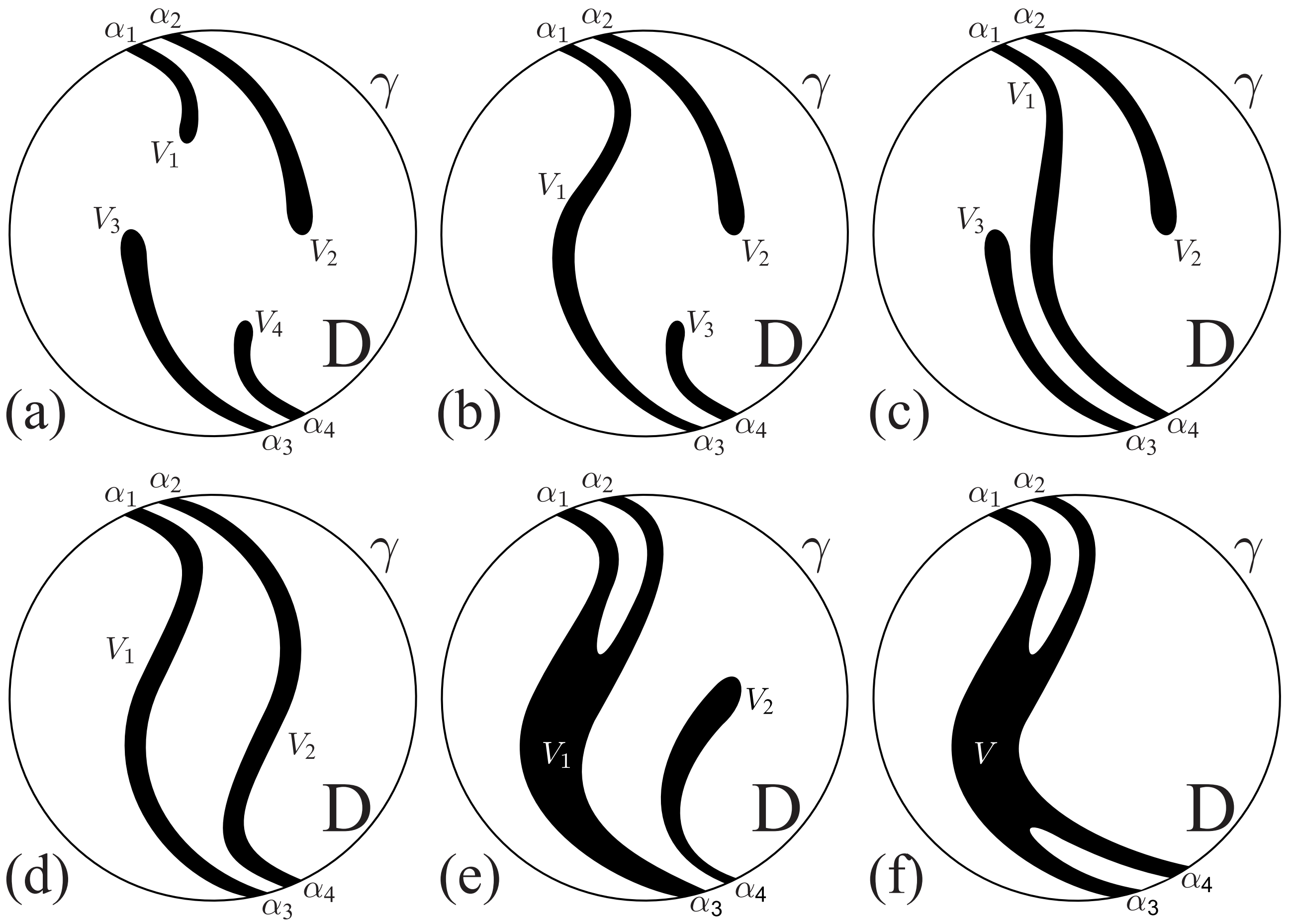

FIGURE 6. All possible configurations of the set $\mathcal{U}_\pi^+$ for equation (1) as $n = 4$. (a): $k = 4$. (b): $k = 3$. (c): $k = 3$. (d): $k = 2$. (e): $k = 2$. (f): $k = 1$.

set $\mathcal{U}_\pi^+$, otherwise, (u_i, u'_j) is included in $\mathcal{U}_\pi^+$. The results obtained for $u_{\max} = 10^5$ and for $u_{\max} = 10^7$ were in a good agreement. The set $\mathcal{U}_\pi^-$ is obtained from $\mathcal{U}_\pi^+$ by the symmetric mapping w.r.t. the axis u.

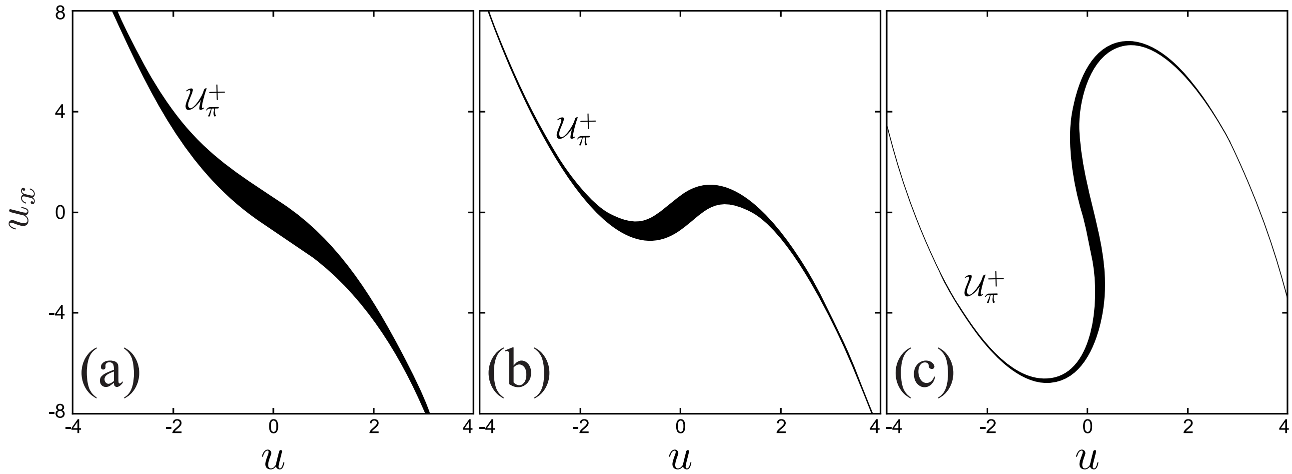

FIGURE 7. The set $\mathcal{U}_\pi^+$ for equation (24) found numerically. The parameters as (a) $\mu_1 = 0, \mu_2 = -2$; (b) $\mu_1 = 1, \mu_2 = 3$; (c) $\mu_1 = 4, \mu_2 = 10$

In what follows we provide the results of the numerical studying of equation (1) for some types of nonlinearity $P(u)$. In all cases the function $Q(x)$ has the same form, $Q(x) = \mu_1 + \mu_2 \cos 2x$, corresponding to a model π-periodic potential.

6.1. Nonlinearity $P(u) = u^3$. The equation

$$u_{xx} + (\mu_1 + \mu_2 \cos 2x)u - u^3 = 0 \tag{24}$$

arises in studying the stationary modes of the form

$$\psi(x,t) = e^{i\mu_1 t} u(x) \tag{25}$$

with a real-valued function $u(x)$ in the Gross-Pitaevskii model

$$i\psi_t = \psi_{xx} + (\mu_2 \cos 2x)\psi - |\psi|^2\psi = 0. \tag{26}$$

The latter equation is one of the most demanded equations in the considered class since it arises in various problems in the nonlinear optics and low temperature physics [1, 2]. Paper [3] was devoted to classification of solution to equation (24). In particular, in [3] numerically constructed sets $\mathcal{U}_{k\pi}^\pm$, $k = 1, 2, \ldots$, are given. The examples of the sets $\mathcal{U}_\pi^+$ for three sets of the parameters $Q(x)$ are given in Figure 7. It is obvious that the configuration of these sets correspond to the scheme in Figure 4(c), case $n = 2$.

6.2. Nonlinearity $P(u) = u^3 + \nu/u^3$, $\nu > 0$. The equation

$$u_{xx} + (\mu_1 + \mu_2 \cos 2x)u - u^3 - \frac{\nu}{u^3} = 0, \quad \nu > 0 \tag{27}$$

also arises in studying the stationary modes for equation (26). In this case the stationary modes are assumed to be complex-valued of the form $\psi(x,t) = e^{i\mu_1 t + i\varphi(x)} u(x)$ with real-valued $u(x)$ and $\varphi(x)$. It is obvious that (24) is a limiting case of (27) as $\nu \to 0$.

The considered nonlinearity does not satisfy condition (P1) and this is why, saying rigorously, the above proven facts are not applicable to this nonlinearity. Nevertheless, the configuration $\mathcal{U}_\pi^+$ given in Figure 8 corresponds to the scheme in Figure 4(b) ($n = 2$).

6.3. Nonlinearity $P(u) = -u^3 + \nu u^5$, $\nu > 0$. The equation

$$u_{xx} + (\mu_1 + \mu_2 \cos 2x)u + u^3 - \nu u^5 = 0, \quad \nu > 0, \tag{28}$$

arises, in particular, in the problems of the Bose-Einstein condensate theory while taking into consideration three-particle interactions [9]. The examples of the sets $\mathcal{U}_\pi^+$ for three sets of the parameters $Q(x)$ are given in Figure 9. The configuration of $\mathcal{U}_\pi^+$ in Figure 9(a) corresponds to

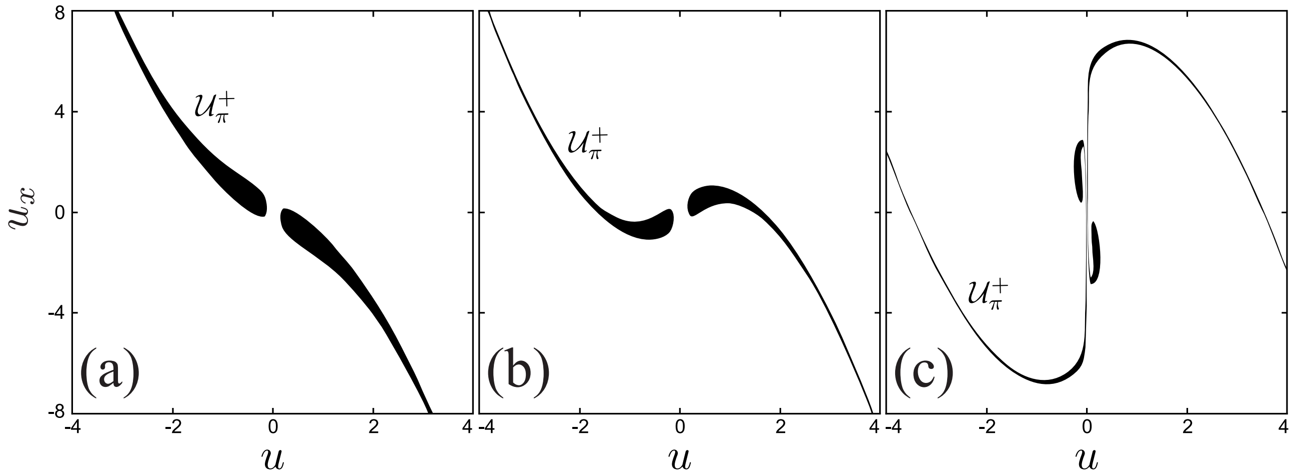

FIGURE 8. The set $\mathcal{U}_\pi^+$ for equation (27) found numerically. The parameters are (a) $\mu_1 = 0, \mu_2 = -2, \nu = 0.1$; (b) $\mu_1 = 1, \mu_2 = 3, \nu = 0.1$; (c) $\mu_1 = 4, \mu_2 = 10, \nu = 0.1$

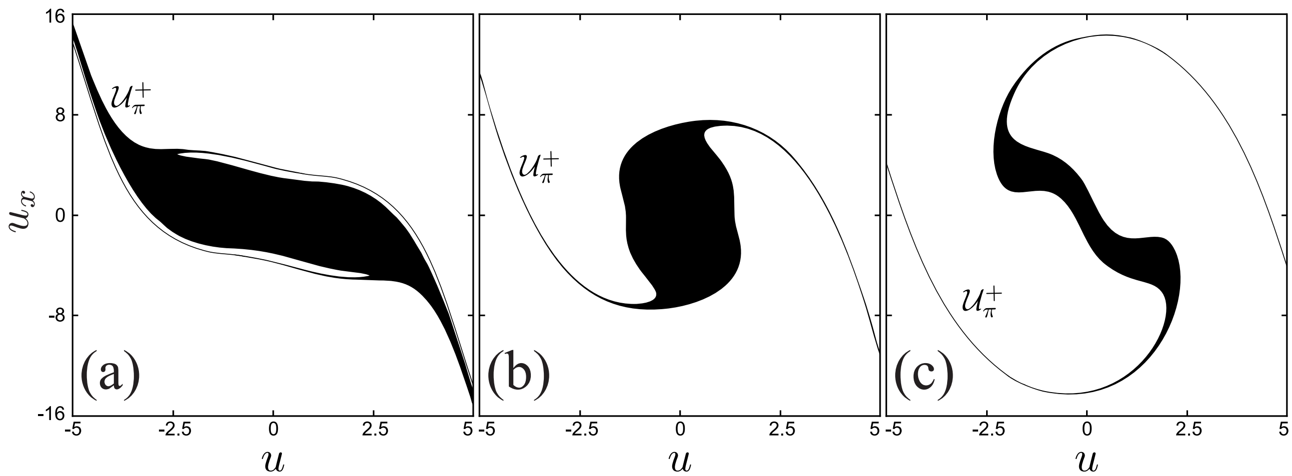

FIGURE 9. The set $\mathcal{U}_\pi^+$ for equation (28) found numerically.
The parameters are (a) $\mu_1 = 0, \mu_2 = -2, \nu = 0.3$; (b) $\mu_1 = 1, \mu_2 = 3, \nu = 0.3$; (c) $\mu_1 = 4, \mu_2 = 10, \nu = 0.3$.

the scheme in Figrue 6(f) (the case $n = 4$), Figures 9(b) and 9(c) correspond to the scheme in Figure 4(c) ($n = 2$).

6.4. Nonlinearity $P(u) = u^2$. To give a complete picture, we consider the equation

$$u_{xx} + (\mu_1 + \mu_2 \cos 2x)u - u^2 = 0. \tag{29}$$

In this case, the nonlinearity differs from ones considered above since now it is not odd. The examples of the sets $\mathcal{U}_\pi^+$ for this case are given in Figure 10. It is interesting that by varying parameters μ_1 and μ_2, we change essentially the geometry of the set $\mathcal{U}_\pi^+$. The configuration of $\mathcal{U}_\pi^+$ in Figure 10(a) corresponds to the scheme in Figure 4(a), ($n = 1$), that in Figure 10(b) corresponds to the scheme in Figure 4(c), ($n = 2$), that in Figure 10(c) corresponds to the scheme in Figure 5(b) ($n = 3$).

7. CONCLUSION

In the work we study all possible configurations of the sets $\mathcal{U}_L^\pm$ in the plane of the initial data for equation (1). By the definition in Section 2, a points in the plane $p = (u_0, u_0')$ belongs to

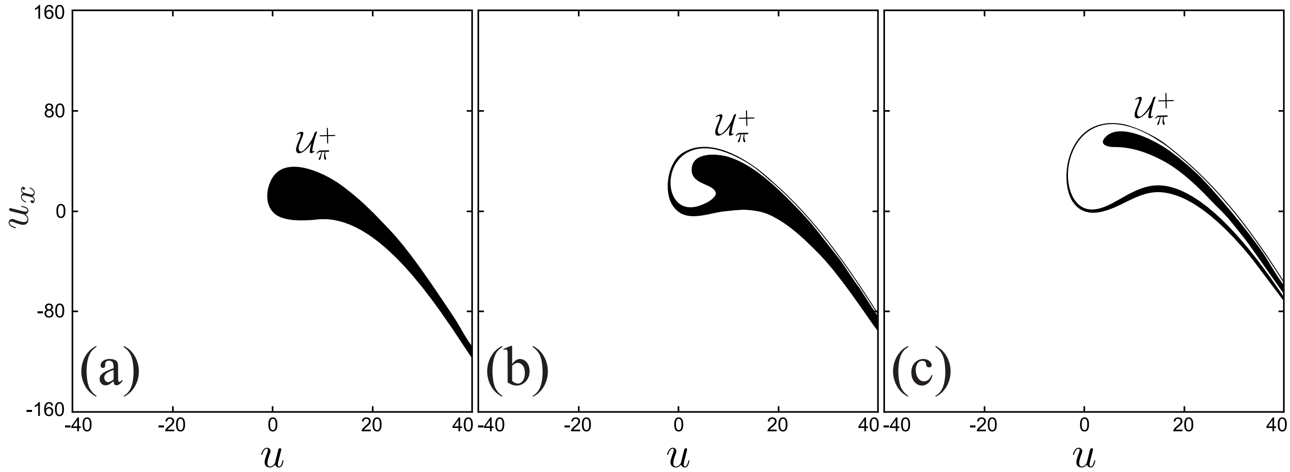

FIGURE 10. The set $\mathcal{U}_\pi^+$ for equation (29) found numerically. The parameters are (a) $\mu_1 = 3, \mu_2 = 20$; (b) $\mu_1 = 3, \mu_2 = 25$; (c) $\mu_1 = 3, \mu_2 = 30$

$\mathcal{U}_L^+$ if and only if the solution to the Cauchy problem $u(0) = u_0$, $u_x(0) = u_0'$ for (1) is extended to the interval $[0; L]$, $L > 0$. In the same way, a point in the plane $p = (u_0, u_0')$ belongs to $\mathcal{U}_L^-$ if and only if the solution to Cauchy problem $u(0) = u_0$, $u_x(0) = u_0'$ for (1) is extended to the interval $[-L; 0]$, $L > 0$. The study of the sets $\mathcal{U}_L^\pm$ turns out to be important for describing of the set of solutions to (1) bounded on the whole real line. In particular, since the equation is invariant w.r.t. the shift by π, in some cases we succeed to describe the set of the solutions to (1) in terms of the dynamics generated by the Poincaré map on the set $\mathcal{U}_\pi = \mathcal{U}_\pi^+ \cap \mathcal{U}_\pi^-$. This is why in studying the bounded solutions to (1), the sets $\mathcal{U}_\pi^\pm$ as well as related sets $\mathcal{U}_{n\pi}^\pm$, $n = 2, 3, \ldots$ play a key role.

The main result of the work is Theorem 3. It imposes rather strict restrictions for the shape of the set $\mathcal{U}_\pi^+$. On the base of Theorem 3, in this work we present possible shapes of the set $\mathcal{U}_\pi^+$. The numerical calculations made for equation (1) with various types of nonlinearities allow us to find the shapes of $\mathcal{U}_\pi^+$ in several particular cases. In all considered cases the shapes of $\mathcal{U}_\pi^+$ correspond to the geometries predicted by the developed theory.

However, the made study gives rise to new questions, which, in our opinion, deserve further studying.

First, it is unknown whether all the predicted geometries of the set $\mathcal{U}_\pi^+$ can be realized under a proper choice of inhomogeneity $Q(x)$ and nonlinearity $P(u)$. In particular, it is interesting whether the shapes of $\mathcal{U}_\pi^+$ with infinitely many connected components can be realized.

Second, the issue on possible shapes of the set $\mathcal{U}_\pi$ is left unstudied. As it follows from work [3], the most interesting situations are those when $\mathcal{U}_\pi$ has an "island" structure, that is, consists of finitely many or infinitely many curvilinear quadrilateral having no common points. It is interesting to employ the results of the present work to identify such situations.

Third, for further purposes it can be useful to study the bifurcations of the sets $\mathcal{U}_\pi^\pm$ under variation of the parameters $Q(x)$ and $P(u)$. In particular, in the numerical studying in Subsection 6.4 such bifurcation were found. In our opinion, it is interesting to study the issue on existence or absence of a universal mechanism responsible for restructuring of $\mathcal{U}_\pi^\pm$ under the variation of the parameters.

Fourth, the further progress in studying the problems of such form can be related to the following fact. The mapping IT (I is the reflection w.r.t. the axis u, T is the Poincaré map, see Section 2) is an automorphism of $\mathcal{U}_\pi^+$. Therefore, this mapping transforms the boundary of $\mathcal{U}_\pi^+$ into itself. The study of mapping of the boundary under the action of IT can be useful for many issues related with the symbolic dynamics determined by the Poincaré map.

Finally, in our opinion, it is rather interesting to generalize the developed method for the higher order equations.

A. Comparison Lemma

Lemma (Comparison Lemma [8]). *Let functions $u(x)$ and $z(x)$, $x \in [a;b]$, solve the equations*

$$u_{xx} - f(x,u) = 0, \tag{30}$$

$$z_{xx} - g(x,z) = 0. \tag{31}$$

Assume that

(a) *$f(x,\xi)$ and $g(x,\xi)$ can be defined on $[a;b]\times[A;B]$ and are locally Lipschitz in ξ, $\xi \in [A;B]$, and A and B can be finite or infinite;*
(b) *$g(x,\xi) \geqslant f(x;\xi)$ as $x \in [a;b]$, $\xi \in [A;B]$;*
(c) *$f(x,\xi)$ is monotonically non-decreasing w.r.t. variable ξ.*

Let $A < u(a) \leqslant z(a) < B$ and $u_x(a) \leqslant z_x(a)$. Then $u(x) \leqslant z(x)$ and $u_x(x) \leqslant z_x(x)$ on the whole interval $[a;b]$ or until $A < u(x) \leqslant z(x) < B$

Remark. *The monotonicity condition for $f(x,\xi)$ w.r.t. the variable ξ can not be omitted. Consider the case $g(x,\xi) = -\xi$, $f(x,\xi) = -\xi - \xi^2$. It is obvious that $g(x,\xi) \geqslant f(x,\xi)$ for all ξ, but $f(x,\xi)$ is not monotonous. In Figure 11 we provide the graph of the difference $w(x) = u(x) - z(x)$ of the solutions to equation (30) and (31) with the same initial condition $u(0) = 0$, $u_x(0) = 0.5$. We see that $w(x)$ is negative in some neighbourhood $x = 0$ but changes sign for some $x \in [8;9]$.*

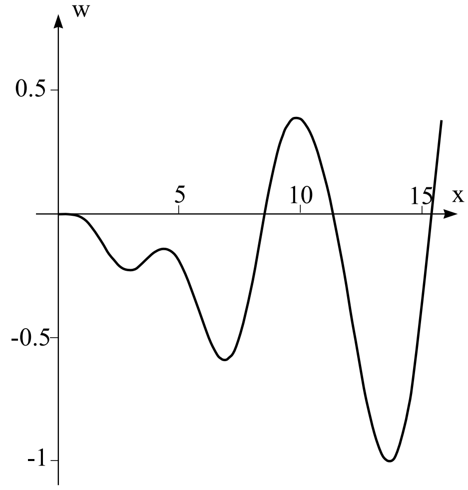

Figure 11. Comment on Comparison Lemma: the importance of the monotonicity of $f(x,\xi)$. The graph of $w(x)$, the difference of solutions to the equations $u_{xx} + u = 0$ and $z_{xx} + z + z^2 = 0$ with the same initial condition $u(0) = z(0) = 0$, $u_x(0) = z_x(0) = 0.5$

BIBLIOGRAPHY

1. Yu.S. Kivshar, G.P. Agraval. *Optical solitons. From fibers to photonic crystals.* Academic press, Amsterdam (2003).
2. L. Pitaevskii, S. Stringari. *Bose–Einstein condensation.* Clarendon Press, Oxford (2003).
3. G.L. Alfimov, A.I. Avramenko. *Coding of nonlinear states for the Gross-Pitaevskii equation with periodic potential* // Physica D. **254**, 29–45 (2013).

4. J. Guckenheimer, P.J. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields.* Springer, New York (1983).
5. G.L. Alfimov, M.E. Lebedev. *On regular and singular solutions for equation* $u_{xx}+Q(x)u+P(x)u^3 = 0$ // Ufimskij Matem. Zhurn. **7**:2, 3–18 (2015). [Ufa Math. J. **7**:2, 3–16 (2015).]
6. P. Hartman. *Ordinary differential equations.* John Wiley and Sons, New York (1964).
7. P.S. Aleksandrov. *Introduction to set theory and general topology*, Nauka, Moscow (1977). (in Russian.)
8. G.L. Alfimov, D.A. Zezyulin. *Nonlinear modes for the Gross-Pitaevskii equation – a demonstrative computational approach* // Nonlinearity. **20**:9, 2075–2092 (2007).
9. F.K. Abdullaev, M. Salerno. *Gap-Townes solitons and localized excitations in low-dimensional Bose-Einstein condensates in optical lattices* // Phys. Rev. A. **72**:3, id 033617 (2005).

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