ISSN 2304-0122 Ufa Mathematical Journal. Vol. 8. No 2 (2016). P. 44-57.

doi:10.13108/2016-8-2-44 UDC 517.954+517.956.45+517.958:531.72

UNIQUENESS OF THE RENORMALIZED SOLUTIONS TO THE CAUCHY PROBLEM FOR AN ANISOTROPIC PARABOLIC EQUATION

F.KH. MUKMINOV

Abstract. We consider the Cauchy problem for a certain class of anisotropic parabolic second-order equations with double non-power nonlinearities. The equation contains an "inhomogeneity" in the form of a non-divergent term depending on the sought function and spatial variables. Non-linearities are characterized by N-functions, for which Δ_2-condition is not imposed. The uniqueness of renormalized solutions in Sobolev-Orlich spases is proved by the S.N.Kruzhkov method of doubling the variables.

Keywords: anisotropic parabolic equation, renormalized solution, non-power nonlinearities, N-functions, uniqueness of solution.

Mathematics Subject Classification: 35D05; 35K55; 35B50; 35B45; 35B05

1. Introduction

In a layer $D^T = (0,T)\times \mathbb{R}^n,\ n \geqslant 2$, we consider Cauchy problem for the equations of the form

$$(\beta(x,u))'_t = \operatorname{div} a(x,u,\nabla u) + b(t,x,u,\nabla u),\ a = (a_1,\ldots,a_n), \tag{1.1}$$

$$\beta(x,u(0,x)) = \beta(x,u_0(x)), \tag{1.2}$$

where $\beta(x,u)$ is a non-decreasing and continuous w.r.t. u function measurable w.r.t. x.

As a model example of the considered equations, the equation

$$(\beta(u))_t = \sum_{i=1}^{n}(B'_i(u_{x_i}) + \Psi_i(x))_{x_i} + \Phi(x) \tag{1.3}$$

serves, where B_i are N-functions (see [1]).

In work [2] P.A. Raviart showed first the existence of a solution with a double nonlinearity:

$$(|u|^{\alpha-2}u)_t = \sum_{i=1}^{n}(|u_{x_i}|^{p-2}u_{x_i})_{x_i} + f(t,x), \quad \alpha, p > 1, \nabla u_0(x) \in L_p, \tag{1.4}$$

in a bounded domain $D^T = (0,T)\times\Omega$.

A. Bamberger [3] proved the uniqueness of solution to equation (1.4) in the case, when $\alpha \in (1,2)$ under the assumption $(\beta)'_t \in L_1(D^T)$, $u_0 \geqslant 0$. However, he did not succeed to prove the existence of such solution for $\alpha \in (1,2)$.

H.W. Alt, S. Luckhaus [4] proved the existence and the uniqueness of the solution to the equation $(\beta(u))'_t = \operatorname{div} a(\beta(u),\nabla u)$ in the case $\alpha \geqslant 2$ under the assumption $(\beta)'_t \in L_1(D^T)$. Similar results for equation (1.1) written in another form were established in [5, 6]. The

F.Kh. Mukminov, Uniqueness of the renormalized solutions to the Cauchy problem for an anisotropic parabolic equation.

The work is supported by RFBR (grant no. 13-01-00081-a).

Submitted Feburary 6, 2016.

condition $(\beta)'_t \in L_1(D^T)$ was weakened to $\beta \in L_1(D^T)$ in work by F. Otto [7] in the proof the uniqueness of the solution.

In work [8] authors showed that it is necessary to extend the notion of a solution in the case of the equation $\triangle_p u = F(x,u)$ in L_1 with data $\sup_{|u|<c} F(x,u) \in L_{1,\mathrm{loc}}(\Omega)$. Namely, they considered entropy solution of the Dirichlet problem for an elliptic equation and proved the existence and uniqueness. Authors point out that instead of the entropy solution introduced first by S.N. Kruzhkov [9] for first order equations, one can consider also renormalized solutions. The notion of the renormalized solution was introduced first in work by R.J. DiPerna and P.-L. Lions [10] in studying Cauchy problem for Boltzman equation.

D. Blanchard and F. Murat [11] proved the uniqueness of the renormalized solution to the equation $u_t - \mathrm{div} a(t,x,\nabla u) = f$. For the same equation, A. Prignet [12] proved the uniqueness of the entropy solution and showed its equivalence to the renormalized solution.

A much stronger statement, the uniqueness of the renormalized solution to elliptic-parabolic problem for the equation with power nonlinearities $(\beta(u))'_t = \mathrm{div} a(u,\nabla u)$ was formulated by J. Carrillo, P. Wittbold in [13] but the proof has an essential gap. The proof employs the method of doubling variables proposed by S.N. Kruzhkov in [9].

In the present work by the method of doubling variables we prove the uniqueness of the renormalized solution to Cauchy problem (1.1),(1.2) with non-power nonlinearities determined by N-functions.

In work [14], S.N. Antontsev and S.I. Shmarev proved the existence and uniqueness theorems for the generalized solution of Dirichlet problem for parabolic equations of the form (1.1) in particular case $\beta = u$, $a = u^{\gamma(t,x)}\nabla u$. The uniqueness of the renormalized solution to the first mixed problem in a bounded domain for isotropic equation (1.1) with non-power nonlinearities was proved by H. Redwane in [15] under the strong restriction $0 < c < \beta'_u < C(K)$, $\nabla_x \beta'_u < C(K)$, $|u| < K$. Under the same restrictions, the existence of the renormalized solution was proved in work [16]. The existence and uniqueness of the renormalized solution to the first mixed problem in a bounded domain for equation (1.1) with $\beta = u$ and variable nonlinearities was proved in works by Ch. Zhang, Sh. Zhou [17] and by M. Bendahmane, P. Wittbold [18].

In works [19], [20], there were considered equation of the form (1.1) with non-power nonlinearities in a cylindrical domain with an unbounded base under the assumption that the initial function belongs to some Sobolev-Orlicz space. In [19], as $b \equiv 0$ and β'_u is bounded in the vicinity $u \in (-\delta,\delta)$, the existence of a solution to a model equation was proved and power in t upper and lower bounds for the decay rate of a solution were established for large t. In [20] there was proved the existence of a generalized solution to the first mixed problem under the condition of a strong monotonicity (i.e., of the whole operator in the right hand side of the equation). In the case of power nonlinearities, sharp estimate for the decay rate of a solution to an anisotropic parabolic equation with a double nonlinearity were established in [21].

2. Functional spaces and main assumptions

Here we define functional spaces used in the work and provide some known facts from the theory of Sobolev-Orlicz spaces (see also [22]).

We introduce the following notations:

$$\langle F(t)\rangle = \int\limits_{\Omega} F(t,x)dx, \quad [F] = \int\limits_{D^T} F(t,x)dxdt,$$

where, as a rule, $\Omega = \mathbb{R}^n$, but other domains are also possible. The value of a generalized function σ on an element $\phi \in C_0^\infty(D^T)$ will be written as $\sigma(\phi) = (\sigma,\phi)_{D^T}$.

For convex domains $B(s)$, $s \geqslant 0$, the function

$$\overline{B}(z) = \sup_{s\geqslant 0}(s|z| - B(s))$$

is called additional. The following property of the additional functions

$$|zs| \leqslant B(z) + \overline{B}(s)$$

is obvious (Young inequality). A convex function $B(s)$, $s \geqslant 0$, is called N-function if

$$\lim_{s\to 0} B(s)/s = 0, \quad \lim_{s\to\infty} B(s)/s = \infty.$$

We shall say that N-function $B(s)$ satisfies Δ_2-condition if there exist numbers $s_0, k > 0$ such that $B(2s) \leqslant kB(s)$ for all $s \geqslant s_0$.

In the present work we do not assume that the used N-functions satisfy Δ_2-condition.

All constants in the work are positive.

By $L_B(Q)$ we denote the Orlicz space corresponding to N-function $B(s)$ with the Luxembourgh norm

$$\|u\|_{L_B(Q)} = \|u\|_{B,Q} = \inf\left\{k > 0 : \int\limits_Q B\left(\frac{u(x)}{k}\right) dx \leqslant 1\right\}.$$

In what follows as Q, domains $\mathbb{R}^n$, D^T and others can serve, at that, subscript $Q = \mathbb{R}^n$ can be omitted.

By $Lip_0(Q)$ we denote the space of Lipschitz functions with a compact support lying in Q. The closure of space $Lip_0(Q)$ in $L_B(Q)$ will be denoted by $E_B(Q)$. We define anisotropic Sobolev-Orlicz spaces $W^1_{\mathbf{LB}}(\mathbb{R}^n)$ as the set all the elements $\theta = (v_1, v_2, \ldots, v_n) \in \prod_{i=1}^n L_{B_i}(\mathbb{R}^n)$, for which there exist sequences $\varphi_m \in Lip_0(\mathbb{R}^n)$ such that $\nabla\varphi_m \to \theta$ weakly as sequences of functionals on $\prod_{i=1}^n E_{B_i}(\mathbb{R}^n)$. We shall assume the following condition for the set of N-functions B_i: *for each $\theta \in W^1_{\mathbf{LB}}(\mathbb{R}^n)$, there exists a potential $v \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ such that $\nabla v = \theta$.* A sufficient condition for this is the existence of N-function G such that for all $\varphi \in Lip_0(\mathbb{R}^n)$ the inequality

$$\|\varphi\|_{L_G(\mathbb{R}^n)} \leqslant \sum_{i=1}^n \|\varphi_{x_i}\|_{B_i,\mathbb{R}^n} \tag{2.1}$$

holds true. Indeed, as one can see easily, under the weak convergence $\nabla\varphi_m \to \theta$, this inequality implies $*$-weak convergence $\varphi_m \to v \in L_G(\mathbb{R}^n)$ and identity $\nabla v = \theta$. We note that inequality of the form (2.1) was established in work [23] under the assumption that the following integral

$$\int\limits_0^1 \frac{\Theta(s)}{s} ds, \quad \Theta(s) = s^{-\frac{1}{n}} \prod_{i=1}^n \left(B_i^{-1}(s)\right)^{\frac{1}{n}}$$

converges.

Space $W^1_{\mathbf{LB}}(D^T)$ with the norm

$$\|u\|_{W^1_{\mathbf{B}}(D^T)} = \sum_{i=1}^n \|u_{x_i}\|_{B_i,D^T}$$

is introduced by analogy with one described above and in what follows we denote it by X.

Let $\chi(P)$ stand for a logical function being 1 as P is true and 0 as P is false.

We provide the conditions for the functions involved in equation (1.1). Function $\beta(x,u)$, $\beta(x,0) = 0$, satisfies Carathéodory condition and does not decrease in u. Functions $a_i(x,u,s,)$, $s \in \mathbb{R}^n$ satisfy Caratheodory condition as well.

We suppose the existence of a continuous function $C(R,N)$ such that

$$\overline{B_i}(a_i(x,r,p)) \leqslant C(R,N)(1 + S(p)), S(p) = \sum_{i=1}^n B_i(p_i); \tag{2.2}$$

$$(a(x,r,p) - a(x,\widetilde{r},q)) \cdot (p-q) + C(R,N)(1 + S(p) + S(q))|r - \widetilde{r}| \geqslant 0 \tag{2.3}$$

for all $r, \widetilde{r} \in [-N, N]$, $p, q \in \mathbb{R}^n$, $|x| < R$. This condition is similar to the condition in work [13] with continuous functions $C(N)$ and vector functions Γ, $\widetilde{\Gamma}$:

$$(a(r,p) - a(\widetilde{r}, q)) \cdot (p - q) + C(N)(1 + |p|^k + |q|^k)|r - \widetilde{r}| \geqslant \Gamma(r, \widetilde{r})p + \widetilde{\Gamma}(r, \widetilde{r})q,$$

for all $r, \widetilde{r} \in [-N, N]$, $p, q \in \mathbb{R}^n$.

3. Formulation of main results

We introduce the functions

$$T_k(v) = \begin{cases} k & \text{if } v > k, \\ v & \text{if } |v| \leqslant k, \\ -k & \text{if } v < -k; \end{cases} \qquad \eta(r) = \begin{cases} 0 & \text{if } r > 1, \\ 1 - r & \text{if } 0 \leqslant r \leqslant 1, \\ 1 & \text{if } r < 0; \end{cases}$$

$H_\varepsilon(r) = 1 - \eta(r/\varepsilon)$.

Definition 1. *A renormalized solution to Cauchy problem* (1.1), (1.2) *is a measurable function* $u : D^T \to \mathbb{R}$ *such that*

$$1) \;\; \beta(x, u) \in L_1(D^T), \;\; f(t, x) = b(t, x, u, \nabla u) \in L_1(D^T),$$
$$2) \;\; T_k(u) \in X \quad \textit{for all} \quad k > 0;$$

and function $A(t, x) = a(x, u, \nabla u)$ *satisfies the conditions*

for all $h \in Lip_0(\mathbb{R})$, $\xi \in C_0^1((-1, T) \times \mathbb{R}^n)$ *the identity holds:*

$$[\xi_t \int\limits_{u_0}^{u} h(r) d\beta(x, r) + \xi f h(u)] = [A \cdot \nabla(h(u)\xi)]; \tag{3.1}$$

$$[\chi(m \leqslant |u| \leqslant m + 1)|A \cdot \nabla u|] \to 0 \;\; \textit{as } m \to \infty; \tag{3.2}$$

$$[\chi(|u| \leqslant k)|A(t, x)|\chi(m < |x| < m + 1)] \to 0 \;\; \textit{as } m \to \infty; \tag{3.3}$$

$$\lim_{l \to \infty} l \int\limits_0^{1/l} \langle \eta(|x| - N)|\beta(x, u(t)) - \beta(x, u_0)|\rangle dt = 0. \tag{3.4}$$

for all $k, N > 0$.

Remark 1. *Stiltjes integral in formula* (3.1) *is calculated for fixed* x.

Remark 2. *It follows from conditions* (2.2) *and 2) that*

$$\overline{B_i}(a_i(x, u, \nabla u))\chi(|u| \leqslant k)\chi(|x| \leqslant R) \in L_1(D^T). \tag{3.5}$$

We introduce the multi-valued function: $\text{sign}^+ r = 1$ as $r > 0$, $\text{sign}^+ r = 0$ as $r < 0$ and $\text{sign}^+ r = [0, 1]$ as $r = 0$; $r^+ = \max(r, 0)$.

Theorem 1. *Assume that conditions* (2.2) *and* (2.3) *hold true. Let for* $i = 1, 2$, *the functions* $u_{0i} : \mathbb{R}^n \to \mathbb{R}$ *are such that* $\beta(x, u_{0i}) \in L_1(\mathbb{R}^n)$. *Let* u_i *are renormalized solution to Cauchy problem* (1.1), (1.2) *with* u_{0i}, b_i. *Then there exists a function* $G(t, x) \in \text{sign}^+(u_1 - u_2)$ *such that for all* $\alpha(t) \in Lip_0(-1, T)$, $\alpha(0) = 1$, $\alpha \geqslant 0$ *the inequality*

$$-[\alpha'(\beta(x, u_1(t)) - \beta(x, u_2(t)))^+] \leqslant \langle(\beta(x, u_{01}) - \beta(x, u_{02}))^+\rangle + [\alpha G(f_1 - f_2)], \tag{3.6}$$

holds true, where $f_i = b_i(t, x, u_i, \nabla u_i)$. *In particular, if* $\beta(x, u_{0i}) = \beta(x, u_0)$, $b_i = b = b(x, u)$ *and function* $b(x, u)$ *does not increase w.r.t.* u, *then* $\beta(x, u_1) = \beta(x, u_2)$.

In what follows we provide Lemma 1, which implies easily that a generalized solution satisfying condition (3.3) is renormalized.

As a generalized solution to Cauchy problem (1.1), (1.2) with $\beta(x,u_0)\in L_1(\mathbb{R}^n)$, we call a function $u\in X$ such that

$$\beta(x,u)\in L_1(D^T),\ f(t,x)=b(t,x,u,\nabla u),\overline{B}_i(a_i(x,u,\nabla u))\in L_1(D^T),$$

satisfying condition (3.4) and identity

$$[(\beta(x,u)-\beta(x,u_0))\varphi_t+f\varphi]=\sum_{i=1}^n[a_i\varphi_{x_i}]$$

for all $\varphi\in C_0^1((-1,T)\times\mathbb{R}^n)$. In particular, the last relation yields that $\beta(x,u)_t\in X'+L_1(D^T)$.

The following lemma is close to the corresponding statement in work [13]. In distinction to [13], in the present work domain $\Omega=\mathbb{R}^n$ is unbounded and this is why for the completeness of the presentation, in the appendix we prove it.

Lemma 1. *Let $u\in X$ be a generalized solution to Cauchy problem* (1.1), (1.2). *Then*

$$[(\beta(x,u)-\beta(x,u_0))(h(u)\varphi)_t]=[\varphi_t\int\limits_{u_0}^{u}h(r)d\beta(x,r)] \tag{3.7}$$

for all $h\in Lip_0(\mathbb{R})$ and $\varphi\in C_0^1((-1,T)\times\mathbb{R}^n)$.

4. Auxiliary statements

We provide some notations which will be employed later in the doubling variables method.

We let $\chi(f\neq 0\wedge\varepsilon):=\chi((f\neq 0)\wedge(f\neq\varepsilon))$, $\chi(f=0\vee\varepsilon):=\chi((f=0)\vee(f=\varepsilon))$, where as f, various functions can serve and $\varepsilon>0$. Let ρ_m be the averaging kernel in $\mathbb{R}^n$, $\langle\rho_m\rangle=1$, $|\rho_m|\leqslant Cm^n$, $\rho_m(x)=0$ as $m|x|>1$. For a measurable function v we let $\chi_m(x,v):=\langle\rho_m(x-y)\chi(0<v(y)<\varepsilon)\rangle_y$, $K(x,v):=\lim\limits_{m\to\infty}\chi_m(x,v)$. The limit exists almost everywhere in $\mathbb{R}^n$ and $0\leqslant K(x,v)\leqslant 1$. In the cases when the dependence of function K on the arguments is inessential, we shall not write them.

We recall that x is called a Lebesgue point of a summable function v if

$$\lim_{m\to\infty}\langle m^{-n}\chi(m|x-y|<1)|v(y)-v(x)|)\rangle_y=0.$$

Lemma 2. *If x is a Lebesgue point of a bounded in $\mathbb{R}^n$ measurable function v, then*

$$K(x,v)\chi(v(x)\neq 0\wedge\varepsilon)=\chi(0<v(x)<\varepsilon),\ \varepsilon>0.$$

Proof. Let us find $\delta(x)>0$ such that

$$\chi(0<v(y)<\varepsilon)\chi(|v(y)-v(x)|<\delta)\chi(v(x)\neq 0\wedge\varepsilon)=\chi(0<v(x)<\varepsilon)\chi(|v(y)-v(x)|<\delta),$$

for all $y\in\mathbb{R}^n$. Since x is a Lebesgue point of function v, then

$$\lim_{m\to\infty}\langle\rho_m(x-y)\chi(|v(y)-v(x)|\geqslant\delta)\rangle_y\leqslant\lim_{m\to\infty}\langle\rho_m(x-y)|v(y)-v(x)|/\delta)\rangle_y=0.$$

Therefore,

$$\begin{aligned}&\lim_{m\to\infty}\chi_m(x,v)\chi(v(x)\neq 0\wedge\varepsilon)\\&\quad=\lim_{m\to\infty}\langle\rho_m(x-y)\chi(0<v(y)<\varepsilon)\chi(|v(y)-v(x)|<\delta)\rangle_y\chi(v(x)\neq 0\wedge\varepsilon)\\&\quad=\lim_{m\to\infty}\langle\rho_m(x-y)\chi(|v(x)-v(y)|<\delta)\rangle\chi(0<v(x)<\varepsilon)=\chi(0<v(x)<\varepsilon).\end{aligned}$$

□

In the next statement, Ω is an arbitrary domain in $\mathbb{R}^n$ (it is possible that $\Omega=\mathbb{R}^n$).

Lemma 3. *Let v_1, v_2 be bounded in Ω measurable functions and $f \in C_0^\infty(\Omega)$, $g \in L_{1,\mathrm{loc}}(\Omega)$. Then*

$$\begin{aligned}\lim_{m\to\infty} &\langle\langle \rho_m(x-y)f(y)g(x)\chi(0 < v_1(y) - v_2(x) < \varepsilon)\rangle_y\rangle_x \\ =&\langle K(x, v_1 - v_2(x))f(x)g(x)\chi(v_1(x) - v_2(x) = 0 \vee \varepsilon\rangle_x \\ &+ \langle f(x)g(x)\chi(0 < v_1(x) - v_2(x) < \varepsilon)\rangle_x.\end{aligned}$$

Proof. We have:

$$\begin{aligned}&\langle\langle \rho_m(x-y)f(y)g(x)\chi(0 < v_1(y) - v_2(x) < \varepsilon)\rangle_y\rangle_x \\ =&\langle\langle \rho_m(x-y))(f(y) - f(x))\chi(0 < v_1(y) - v_2(x) < \varepsilon)\rangle_y g(x)\rangle_x \\ &+ \langle \chi_m(x, v_1 - v_2(x))f(x)g(x)\rangle_x = I_1 + I_2.\end{aligned}$$

In view of the continuity and boundedness of function f, by the Lebesgue theorem on the majorized convergence,

$$\lim_{m\to\infty} |I_1| \leqslant \lim_{m\to\infty} \langle\langle \rho_m(x-y)|f(y) - f(x)|\rangle_y |g(x)|\rangle_x = 0.$$

Letting then $\Delta(x) = v_1(x) - v_2(x)$, we have

$$I_2 = \langle \chi_m(x, v_1 - v_2(x))f(x)g(x)(\chi(\Delta \neq 0 \wedge \varepsilon) + \chi(\Delta = 0 \vee \varepsilon))\rangle_x = I_{21} + I_{22}.$$

It remains to note that by Lemma 2,

$$\begin{aligned}\lim_{m\to\infty} I_{21} = &\lim_{m\to\infty} \langle \chi_m(x, v_1 - v_2(x))f(x)g(x)(\chi(\Delta \neq 0 \wedge \varepsilon)\rangle_x \\ =&\langle f(x)g(x)\chi(0 < v_1(x) - v_2(x) < \varepsilon)\rangle_x.\end{aligned}$$

□

Remark 3. *By passing to the limit, we establish Lemma 3 also for functions $f \in E_B(\Omega)$, $g \in L_{\overline{B}}(\Omega)$ for arbitrary N-function B.*

Remark 4. *Lemma 3 remains true as $\varepsilon = \infty$ and $\Omega = (0, T)$.*

Lemma 4. *Let u_i, $i = 1, 2$, be a renormalized solution to Cauchy problem with initial functions u_{0i} and $A_i = a(x, u_i, \nabla u_i)$, $f_i = b_i(t, x, u_i, \nabla u_i)$. Then there exists a function $G(t, x) \in \operatorname{sign}^+(u_1 - u_2)$ such that*

$$\begin{aligned}-\Big[\chi(u_1 > u_2)\xi_t \int_{u_2}^{u_1} h(r)d\beta(x, r)] + [\chi(u_1 > u_2)((h(u_1)A_1 - -h(u_2)A_2) \cdot \nabla\xi\Big] \\ + [\chi(u_1 > u_2)\xi(h'(u_1)A_1 \cdot \nabla u_1 - h'(u_2)A_2 \cdot \nabla u_2)] \leqslant [\xi G(h(u_1)f_1 - h(u_2)f_2)]\end{aligned} \tag{4.1}$$

for all non-negative $h \in Lip_0(\mathbb{R})$ and non-negative $\xi \in Lip_0((0, T) \times \mathbb{R}^n)$.

Proof. We choose two different pairs of variabls (t, x), (s, y) and we consider functions u_1, A_1, f_1 as functions of (s, y) and u_2, A_2, f_2 as functions of (t, x). Let ρ_m be the averaging kernel in $\mathbb{R}^n$, ϱ_l be the averaging kernel in $\mathbb{R}$. We let

$$\rho_{lm}(z) = \rho_{lm}(t, x, s, y) = \rho_m(x - y)\varrho_l(s - t), \quad \xi_{lm}(z) = \xi(t, x)\rho_{lm}(z).$$

We substitute function $h(r)H_\varepsilon(r - u_2(t, x))$ in definition (3.1) of renormalized solution u_1 instead of h:

$$\begin{aligned}&[(\xi_{lm})_s \int_{u_{01}}^{u_1} h(r)H_\varepsilon(r - u_2(t, x))d\beta(y, r) + \xi_{lm}f_1 h(u_1)H_\varepsilon(u_1 - u_2(t, x))]_{s,y} \\ &\quad= [A_1 \cdot \nabla_y(h(u_1)H_\varepsilon(u_1 - u_2(t, x))\xi_{lm})]_{s,y}.\end{aligned}$$

For u_2 a similar relation

$$\left[(\xi_{lm})_t\int\limits_{u_{02}}^{u_2} h(r)H_\varepsilon(u_1(s,y)-r)d\beta(x,r)+\xi_{lm}f_2h(u_2)H_\varepsilon(u_1(s,y)-u_2)\right]_{t,x}$$
$$=[A_2\cdot\nabla_x(h(u_2)H_\varepsilon(u_1(s,y)-u_2)\xi_{lm})]_{t,x}$$

holds true. Integrating these relations in (t,x) and (s,y), respectively, and calculating their difference, by using the notation $\{g\}=\int_{D^T\times D^T} gdtdxdsdy$ we obtain

$$\begin{aligned}
&\{(\xi_{lm})_s\int\limits_{u_{01}}^{u_1} h(r)H_\varepsilon(r-u_2)d\beta(y,r)-(\xi_{lm})_t\int\limits_{u_{02}}^{u_2} h(r)H_\varepsilon(u_1-r)d\beta(x,r)\}\\
&+\{\xi_{lm}H_\varepsilon(u_1-u_2)(f_1h(u_1)-f_2h(u_2))\}\\
&=\{(h(u_1)A_1-h(u_2)A_2)\cdot(\nabla_x+\nabla_y)(H_\varepsilon(u_1-u_2)\xi_{lm})\}\\
&+\{(h'(u_1)A_1\cdot\nabla_y u_1-h'(u_2)A_2\cdot\nabla_x u_2)H_\varepsilon(u_1-u_2)\xi_{lm}\}.
\end{aligned}\tag{4.2}$$

In the last relation we used the identities of the form $\{h(u_1)A_1\cdot\nabla_x(H_\varepsilon(u_1-u_2)\xi_{lm})\}=0$.

We denote two integrals in the left hand side by I_1, I_2 and I_3, I_4 stand for the integrals in the right hand side. For these integrals we make the passages to limit in the following order: $m\to\infty$, $\varepsilon\to 0$ and $l\to\infty$. In particular, for I_2 we have

$$\lim_{m,\varepsilon} I_2=[\int\limits_0^T \xi(t,x)\varrho_l\chi(u_1(s,x)>u_2(t,x))(f_1h(u_1)-f_2h(u_2))ds]_{t,x}.$$

By Lemma 3 with $\varepsilon=\infty$ and interval $(0,T)$ instead of Ω we obtain

$$\begin{aligned}
\lim_{m,\varepsilon,l} I_2 =&[\xi(t,x)(K_1(t,x,u_1-u_2)\chi(u_1=u_2)+\chi(u_1>u_2))(f_1h(u_1)\\
&-f_2h(u_2))]_{t,x}=[\xi(t,x)G(f_1h(u_1)-f_2h(u_2))]_{t,x}.
\end{aligned}\tag{4.3}$$

Here we used the notation $G=K_1\chi(u_1=u_2)+\chi(u_1>u_2)$, where K_1 is a function in Lemma 3.

In the same way, taking into consideration that the gradient vanishes almost everywhere on the level set of the function,

$$\lim_{m,\varepsilon,l} I_4=[\chi(u_1>u_2)\xi(h'(u_1)A_1\cdot\nabla u_1-h'(u_2)A_2\cdot\nabla u_2)].\tag{4.4}$$

We proceed to I_1. Consider the integral

$$\begin{aligned}
J=&\{(\xi_{lm})_s\int\limits_{u_{01}}^{u_1} h(r)H_\varepsilon(r-u_2)d(\beta(y,r)-\beta(x,r))\}\\
=&\{(\xi_{lm})_s\int\limits_{u_2}^{u_1} h(r)H_\varepsilon(r-u_2)d(\beta(y,r)-\beta(x,r))\}\\
&+\{(\xi_{lm})_s\int\limits_{u_{01}}^{u_2} h(r)H_\varepsilon(r-u_2)d(\beta(y,r)-\beta(x,r))\}=J_1+J_2.
\end{aligned}$$

It is clear that $J_2=0$ since $\xi_{lm}\Big|_{s=0}^{s=T}=0$ for sufficiently large l, while the other factor in integral is independent of s. We let $\Phi(y,v,k)=\int\limits_k^v h(r)H_\varepsilon(r-k)d\beta(y,r)$. Then

$$J_1=\{(\xi_{lm})_s(\Phi(y,u_1,u_2)-\Phi(x,u_1,u_2))\}$$

$$
\begin{aligned}
=&\{(\xi_{lm})_s(\Phi(y,u_1(s,y),u_2(t,x))-\Phi(y,u_1(s,y),u_2(t,y)))\}\\
&+\{(\xi_{lm})_s(\Phi(y,u_1(s,y),u_2(t,y))-\Phi(x,u_1(s,x),u_2(t,x)))\}\\
&+\{(\xi_{lm})_s(\Phi(x,u_1(s,x),u_2)-\Phi(x,u_1(s,y),u_2))\}=J_{11}+J_{12}+J_{13}.
\end{aligned}
$$

In view of the boundedness and continuity of function Φ w.r.t. the second and third arguments

$$\lim_m J_{11}=\lim_m J_{12}=\lim_m J_{13}=0.$$

Since $(\partial_s+\partial_t)\rho_{lm}=0$, then

$$
\begin{aligned}
&\lim_{m,\varepsilon,l} I_1=\lim_{m,\varepsilon,l}(I_1-J)=\lim_{m,\varepsilon,l}\Big(\{(\xi_{lm})_s\int\limits_{u_2}^{u_1} h(r)H_\varepsilon(r-u_2)d\beta(x,r)\}\\
&-\{(\xi_{lm})_t\int\limits_{u_1}^{u_2} h(r)H_\varepsilon(u_1-r)d\beta(x,r)\}\Big)\qquad (4.5)\\
&=[\chi(u_1>u_2)\xi_t\int\limits_{u_2}^{u_1} h(r)d\beta(x,r)]_{t,x}.
\end{aligned}
$$

It remains to consider I_3. We have

$$
\begin{aligned}
I_3=&\{\varrho_l\rho_m H_\varepsilon(u_1-u_2)(h(u_1)A_1-h(u_2)A_2)\cdot\nabla_x\xi(t,x)\}\\
&+1/\varepsilon\{\chi(0<u_1-u_2<\varepsilon)\xi_{lm}(h(u_1)A_1-h(u_2)A_2)\cdot(\nabla_y u_1-\nabla_x u_2)\}\\
=&I_{31}+I_{32}.
\end{aligned}
$$

It is obvious that

$$\lim_{m,\varepsilon,l} I_{31}=[\chi(u_1>u_2)(h(u_1)A_1-h(u_2)A_2)\cdot\nabla\xi(t,x)]. \tag{4.6}$$

Employing Fubini theorem and Lemma 3, we establish that

$$
\begin{aligned}
&\lim_{m\to\infty} I_{32}-1/\varepsilon\int\limits_0^T ds[\chi(u_1(s,x)-u_2(t,x)=0\vee\varepsilon)K(x,u_1-u_2)\\
&\quad\times\xi\varrho_l(h(u_1)A_1-h(u_2)A_2)\cdot\nabla_x(u_1-u_2)]_{t,x}=\lim_{m\to\infty} I_{32}=1/\varepsilon\int\limits_0^T ds\\
&\quad\times[\chi(0<u_1(s,x)-u_2(t,x)<\varepsilon)\xi\varrho_l(h(u_1)A_1-h(u_2)A_2)\cdot\nabla_x(u_1-u_2)].
\end{aligned}
$$

We denote by M the last expression. By (4.2)–(4.6) it is sufficient to show that $\liminf_{\varepsilon,l} M\geqslant 0$. In order to do it, we note the identity

$$
\begin{aligned}
M=&1/\varepsilon\int\limits_0^T ds[\chi(0<u_1(s,x)-u_2(t,x)<\varepsilon)\xi\varrho_l((h(u_1)-h(u_2))A_1\\
&+h(u_2)(A_1-A_2))\cdot\nabla_x(u_1-u_2)]_{t,x}=M_1+M_2,
\end{aligned}
$$

in which both the integrals in the right hand side are well-defined. Indeed, since functions u_1, u_2 are close on the integration set, cutting of one function implies the cutting of the other.

Let number $N > 0$ be such that $\operatorname{supp} h \subset (-N, N)$. Then for sufficiently small ε, employing (2.2), we obtain

$$M_1 \leqslant \int_0^T ds[\chi(0 < u_1(s,x) - u_2(t,x) < \varepsilon)\xi\varrho_l$$
$$\times L_h \sum_{i=1}^n \overline{B}_i^{-1}(C(R,N)(1 + S(\nabla T_N(u_1))))(|T_N(u_1)_{x_i}| + |T_N(u_2)_{x_i}|)]_{t,x},$$

where $L_h = \sup h'$, R is the radius of the support of function ξ. For a fixed l, employing Young inequality, we establish that the integrand in the last integral belongs $L_1((0,T) \times D^T)$ and this is why integral M_1 tends to zero as $\varepsilon \to 0$. For the remaining part, in view of condition (2.3), we have

$$1/\varepsilon \int_0^T ds[\chi(0 < u_1(s) - u_2(t) < \varepsilon)\xi\varrho_l h(u_2)(A_1 - A_2) \cdot \nabla_x(u_1 - u_2)]$$
$$\geqslant -1/\varepsilon \int_0^T ds[\chi(0 < u_1(s,x) - u_2(t,x) < \varepsilon)\xi\varrho_l h(u_2)C(R,N)$$
$$\times (1 + S(\nabla_x T_N(u_1)) + S(\nabla_x T_N(u_2)))|u_1 - u_2|]_{t,x} = M_{21}.$$

Arguing as above, we obtain that $\lim\limits_{\varepsilon \to 0} M_{21} = 0$. The proof is complete. $\square$

5. Proof of uniqueness theorem

Let u_1, u_2 be renormalized solution to problem (1.1),(1.2) with initial functions u_{01}, u_{02} and functions b_1, b_2 in the right hand side. We substitute function $\xi(t,x) = \alpha_l(t)\eta(|x| - N)$, $\alpha_l(t) \in Lip_0(0,T)$, $h(r) = \eta(|r| - m)$ into (4.1). The obtained inequality can be written in the form

$$I_1 + I_2 + I_3 \leqslant I_4. \tag{5.1}$$

We let $\alpha_l(t) = \alpha(t)(1 - \eta(lt))$, $\alpha(t) \in Lip_0(-1,T)$, $0 \leqslant \alpha(t) \leqslant 1$, $\alpha(0) = 1$. We pass to the limit in (5.1) as $l \to \infty$, $N \to \infty$, $m \to \infty$ in each of the integrals. For the second integral in the left hand side we have

$$|I_2| \leqslant [(\chi(|u_1| < m+1)|A_1| + \chi(|u_2| < m+1)|A_2|)\chi(N < |x| < N+1)].$$

Employing (3.3), we establish that $\lim\limits_{l,N} I_2 = 0$. For I_3 we have

$$|I_3| = |[\xi\chi(u_1 > u_2)(h'(u_1)A_1 \cdot \nabla u_1 - h'(u_2)A_2 \cdot \nabla u_2)]|$$
$$\leqslant [\chi(m < |u_1| < m+1)|A_1 \cdot \nabla u_1|] + [\chi(m < |u_2| < m+1)|A_2 \cdot \nabla u_2|].$$

In view of (3.2), two integrals in the right hand side tends to zero as $m \to \infty$. Thus, $I_3 \to 0$.

Consider integral I_1. We assume first that $\alpha(t) = 1$ as $|t| < \delta$. Then for $l\delta > 1$ we have

$$I_1 = -[\chi(u_1 > u_2)\alpha'(t)(1 - \eta(lt))\eta(|x| - N) \int_{u_2}^{u_1} h(r)d\beta(x,r)]$$
$$- l \int_0^{1/l} \langle \chi(u_1 > u_2)\eta(|x| - N) \int_{u_2}^{u_1} h(r)d\beta(x,r)\rangle dt = I_{11} - I_{12}.$$

We introduce the notation $\Phi_N(x,v) = \eta(|x|-N)\int\limits_0^v h(r)d\beta(x,r)$. Then

$$I_{12} = l\int\limits_0^{1/l} \langle(\Phi_N(x,u_1) - \Phi_N(x,u_2))^+\rangle dt.$$

It is obvious that

$$\begin{aligned}(\Phi_N(x,u_1) - \Phi_N(x,u_2))^+ \leqslant &(\Phi_N(x,u_1) - \Phi_N(x,u_{10}))^+ \\ &+ (\Phi_N(x,u_{10}) - \Phi_N(x,u_{20}))^+ + (\Phi_N(x,u_{20}) - \Phi_N(x,u_2))^+.\end{aligned}$$

Therefore,

$$\begin{aligned}&I_{12} - \langle(\Phi_N(x,u_{10}) - \Phi_N(x,u_{20}))^+\rangle \\ &\qquad \leqslant l\int\limits_0^{1/l} \langle(\Phi_N(x,u_1) - \Phi_N(x,u_{10}))^+ + (\Phi_N(x,u_{20}) - \Phi_N(x,u_2))^+\rangle dt.\end{aligned}$$

Then

$$\begin{aligned}l\int\limits_0^{1/l} \langle(\Phi_N(x,u_1) - \Phi_N(x,u_{10}))^+\rangle dt &\leqslant l\int\limits_0^{1/l} \langle\eta(|x|-N)\Big|\int\limits_{u_{10}}^{u_1} h(r)d\beta(x,r)\Big|\rangle dt \\ &\leqslant l\int\limits_0^{1/l} \langle\eta(|x|-N)|\beta(x,u_1) - \beta(x,u_{10})|\rangle dt.\end{aligned}$$

This is why by (3.4), passing to the limit as $l\to\infty$, $N\to\infty$ in integral I_1, we get

$$\lim_{l,N} I_1 \geqslant -\,[\chi(u_1>u_2)\alpha'(t)\int\limits_{u_2}^{u_1} h(r)d\beta(x,r)] - \langle\chi(u_{10}>u_{20})\int\limits_{u_{20}}^{u_{10}} h(r)d\beta(x,r)\rangle.$$

Thus, the triple passage to the limit in (5.1) completes the proof of inequality (3.6) of the theorem. The passage to the limit to function α of general form is obvious.

Let us prove now that (3.6) implies the uniqueness of the renormalized solution to Cauchy problem. Indeed, the non-increasing of function $b(x,u)$ in u implies the inequality $G(f_1-f_2)\leqslant 0$ and this is why as $\alpha(t) = \eta(t/T)\eta(-t)$, by (3.6) we obtain that $[(\beta(x,u_1) - \beta(x,u_2))^+]\leqslant 0$ or $\beta(x,u_1)\leqslant\beta(x,u_2)$ almost everywhere in D^T. Swapping u_1 and u_2, we obtain the opposite inequality, i.e., $\beta(x,u_1)=\beta(x,u_2)$. This completes the proof of the uniqueness.

It is easy to see that the uniqueness holds true also under weaker conditions for the renormalized solution:

$$\beta(x,u)\in L_{1,\mathrm{loc}}(D^T),\ \ f(t,x) = b(t,x,u,\nabla u)\in L_{1,\mathrm{loc}}(D^T).$$

6. Appendix

The proof of Lemma 1 is implied easily by the following statement.

Lemma 5. *Let $v\in X$, $\beta(x,v)$ be a Carathéodory function non-decreasing w.r.t. v, $\beta(x,0)=0$, $\beta(x,v)\in L_{1,\mathrm{loc}}(\overline{D^T})$ and $v_0:\mathbb{R}^n\to\mathbb{R}$, $\beta(x,v_0)\in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Let $w\in X' + L_{1,\mathrm{loc}}(\overline{D^T})$ and the inequality*

$$[(\beta(x,v) - \beta(x,v_0))\varphi_t] \geqslant (\text{resp.} \leqslant)(w,\varphi)_{D^T} \tag{6.1}$$

holds true for all non-negative $\varphi \in C_0^1((-1,T)\times\mathbb{R}^n)$. *Then*

$$\left[\varphi_t \int\limits_{v_0}^{v} h(r)d\beta(x,r)\right] \geqslant (\text{resp.} \leqslant)(w, h(v)\varphi)_{D^T} \tag{6.2}$$

for all non-negative $h \in W_\infty^1(\mathbb{R})$ *and* $\varphi \in C_0^1((-1,T)\times\mathbb{R}^n)$.

Proof of Lemma 5. Since $|\int\limits_{v_0}^{v} h(r)d\beta(x,r)| \leqslant \|h\|_\infty|\beta(x,v)-\beta(x,v_0)|$, then $\int\limits_{v_0}^{v} h(r)d\beta(x,r) \in L_{1,\mathrm{loc}}(\overline{D^T})$ and the integrals in (6.2) are well-defined. It is sufficient to prove one of the inequalities of the lemma, since if v satisfies the first inequality in (6.1), then $-v$ satisfies the other with the replacement $\widetilde{\beta}(x,r) = -\beta(x,-r)$, $\widetilde{v}_0 = -v_0$ and $\widetilde{w} = -w$, respectively.

If the first inequality (6.1) holds true, it is also valid for non-negative functions $\varphi \in Y$,

$$Y = \left\{\varphi(t,x) = \int\limits_t^T z(s,x)ds | z \in X \cap L_\infty(D^T),\ \operatorname{supp} z \text{ is compact}\right\}$$

that can be established easily by passing to the limit.

We first assume that $h \geqslant 0$ does not decrease, $h \in W_\infty^1(\mathbb{R})$. It is clear that

$$\int\limits_r^s h(\tau)d\beta(x,\tau) \leqslant h(s)(\beta(x,s)-\beta(x,r))$$

for almost all $r,s \in \mathbb{R}$ and almost all $x \in \mathbb{R}^n$. Therefore, for all $t > 0$

$$\int\limits_{v(t-\eta)}^{v(t)} h(r)d\beta(x,r) \leqslant h(v(t))(\beta(x,v(t)) - \beta(x,v(t-\eta))), \tag{6.3}$$

$$\int\limits_{v(t-\eta)}^{v(t)} h(r)d\beta(x,r) \geqslant h(v(t-\eta))(\beta(x,v(t)) - \beta(x,v(t-\eta))) \tag{6.4}$$

almost everywhere in $\mathbb{R}^n$, where we let $v(t) = v_0$ as $t < 0$. Let $\varphi \in C_0^1((-\infty,T)\times\mathbb{R}^n)$, $\varphi \geqslant 0$, then $\zeta = h(v)\varphi \in X$. We note that for each small $\eta > 0$ the function $\zeta_\eta(t) = 1/\eta \int\limits_t^{t+\eta} \zeta(s)ds$, $\zeta_\eta(T) = 0$ belong to space Y. This is why we can subsitute ζ_η into (6.1). In accordance with

(6.3), we write the chain of relations

$$
\begin{aligned}
(w,\zeta_\eta)_{D^T} \leqslant & [(\zeta_\eta)_t(\beta(x,v)-\beta(x,v_0))] \\
= & \int\limits_{D^T_{-\infty}} \frac{1}{\eta}(\zeta(t+\eta)-\zeta(t))(\beta(x,v(t))-\beta(x,v_0))dxdt \\
= & \int\limits_{D^T_{-\infty}} \frac{1}{\eta}\zeta(t)(\beta(x,v(t-\eta))-\beta(x,v(t)))dxdt \\
= & \int\limits_{D^T_{-\infty}} \frac{\varphi(t)}{\eta}h(v(t))(\beta(x,v(t-\eta))-\beta(x,v(t))) \\
\leqslant & \int\limits_{D^T_{-\infty}} \frac{\varphi(t)}{\eta}\int\limits_{v(t)}^{v(t-\eta)} h(r)d\beta(x,r)dxdt = \left[\frac{\varphi(t+\eta)-\varphi(t)}{\eta}\int\limits_{v_0}^{v(t)} h(r)d\beta(x,r)\right].
\end{aligned}
\tag{6.5}
$$

Since functions $\zeta_\eta \to \zeta = h(v)\varphi$ in X, $(\varphi(t+\eta)-\varphi(t))/\eta \to \varphi_t(t)$ in $L_\infty(D^T)$ as $\eta \to 0$ and

$$[\zeta_\eta, f] = [\zeta, f_{-\eta}] \to [\zeta, f], \forall f \in L_{1,\mathrm{loc}}(D^T),$$

by passing to the limit in (6.5) we obtain (6.2).

Now assume that $h \geqslant 0$ does not increase. Let $v_{0m} \in X$, $\beta(x,v_{0m}) \to \beta(x,v_0)$ in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ as $m\to\infty$ and let m be fixed in following calculations. Substituting $\widetilde{h} = -h(r)$ into (6.4), we have

$$\int\limits_{v(t-\eta)}^{v(t)} h(r)d\beta(x,r) \leqslant h(v(t-\eta))(\beta(x,v(t))-\beta(x,v(t-\eta))) \tag{6.6}$$

for almost each $t>0$, as $\eta>0$, where as $t<0$ we define $v(t)=v_{0m}$. As above, $\zeta = h_1(v)\varphi$. Therefore, for small $\eta>0$ the function $\zeta_{-\eta}(t) = 1/\eta \int\limits_{t-\eta}^{t} \zeta(s)ds$, $\zeta_{-\eta}(T)=0$ belongs to space Y. This is why we can substitute $\zeta_{-\eta}$ into (6.1). Employing (6.6), we write the following relations

$$
\begin{aligned}
(w,\zeta_{-\eta})_{D^T} \leqslant & [(\zeta_{-\eta})_t(\beta(x,v(t))-\beta(x,v_0))] = \left[\frac{1}{\eta}(\zeta(t)-\zeta(t-\eta))(\beta(x,v(t))-\beta(x,v_0))\right] \\
= & \left[\frac{1}{\eta}\zeta(t-\eta)(\beta(x,v(t-\eta))-\beta(x,v(t)))\right] - \frac{1}{\eta}\int\limits_0^\eta \langle\zeta(t-\eta)(\beta(x,v_{0m})-\beta(x,v_0)\rangle dt \\
\leqslant & \left[\frac{\varphi(t-\eta)}{\eta}\int\limits_{v(t)}^{v(t-\eta)} h_1(r)d\beta(x,r)\right] - \frac{1}{\eta}\int\limits_{-\eta}^{0}\langle\varphi(t)h_1(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle dt \\
= & \left[\frac{\varphi(t)-\varphi(t-\eta)}{\eta}\int\limits_{v_0}^{v(t)} h_1(r)d\beta(x,r)\right] + \frac{1}{\eta}\int\limits_{-\eta}^{0}\langle\varphi(t)\int\limits_{v_0}^{v_{0m}} h_1(r)d\beta(x,r)\rangle dt \\
& - \frac{1}{\eta}\int\limits_{-\eta}^{0}\langle\varphi(t)h_1(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle dt.
\end{aligned}
$$

We note that

$$\frac{1}{\eta}\int\limits_{-\eta}^{0}\langle\varphi(t)\int\limits_{v_0}^{v_{0m}}h_1(r)d\beta(x,r)\rangle dt=\langle\varphi(0)\int\limits_{v_0}^{v_{0m}}h_1(r)d\beta(x,r)\rangle$$
$$+\frac{1}{\eta}\int\limits_{-\eta}^{0}\langle(\varphi(t)-\varphi(0))\int\limits_{v_0}^{v_{0m}}h_1(r)d\beta(x,r)\rangle dt,$$

where the last integral tends to zero as $\eta\to 0$. In the same way we have

$$\frac{1}{\eta}\int\limits_{-\eta}^{0}\langle\varphi(t)h_1(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle dt=\langle\varphi(0)h_1(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle$$
$$+\frac{1}{\eta}\int\limits_{-\eta}^{0}\langle(\varphi(t)-\varphi(0))h_1(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle dt,$$

where the last integral tends to zero as $\eta\to 0$. Now, employing that $\zeta_{-\eta}\to h_1(v)\varphi$ in X and $(\varphi(t+\eta)-\varphi(t))/\eta\to\varphi_t(t)$ in $L_\infty(D^T)$ as $\eta\to 0$, we obtain

$$(w,h_1(v)\varphi)_{D^T}\leqslant[\varphi_t\int\limits_{v_0}^{v}h_1(r)d\beta(x,r)]+\langle\varphi(0)\int\limits_{v_0}^{v_{0m}}h_1(r)d\beta(x,r)\rangle$$
$$-\langle\varphi(0)h(v_{0m})(\beta(x,v_{0m})-\beta(x,v_0))\rangle.$$

Since $\beta(x,v_{0m})\to\beta(x,v_0)$ in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$, after the passage to the limit $m\to\infty$, by the boundedness of function h_1 we obtain (6.2) in the case of a non-increasing h_1.

We rewrite (6.2) as

$$(\widetilde{w},\psi)_{D^T}\leqslant[\psi_t(\widetilde{\beta}(x,v)-\widetilde{\beta}(x,v_0))],\tag{6.7}$$

where $\widetilde{w}=wh_1(v)$, $\widetilde{\beta}(x,s)=\int\limits_0^s h_1(r)d\beta(x,r)$. Inequality (6.7) established for $\psi\in C_0^1((-1,T)\times\mathbb{R}^n)$, by the passage to the limit is extended for functions $\psi\in Y$. In particular, as it was shown above, for a non-decreasing function h_2 by (6.7) we obtain the inequality

$$(\widetilde{w},h_2(v)\varphi)_{D^T}\leqslant\left[\varphi_t\int\limits_{v_0}^{v}h_2(r)d\widetilde{\beta}(x,r)\right]$$

for each function $\varphi\in C_0^1((-1,T)\times\mathbb{R}^n)$, $\varphi\geqslant 0$ and this inequality is equivalent to (6.2) with $h=h_1h_2$. Each non-negative function $h\in W^1_\infty(\mathbb{R})$ can be approximated by a convex combination of such products. This is the lemma is true for such functions. $\square$

BIBLIOGRAPHY

1. M.A. Krasnosel'skii, Ya.B. Rutitskii. *Convex functions and Orlicz spaces.* Gostekhizdat, Moscow (1958). (in Russian).
2. P.A. Raviart. *Sur la résolution de certaines equations paraboliques non linéaires* // J. Funct. Anal. **5**:2, 209-328 (1970).
3. A. Bamberger. *Étude d'une équation doublement non linéaire* // J. Funct. Anal. **24**:2, 148-155 (1977).
4. H.W. Alt, S. Luckhaus. *Quasilinear elliptic-parabolic differential equations* // Math. Z. **183**:3, 311-341 (1983).

5. A.V. Ivanov, P.Z. Mkrtychan, W. Jäger. *Existence and uniqueness of a regular solution of the Cauchy-Dirichlet problem for a class of doubly nonlinear parabolic equations* // Zap. Nauchn. Sem. POMI. **123**, 48-65 (1994). [J. Math. Sci. **84**:1, 845-855 (1997)]
6. P.Z. Mkrtychan. *On uniqueness of solution of the second initial-boundary value problem for non-Newtonian politropical filtration equation* // Zap. Nauchn. Sem. POMI. **200**, 110-117 (1992). [J. Math. Sci. **77**:3, 3207-3211 (1995).]
7. F. Otto. *L^1-Contraction and uniqueness for quasilinear elliptic-parabolic equations* // J. Diff. Equat. 131:1, 20-38 (1996).
8. Ph. Benilan, L. Boccardo, Th. Galluet, M. Pierre, J.L. Vazquez. *An L^1-theory of existence and Uniqueness of solutions of nonlinear elliptic equations* // Ann. del. Scu. Norm. Super. di Pisa. Classe di Scienze. **22**:2, 241-273 (1995).
9. S.N. Kružkov. *First order quasilinear equations in several independent variables* // Matem. Sborn. **81(123)**:2, 228-255 (1970). [Math. USSR-Sb. **10**:2, 217-243 (1970).]
10. R.J. DiPerna and P.-L. Lions. *On the Cauchy problem for Boltzmann equations: global existence and weak stability* // Ann. Math. **130**:2, 321-366 (1989).
11. D. Blanchard, F. Murat. *Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness* // Proc. Royal Soc. Edinburgh. A. **127**:6, 1137-1152 (1997).
12. A. Prignet. *Existence and uniqueness of "entropy" solutions of parabolic problems with L^1 data* // Nonl. Anal. Th. Math. Appl. **28**:12, 1943-1954 (1997).
13. J. Carrillo, P. Wittbold. *Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems* // J. Diff. Equat. **156**:1, 93-122 (1999).
14. S.N. Antontsev , S.I. Shmarev. *Existence and uniqueness of solutions of degenerate parabolic equations with variable exponents of nonlinearity* // Fundam. Prikl. Matem. **12**:4, 3-19 (2006). [J. Math. Sci. **150**:5, 2289-2301 (2008).]
15. H. Redwane. *Uniqueness of renormalized solutions for a class of parabolic equations with unbounded nonlinearities* // Rend. Matem. Ser. VII. Roma. **28**:2, 189-200 (2008).
16. H. Redwane. *Existence results for a class of nonlinear parabolic equations in Orlicz spaces* // Elect. J. Qualit. Theor. Diff. Equat. id 2 (19 pp.) (2010).
17. Chao Zhang, Shulin Zhou. *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L_1 data* // J. Diff. Equat. **248**:6, 1376-1400 (2010).
18. M. Bendahmane, P. Wittbold, A. Zimmermann. *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L_1 data* // J. Diff. Equat. **249**:6, 1483-1515 (2010).
19. E.R. Andriyanova. *Estimates of decay rate for solution to parabolic equation with non-power nonlinearities* // Ufimskij Matem. Zhurn. **6**:2, 3-25 (2014). [Ufa Math. J. **6**:2, 3-24 (2014).]
20. E.R. Andriyanova, F.Kh. Mukminov. *Existence of solution for parabolic equation with non-power nonlinearities* // Ufimskij Matem. Zhurn. **6**:4, 32-49 (2014). [Ufa Math. J. **6**:4, 31-47 (2014).]
21. L.M. Kozhevnikova, A.A. Leontiev. *Decay of solution of anisotropic doubly nonlinear parabolic equation in unbounded domains* // Ufimskij Matem. Zhurn. **5**:1, 63-82 (2013). [Ufa Math. J. **5**:1, 63-82 (2013).]
22. L.M. Kozhevnikova, A.A. Khadzhi. *Existence of solutions of anisotropic elliptic equations with nonpolynomial nonlinearities in unbounded domains* // Matem. Sborn. **206**:8, 99-126 (2015). [Sb. Math. **206**:8, 1123-1149 (2015).]
23. A.G. Korolev. *Embedding theorems for anisotropic Sobolev-Orlicz spaces* // Vestn. Mosk. Univ. Ser. I. 1, 32-37 (1983). [Mosc. Univ. Math. Bull. **38**:1, 37-42 (1983).]

Mukminov Farit Khamzaevich
Institute of Mathematics with Computer Center,
Ufa Scientific Center, RAS
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: `mfkh@rambler.ru`