ISSN 2304-0122 Ufa Mathematical Journal. Vol. 8. No 2 (2016). P. 3-13.

doi:10.13108/2016-8-2-3 UDC 517.938

TO THE SOLUTION OF A BOUNDARY VALUE PROBLEM WITH A PARAMETER FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We propose a method for solving a boundary value problem with a parameter under the presence of constraints for state and integral constraints. We obtain the necessary and sufficient conditions solvability for the boundary value problem with a parameter for ordinary differential equations. A method for constructing the solution to the boundary value problem with a parameter and constraints is developed by constructing minimizing sequences. The basis of the proposed method for solving the boundary value problem is the immersion principle. The immersion principle is created by finding the general solution for a class of the first kind Fredholm integral equations. As an example, the solution of the Sturm-Liouville problem for a parameter value in a prescribed interval is given.

Keywords: immersion principle, optimization problem, minimizing sequences, integral equation, Sturm-Liouville problem

Mathematics Subject Classification: 34H05, 49J15

1. Formulation of the problem

We consider the following problem with a parameter

$$\dot{x} = A(t)x + B(t)f(x, \lambda, t) + \mu(t), \quad t \in I = [t_0, t_1], \tag{1}$$

subject to the boundary conditions

$$(x(t_0)) = x_0, x(t_1) = x_1) \in S \subset \mathbb{R}^{2n}, \tag{2}$$

under the presence of the constraints on the state

$$x(t) \in G(t) : G(t) = \{x \in \mathbb{R}^n / \omega(t) \leqslant F(x, \lambda, t) \leqslant \varphi(t), \quad t \in I\}, \tag{3}$$

as well as of integral constraints

$$g_j(u(x_0, x_1, \lambda) \leqslant c_j, \quad j = \overline{1, m_1}; \quad g_j(x_0, x_1, \lambda) = c_j, \quad j = \overline{m_1 + 1, m_2}, \tag{4}$$

$$g_j(x_0, x_1, \lambda) = \int\limits_{t_0}^{t_1} f_{0j}(x(t), x_0, x_1, \lambda, t)dt, \quad j = \overline{1, m_2}, \tag{5}$$

with the parameter

$$\lambda \in \Lambda \subset \mathbb{R}^s, \quad \lambda = (\lambda_1, \dots, \lambda_s). \tag{6}$$

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Here $A(t)$, $B(t)$ are $n\times n$ and $n\times m$ matrices with piece-wise continuous entries, respectively, vector function $f(x,\lambda,t)=(f_1(x,\lambda,t),\dots,f_r(x,\lambda,t))$ is continuous w.r.t. variables $(x,\lambda,t)\in\mathbb{R}^n\times\mathbb{R}^s\times I$ and satisfies Lipschitz condition in variable x, i.e.,

$$|f(x,\lambda,t)-f(y,\lambda,t)|\leqslant l(t)|x-y|,\ \ \forall(x,\lambda,t),(y,\lambda,t)\in\mathbb{R}^n\times\mathbb{R}^s\times I, \tag{7}$$

and the condition

$$|f(x,\lambda,t)|\leqslant c_0(|x|+|\lambda|^2)+c_1(t),\ \ \forall(x,\lambda,t), \tag{8}$$

where $l(t)\geqslant 0$, $l(t)\in L_1(I,\mathbb{R}^1)$, $c_0=const>0$, $c_1(t)\geqslant 0$, $c_1(t)\in L_1(I,\mathbb{R}^1)$.

We note that under conditions (7), (8), for fixed $x_0=x(t_0)\in\mathbb{R}^n$, $\lambda\in\mathbb{R}^s$ differential equation (1) has the unique solution for $t\in I$.

Vector function $F(x,\lambda,t)=(F_1(x,\lambda,t),\dots,F_S(x,\lambda,t))$ is continuous w.r.t. variables $(x,\lambda,t)\in\mathbb{R}^n\times I$. Function $f_0(x(t),x_0,x_1,\lambda,t)=(f_{01}(x,x_0,x_1,\lambda,t),\dots,f_{0m_2}(x,x_0,x_1,\lambda,t))$ is continuous and satisfies the condition

$$\begin{aligned}&|f_0(x,x_0,x_1,\lambda,t)|\leqslant c_2(|x|+|x_0|+|x_1|+|\lambda|^2)+c_3(t),\\&\forall(x,x_0,x_1,\lambda,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^s\times I,\\&c_2=const\geqslant 0,\ \ c_3(t)\geqslant 0,\ \ c_3(t)\in L_1(I,\mathbb{R}^1),\end{aligned}$$

$\omega(t)$, $\varphi(t)$, $t\in I$, are given r-dimensional continuous functions, S is a given bounded convex closed set in $\mathbb{R}^{2n}$, Λ is a given bounded convex closed set in $\mathbb{R}^s$, times t_0, t_1 are fixed, $t_1>t_0$.

We observe that if $A(t)\equiv 0$, $m=n$, $B(t)=I_n$, where I_n is the unit $n\times n$ matrix, equation (1) becomes

$$\dot{x}=f(x,\lambda,t)+\mu(t),\ \ t\in I. \tag{9}$$

This is why the results obtained in what follows remain true for equations of the form (9) under conditions (2)–(6). In particular, set S is determined by the identity

$$S=\big\{(x_0,x_1)\in\mathbb{R}^{2n}/H_j(x_0,x_1)\leqslant 0,\ j=\overline{1,p};\ <a_j,x_0>+<b_j,x_1>-e_j=0,\ j=\overline{p+1,s_1}\big\},$$

where $H_j(x_0,x_1)$, $j=\overline{1,p}$ are functions convex in variables (x_0,x_1), $x_0=x(t_0)$, $x_1=x(t_1)$, $a_j\in\mathbb{R}^n$, $b_j\in\mathbb{R}^n$, $e_j\in\mathbb{R}^1$, $j=\overline{p+1,s}$, are given vectors and numbers, $<\cdot,\cdot>$ is the scalar product. In particular,

$$\Lambda=\{\lambda\in\mathbb{R}^s/h_j(\lambda)\leqslant 0,\ \ j=\overline{1,p_1};\ \ <\overline{a}_j,\lambda>-\overline{e}_j=0,\ \ j=\overline{p_1+1,s_1}\},$$

where $h_j(\lambda)$, $j=\overline{1,p_1}$, are functions convex in λ, $\overline{a}_j\in\mathbb{R}^s$, $\overline{e}_j\in\mathbb{R}^1$, $j=\overline{p_1+1,s_1}$, are given vectors and numbers.

We pose the following problems.

Problem 1. *Find necessary and sufficient solvability conditions for problem (1)–(6).*

As it follows from the formulation of the problem, we need to the prove the existence of a pair $(x_0,x_1)\in S$ and parameter $\lambda\in\Lambda$ such that the solution to system (1) leaving point x_0 at time t_0 passes point x_1 at time t_1; at that, along the solution to system (1), where $x(t)=x(t;x_0,t_0,\lambda)$, $t\in I$, $x(t_0)=x_0$, $x(t_1)=x_1$, constraint (3) on the state holds true at each time and integrals (5) satisfy conditions (4).

Problem 2. *Construct solution to problem (1)–(6).*

In particular, once constraints on the state and integral constraints are absent, boundary value problem (1)–(6) becomes Sturm-Liouville problem. By applying Fourier method to the problems of mathematical physics we arrive at the following problem [1]: find values λ, for which there exists a non-zero solution to the homogeneous equation

$$L[y]+\lambda r(t)y(t)\equiv 0, \tag{10}$$

in the finite interval $[t_0,t_1]$ satisfying the conditions at the end-points:

$$\alpha_1 y(t_0)+\alpha_2\dot{y}(t_0)=0,\ \ \beta_1 y(t_1)+\beta_2\dot{y}(t_1)=0, \tag{11}$$

where $L[y] = \dfrac{d}{dt}[p(t)\dot{y}(t)] - q(t)y(t)$, $p(t) > 0$, $t \in [t_0, t_1]$.

Introducing the notations $y(t) = x_1(t)$, $\dot{x}_1(t) = x_2(t)$, $t \in [t_0, t_1]$, equation (10) can be represented as

$$\dot{x} = A(t)x + B(t)f(x_1, \lambda, t), \quad t \in I = [t_0, t_1], \tag{12}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ \dfrac{q(t)}{p(t)} & -\dfrac{\dot{p}(t)}{p(t)} \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{r(t)}{p(t)} \end{pmatrix}, \quad f(x_1, \lambda, t) = \begin{pmatrix} 0 \\ \lambda x_1 \end{pmatrix}.$$

Boundary condition (11) casts into the form

$$\alpha_1 x_{10} + \alpha_2 x_{20} = 0, \quad \beta_1 x_{11} + \beta_2 x_{21} = 0, \tag{13}$$

where $x(t_0) = (x_{10}, x_{20})$, $x(t_1) = (x_{11}, x_{21})$, and $\lambda \in \mathbb{R}^1$. Equation (12), boundary condition (13), $\lambda \in \mathbb{R}^1$ are particular cases of (1), (2), (6), respectively.

As it is known [2], Sturm-Liouville problem (10), (11) is reduced to a homogeneous second kind integral Fredholm equation:

$$y(t) = -\lambda \int_{t_0}^{t_1} G(t, \xi) r(\xi) y(\xi) d\xi, \tag{14}$$

where $G(t, \xi)$ is the Green function. We note that it is rather complicated to construct Green function $G(t, \xi)$ and to solve integral equation (14). This is why it is topical to develop new methods for solving boundary value problems (1)–(6).

In works [3–5], there were made attempts to extend the methods of studying boundary value problems for second order linear equations to high order systems and to nonlinear systems with complicated boundary conditions. In work [3] there were proposed sufficient solvability conditions for a two-point homogeneous boundary value problem for a system of two nonlinear second differential equations; apriori estimates for the solutions were obtained. In paper [4] eigenvalue problems were considered for a quasilinear second order differential equation. The conditions for the nonlinearity ensuring the existence of multiple eigenvalues were studied. Work [5] was devoted to studying nonlinear eigenvalue problem for Sturm-Liouville operator. Boundary conditions at both end-points depend on the spectral parameter; in this case there was established the existence of the system of eigenfunctions formed a basis in space $L_p(0, 1)$, $p > 1$.

It is a topical problem to develop a general theory of boundary value problems for ordinary differential equations of arbitrary order with complicated boundary conditions under the presence of constraints on the state and integral constraints.

In many cases in practice, the studied process is described by an equation of the form (1) in a domain in the state space of the system determined by constraints on the state (3). Outside this domain the process is described by completely different equations or it does not exist. In particular, such phenomenon arises in the studies of the dynamics of nuclear and chemical reactors (outside domain (3) the reactor does not exist). Integral constraints (4) characterize the total load on the elements of the system (for instance, the total load for spacemen), which should not exceed prescribed values. Identities (4) describe the total restriction for the system, for instance, the fuel consumption is equal to a given quantity.

The basis of the proposed method for solving the boundary value problem with the parameter is the immersion principle. The matter of this principle is that the original boundary value problem with constraints is replaced by an equivalent optimal control problem with a free right end of a trajectory. Such approach is possible thanks to finding the general solution to one class of first kind Fredholm integral equation. The solvability of the original problem and constructing its solutions are made by solving a special optimal control problem. Under such approach necessary and sufficient solvability conditions for boundary value problem (1)–(6) can be obtained by the conditions of attaining the lower bound of a functional on a prescribed set, while the solution to the original boundary value problem is determined by the limiting points of the minimizing sequences. This is the principal difference of the proposed method in comparison with known methods. This work is a continuation of the studies presented in [6–12].

2. Immersion principle

Consider integral constraints (4), (5). By introducing additional variables $d = (d_1, \dots, l_{m_1}) \in \mathbb{R}^{m_1}$, $d \geqslant 0$, relations (4), (5) can be represented as

$$g_j(x_0, x_1, \lambda) = \int_{t_0}^{t_1} f_{0j}(x(t), x_0, x_1, \lambda, t)dt = c_j - d_j, \quad j = \overline{1, m_1},$$

where $d \in \Gamma = \{d \in \mathbb{R}^{m_1} / d \geqslant 0\}$. Let $\bar{c} = (\bar{c}_1, \dots, \bar{c}_{m_2})$, where $\bar{c}_j = c_j - d_j$, $j = \overline{1, m_1}$, $\bar{c}_j = c_j$, $j = \overline{m_1 + 1, m_2}$.

We introduce vector function $\eta(t) = (\eta_1(t), \dots, \eta_{m_2}(t))$, $t \in I$, where

$$\eta(t) = \int_{t_0}^{t} f_0(x(\tau), x_0, x_1, \lambda, \tau)d\tau, \quad t \in I.$$

It follows that

$$\dot{\eta}(t) = f_0(x(t), x_0, x_1, \lambda, t), \quad t \in I = [t_0, t_1].$$

Hence,

$$\dot{\eta} = f_0(x(t), x_0, x_1, \lambda, t), \quad t \in I,$$

$$\eta(t_0) = 0, \quad \eta(t_1) = \bar{c}, \quad d \in \Gamma.$$

Now original boundary value problem (1)–(6) is written as

$$\dot{\xi} = A_1(t)\xi + B_1(t)f(P\xi, \lambda, t) + B_2 f_0(P\xi, x_0, x_1, \lambda, t) + B_3\mu(t), \quad t \in I, \tag{15}$$

$$\xi(t_0) = \xi_0 = (x_0, O_{m_2}), \quad \xi(t_1) = \xi_1 = (x_1, \bar{c}), \tag{16}$$

$$(x_0, x_1) \in S, \quad d \in \Gamma, \quad P\xi(t) \in G(t), \quad t \in I, \quad \lambda \in \Lambda, \tag{17}$$

where

$$\xi = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} A(t) & O_{n,m_2} \\ O_{m_2,n} & O_{m_2,m_2} \end{pmatrix}, \quad B_1(t) = \begin{pmatrix} B(t) \\ O_{m_2,m} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} O_{n,m_2} \\ I_{m_2} \end{pmatrix}, \quad B_3 = \begin{pmatrix} I_n \\ O_{m_2,n} \end{pmatrix}, \quad P = (I_n, O_{n,m_2}), \quad P\xi = x,$$

where $O_{j,k}$ are $j \times k$ zero matrix, $O_q \in \mathbb{R}^q$ is $q \times 1$ zero vector, $\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{n+m_2})$.

The basis of the proposed method for solving Problems 1, 2 are the following theorems on the properties of solution to first kind Fredholm integral equations

$$Ku = \int_{t_0}^{t_1} K(t_0, t)u(t)dt = a, \tag{18}$$

where $K(t_0, t) = \|K_{ij}(t_0, t)\|$, $i = \overline{1, n_1}$, $j = \overline{1, s_1}$, is a given $n_1 \times s$ matrix with piece-wise continuous in t entries for a fixed t_0, $u(\cdot) \in L_2(I, \mathbb{R}^{s_1})$ is a sought function, $I = [t_0, t_1]$, $a \in \mathbb{R}^{n_1}$ is a given n_1-dimensional vector.

Theorem 1. *For each fixed $a \in \mathbb{R}^{n_1}$ integral equation (18) is solvable if and only if the $n_1 \times n_1$ matrix*

$$C(t_0, t_1) = \int_{t_0}^{t_1} K(t_0, t) K^*(t_0, t) dt, \tag{19}$$

is positive definite, where $(*)$ denotes the transposition.

Theorem 2. *Suppose that matrix $C(t_0, t_1)$ introduced by formula (19) is positive definite. Then the general solution to equation (18) has the form*

$$u(t) = K^*(t_0, t) C^{-1}(t_0, t_1) a + v(t) - K^*(t_0, t) C^{-1}(t_0, t_1) \int_{t_0}^{t_1} K(t_0, t) v(t) dt, \ \ t \in I, \tag{20}$$

where $v(\cdot) \in L_2(I, \mathbb{R}^{s_1})$ is an arbitrary function, $a \in \mathbb{R}^{n_1}$ is an arbitrary vector.

The proof of Theorems 1, 2 was given in works [6,7]. Application of Theorems 1, 2 to control problems was presented in [8-10], while the boundary value problems for ordinary differential equations were discussed in [11, 12].

Together with differential equation (15) subject to boundary conditons (16), we consider the linear controlled system

$$\dot{y} = A_1(t) y + B_1(t) w_1(t) + B_2 w_2(t) + \mu_1(t), \ \ t \in I, \tag{21}$$

$$y(t_0) = \xi_0 = (x_0, O_{m_2}), \ \ y(t_1) = \xi_1 = (x_1, \bar{c}), \tag{22}$$

$$w_1(\cdot) \in L_2(I, \mathbb{R}^m), \ \ w_2(\cdot) \in L_2(I, \mathbb{R}^m), \tag{23}$$

$$(x_0, x_1) \in S, \ \ d \in \Gamma, \tag{24}$$

where $\mu_1(t) = B_3 \mu(t)$, $t \in I$.

Assume that $\overline{B}(t) = (B_1(t), B_2)$ has the size $(n + m_2) \times (m_2 + m)$, and vector function $w(t) = (w_1(t), w_2(t))$ belongs to $L_2(I, \mathbb{R}^{m+m_2})$. It is easy to make sure that the set of all controls, each element of which maps a trajectory of system (21) from point $\xi_0 \in \mathbb{R}^n$ into point $\xi_1 \in \mathbb{R}^n$, solves the integral equation

$$\int_{t_0}^{t_1} \Phi(t_0, t) \overline{B}(t) w(t) dt = a, \tag{25}$$

where $\Phi(t_0, t) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is the fundamental matrix of solutions to the linear homogeneous system $\dot{\zeta} = A_1(t)\zeta$, and

$$a = a(\xi_0, \xi_1) = \Phi(t_0, t_1)[\xi_1 - \Phi(t_1, t_0)\xi_0] - \int_{t_0}^{t_1} \Phi(t_0, t) \mu_1(t) dt.$$

As it follows from (18), (25), integral equation (25) coincides with (18) if $K(t_0, t) = \Phi(t_0, t)\overline{B}(t)$. We introduce the notations

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t) \overline{B}(t) \overline{B}^*(t) \Phi^*(t_0, t) dt,$$

$$W(t_0, t) = \int_{t_0}^{t} \Phi(t_0, \tau) \overline{B}(\tau) \overline{B}^*(\tau) \Phi^*(t_0, \tau) dt,$$

$$W(t, t_1) = W(t_0, t_1) - W(t_0, t), \ \ E(t) = \overline{B}^* \Phi^*(t_0, t) W^{-1}(t_0, t_1),$$

$$\mu_2(t) = -E(t)\int_{t_0}^{t_1} \Phi(t_0,t)\mu_1(t)dt, \quad E_1(t) = \Phi(t,t_0)W(t,t_1)W^{-1}(t_0,t_1),$$
$$E_2(t) = \Phi(t,t_0)W(t_0,t)W^{-1}(t_0,t_1)\Phi(t_0,t_1),$$
$$\mu_3(t) = \Phi(t_0,t_1)\int_{t_0}^{t_1} \Phi(t_0,\tau)\mu_1(\tau)d\tau - E(t)\int_{t_0}^{t_1} \Phi(t_1,t)\mu_1(t)dt.$$

We calculate functions $\lambda_1(t,\xi_0,\xi_1)$, $\lambda_2(t,\xi_0,\xi_1)$, $N_1(t)$, $N_2(t)$ by the formulae:

$$\begin{aligned}
&\lambda_1(t,\xi_0,\xi_1) = E(t)a = T_1(t)\xi_0 + T_2(t)\xi_1 + \mu_2(t),\\
&\lambda_2(t,\xi_0,\xi_1) = E_1(t)\xi_0 + E_2(t)\xi_1 + \mu_3(t),\\
&N_1(t) = -E(t)\Phi(t_0,t_1), \quad N_2(t) = -E_2(t), \quad t \in I.
\end{aligned}$$

Theorem 3. *Assume that $W(t_0,t_1) > 0$. Control $w(\cdot) \in L_2(I,\mathbb{R}^{m+m_2})$ maps a trajectory of system (21) from each initial point $\xi_0 \in \mathbb{R}^{n+m_2}$ into the final state $\xi_1 \in \mathbb{R}^{n+m_2}$ if and only if*

$$\begin{aligned}
w(t) \in W =& W(v,\xi_0,\xi_1,z(t_1,v)) = \{w(\cdot) \in L_2(I,\mathbb{R}^{m+m_2})/w(t)\\
=& v(t) + \lambda_1(t,\xi_0,\xi_1) + N_1(t)z(t_1,v), \quad t \in I, \quad \forall v(\cdot) \in L_2(I,\mathbb{R}^{m+m_2})\},
\end{aligned} \tag{26}$$

where function $z(t) = z(t,v)$, $t \in I$, solves the differential equation

$$\dot z = A_1 z + \overline{B}(t)v(t), \quad z(t_0) = 0, \quad t \in I, \quad v(\cdot) \in L_2(I,\mathbb{R}^{m+m_2}. \tag{27}$$

The solution to differential equation (21) corresponding to control $w(t) \in W$ is determined by the formula

$$y(t) = z(t) + \lambda_2(t,\xi_0,\xi_1) + N_2(t)z(t_1,v), \quad t \in I. \tag{28}$$

Proof. As it follows from Theorem 1, integral equation (25) is solvable if and only if $W(t_0,t_1)$=$C(t_0,t_1) > 0$, where $K(t_0,t) = \Phi(t_0,t)\overline{B}(t)$. Then relation (20) casts into the form (26). The solution to system (21) corresponding to control (26) is determined by formula (28), where $z(t) = z(t,v)$, $t \in I$, is a solution to differential equation (27). The proof is complete. □

Lemma 1. *Let $W(t_0,t_1) > 0$. Then boundary value problem (1)-(6) (or (15)-(17)) is equivalent to the following problem*

$$w(t) = (w_1,w_2) \in W, \; w_1(t) = f(Py(t),\lambda,t), \; w_2(t) = f_0(Py(t),x_0,x_1,\lambda,t), \tag{29}$$
$$\dot z = A_1(t)z + B_1(t)v_1(t) + B_2 v_2(t), \; z(t_0) = 0, \; t \in I, \tag{30}$$
$$v(t) = (v_1(t),v_2(t)), \;\; v_1(\cdot) \in L_2(I,\mathbb{R}^m), \; v_2(\cdot) \in L_2(I,\mathbb{R}^{m_2}), \tag{31}$$
$$(x_0,x_1) \in S, \; \lambda \in \Lambda, \; d \in \Gamma, \; Py(t) \in G(t), \; t \in I, \tag{32}$$

where $v(\cdot) = (v_1(\cdot),v_2(\cdot)) \in L_2(I,\mathbb{R}^{m+m_2})$ is an arbitrary function, $y(t)$, $t \in I$ are determined by formula (28).

Proof. Under relations (29)-(32) we have $y(t) = \xi(t)_0$, $t \in I$, $Py(t) = P\xi(t) \in G(t)$, $t \in I$, $w(t) = (w_1(t),w_2(t)) \in W$. The proof is complete. □

We consder the following optimization problem: minimize the functional

$$J_1(v_1, v_2, p, d, x_0, x_1, \lambda) = \int_{t_0}^{t_1} [|w_1(t) - f(Py(t), \lambda, t)|^2$$
$$+ |w_2(t) - f_0(Py(t), x_0, x_1, \lambda, t)|^2 + |p(t) - F(Py(t), \lambda, t)|^2]dt \quad (33)$$
$$= \int_{t_0}^{t_1} F_0(t, v_1(t), v_2(t), p(t), d, x_0, x_1, \lambda, z(t), z(t_1))dt \to \infty$$

under the conditions

$$\dot{z} = A(t)z + B_1(t)v_1(t) + B_2 v_2(t), \ z(t_0) = 0, \ t \in I, \quad (34)$$
$$v_1(\cdot) \in L_2(I, \mathbb{R}^m), \ v_2(\cdot) \in L_2(I, \mathbb{R}_2^m), \ (x_0, x_1) \in S, \ \lambda \in \Lambda, \ d \in \Gamma, \quad (35)$$
$$p(t) \in V(t) = \{\omega(t) \leqslant p(t) \leqslant \varphi(t), \ \ t \in I\}, \quad (36)$$

where

$$w_1(t) = v_1(t) + \lambda_{11}(t, \xi_0, \xi_1) + N_{11}(t)z(t_1, v), \ \ t \in I, \quad (37)$$
$$w_2(t) = v_2(t) + \lambda_{12}(t, \xi_0, \xi_1) + N_{12}(t)z(t_1, v), \ \ t \in I, \quad (38)$$
$$N_1(t) = (N_{11}(t), N_{12}(t)), \ \ \lambda_1(t, \xi_0, \xi_1) = (\lambda_{11}(t, \xi_0, \xi_1), \lambda_{12}(t, \xi_0, \xi_1)).$$

We denote

$$X = L_2(I, \mathbb{R}^{m+m_2}) \times V(t) \times \Gamma \times S \times \Lambda \subset H$$
$$= L_2(I, \mathbb{R}^m) \times L_2(I, \mathbb{R}^{m_2}) \times L_2(I, \mathbb{R}^r) \times \mathbb{R}^{m_1} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s,$$
$$J_* = \inf_{\theta \in X} J(\theta), \quad \theta = (v_1, v_2, p, d, x_0, x_1, \lambda) \in X, \quad X_* = \{\theta_* \in X / J(\theta_*) = 0\}.$$

Theorem 4. *Assume that $W(t_0, t_1) > 0$, $X_* \neq \emptyset$, $\emptyset$ is the empty set. Boundary value problem (1)–(6) is solvable if and only if $J(\theta_*) = 0 = J_*$, where $\theta_* = (v_1^*, v_2^*, p_*, d_*, x_0^*, x_2^*, \lambda_*) \in X$ is the optimal control for problem (33) – (36).*

If $J_* = J(\theta_*) = 0$, then the function

$$x_*(t) = P[z(t, v_1^*, v_2^*) + \lambda_2(t, \xi_0^*, \xi_1^*) + N_2(t)z(t_1, v_1^*, v_2^*)], \ \ t \in I,$$

solves boundary value problem (1)–(6). If $J_* > 0$, then boundary value problem (1)–(6) has no solution.

Proof. Necessity. Assume that boundary value problem (1)–(6) is solvable. Then, as it follows from Lemma 1, the value $w_1^*(t) = f(Py_*(t), \lambda_*, t)$, $w_2^*(t) = f_0(Py_*(t), x_0^*, x_1^*, \ \lambda_*, t)$, where $w_*(t) = (w_1^*(t), w_2^*(t)) \in W$, $y_*(t)$, $t \in I$, is determined by the formula (28), $\xi_0^* = (x_0^*, O_{m_2})$, $\xi_1^* = (x_1^*, \overline{c}_*)$, $\overline{c}_* = (c_j - d_j^*)$, $j = \overline{1, m_1}$; c_j, $j = \overline{m_1 + 1, m_2}$. The inclusion $Py_*(t) \in G(t)$, $t \in I$ is equivalent to $p_*(t) = F(Py_*(t), \lambda_*, t)$, $t \in I$, where $\omega(t) \leqslant p_*(t) = F(Py_*(t), \lambda_*, t) \leqslant \varphi(t)$, $t \in I$. Therefore, $J(\theta_*) = 0$. The necessity is proven.

Sufficiency. Assume that $J(\theta_*) = 0$. It is possible if and only if $w_1^*(t) = f(Py_*(t), \lambda_*, t)$, $w_2^*(t) = f_0(Py_*(t), x_0^*, x_1^*, \lambda_*, t)$, $p_*(t) = F(Py_*(t), \lambda_*, t)$, $(x_0^*, x_1^*) \in S$, $v_1^*(\cdot) \in L_2(I, \mathbb{R}^m)$, $v_2^*(\cdot) \in L_2(I, \mathbb{R}^{m_2})$. The proof is complete. □

The passage from boundary value problem (1)–(6) to problem (33)–(36) is called immersion principle.

3. Optimization problem

Consider a solution to optimization problem (33)–(36). We observe that the function

$$\begin{aligned}F_0(t, v_1, v_2, p, d, x_0, x_1, \lambda) =& |w_1(t) - f(Py(t), \lambda, t)|^2 + |w_2(t) - f_0(Py(t), x_0, x_1, \lambda, t)|^2 \\ &+ |p(t) - F(Py(t), \lambda, t)|^2 = F_0(t, q), \quad q = (\theta, z, \overline{z}),\end{aligned}$$

where w_1, w_2 is determined by formulae (37), (38), respectively, and

$$y = z + \lambda_2(t, x_0, x_1, d) + N_2(t)\overline{z}, \quad Py = x.$$

Theorem 5. *Suppose that $W(t_0, t_1) > 0$, function $F_0(t, q)$ is defined and is continuously differentiable in $q = (\theta, z, \overline{z})$ and the following conditions hold:*

$$\begin{aligned}&|F_{0z}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0z}(t, \theta, z, \overline{z})| \leqslant L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|), \\ &|F_{0\overline{z}}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0\overline{z}}(t, \theta, z, \overline{z})| \leqslant L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|), \\ &|F_{0\theta}(t, \theta + \Delta\theta, z + \Delta z, \overline{z} + \Delta\overline{z}) - F_{0\theta}(t, \theta, z, \overline{z})| \leqslant L(|\Delta z| + |\Delta\overline{z}| + |\Delta\theta|), \\ &\forall\theta \in \mathbb{R}^{m+m_2+r+m_1+2n+s}, \quad \forall z \in \mathbb{R}^{n+m_2}, \quad \forall\overline{z} \in \mathbb{R}^{n+m_2}.\end{aligned}$$

Then under conditions (34) – (36) functional (33) is continuous and Fréchet differentiable at each point $\theta \in X$, and

$$J'(\theta) = (J'_{v_1}(\theta), J'_{v_2}(\theta), J'_p(\theta), J'_d(\theta), J'_{x_0}(\theta), J'_{x_1}(\theta), J'_\lambda(\theta)) \in H,$$

where

$$\begin{aligned}&J'_{v_1}(\theta) = F_{0v_1}(t, q) - B_1^*(t)\psi(t), \quad J'_{v_2}(\theta) = F_{0v_2}(t, q) - B_2^*\psi(t), \quad J'_p(\theta) = F_{0p}(t, q), \\ &J'_d(\theta) = \int_{t_0}^{t_1} F_{0d}(t, q)dt, \quad J'_{x_0}(\theta) = \int_{t_0}^{t_1} F_{0x_0}(t, q)dt, \quad J'_{x_1}(\theta) = \int_{t_0}^{t_1} F_{0x_1}(t, q)dt, \\ &J'_\lambda(\theta) = \int_{t_0}^{t_1} F_{0\lambda}(t, q)dt, \quad q = (\theta, z(t), z(t_1)),\end{aligned} \tag{39}$$

function $z(t)$, $t \in I$, is a solution to differential equation (34) as $v_1(\cdot) \in L_2(I, \mathbb{R}^m)$, $v_2(\cdot) \in L_2(I, \mathbb{R}^{m_2})$, while function $\psi(t)$, $t \in I$, is a solution to the dual system

$$\dot{\psi} = F_{0z}(t, q(t)) - A_1^*(t)\psi, \quad \psi(t_1) = -\int_{t_0}^{t_1} F_{0z(t_1)}(t, q(t))dt. \tag{40}$$

Moreover, gradient $J'(\theta)$, $\theta \in X$, satisfies the Lipschitz condition

$$\|J'(\theta_1) - J'(\theta_2)\| \leqslant K\|\theta_1 - \theta_2\|, \quad \forall\, \theta_1,\ \theta_2 \in X, \tag{41}$$

where $K > 0$ is the Lipschitz constant.

Proof. Let θ, $\theta + \Delta\theta \in X$, where $\Delta\theta = (\Delta v_1, \Delta v_2, \Delta p, \Delta d, \Delta x_0, \Delta x_1, \Delta\lambda)$. One can show that

$$\Delta\dot{z} = A_1(t)\Delta z + B_1(t)\Delta v_1 + B_2\Delta v_2, \quad \Delta z(t_0) = 0,$$

and for the increment of the functional we have

$$\begin{aligned}\Delta J =& J(\theta + \Delta\theta) - J(\theta) = < J'_{v_1}(\theta), \Delta v_1 >_{L_2} + < J'_{v_2}(\theta), \Delta v_2 >_{L_2} \\ &+ < J'_p(\theta), \Delta p >_{L_2} + < J'_d(\theta), \Delta d >_{R_{m_1}} + < J'_{x_0}(\theta), \Delta x_0 >_{R_n} \\ &+ < J'_{x_1}(\theta), \Delta x_1 >_{R_n} + < J'_\lambda(\theta), \Delta\lambda >_{R_s} + R, \quad R = \sum_{i=1}^{7} R_i,\end{aligned}$$

where $|R| \leqslant c_* \|\Delta\theta\|_X^2$, $|R|/\|\Delta\theta\|_X \to 0$ as $\|\Delta\theta\|_X \to 0$, $c_* = const > 0$. This implies relations (39), where $\psi(t)$, $t \in I$, is a solution to equation (40).

Let $\theta_1 = (v_1 + \Delta v_1, v_2 + \Delta v_2, p + \Delta p, d + \Delta d, x_0 + \Delta x_0, x_1 + \Delta x_1, \lambda + \Delta\lambda) \in X$, $\theta_2 = (v_1, v_2, p, d, x_0, x_1, \lambda)$. Since

$$|J'(\theta_1) - J'(\theta_2)| \leqslant c_0 |\Delta q(t)| + c_1 |\Delta\psi(t)| + c_2 \|\Delta\theta\|,$$
$$\Delta\dot{\psi} = [F_{0z}(t, q + \Delta q) - F_{0z}(t, q)] - A_1^*(t)\Delta\psi,$$
$$\Delta\psi(t_1) = -\int [F_{0z(t_1)}(t, q + \Delta q) - F_{0z(t_1)}(t, q)]dt,$$

the estimates $\|\Delta q\| \leqslant c_3 \|\Delta\theta\|$, $|\Delta\psi(t)| \leqslant c_4 \|\Delta\theta\|$ hold true. Then

$$\|J'(\theta_1) - J'(\theta_2)\|^2 = \int_{t_0}^{t_1} |J'(\theta_1) - J'(\theta_2)|^2 dt \leqslant K \|\Delta\theta\|^2.$$

The proof is complete. □

We employ relations (39)–(41) to construct the sequence $\{\theta_n\} = \{v_{1n}, v_{2n}, p_n, d_n, x_{0n}, x_{1n}, \lambda_n\} \subset X$ by the rule

$$\begin{aligned} & v_{1n+1} = v_{1n} - \alpha_n J'_{v_1}(\theta_n), \quad v_{2n+1} = v_{2n} - \alpha_n J'_{v_2}(\theta_n), \\ & p_{n+1} = P_V[p_n - \alpha_n J'_p(\theta_n)], \quad d_{n+1} = P_\Gamma[d_n - \alpha_n J'_d(\theta_n)], \\ & x_{0n+1} = P_S[x_{0n} - \alpha_n J'_{x_0}(\theta_n)], \quad x_{1n+1} = P_S[x_{1n} - \alpha_n J'_{x_1}(\theta_n)], \\ & \lambda_{n+1} = P_\Lambda[d_n - \alpha_n J'_\lambda(\theta_n)], \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{42}$$

where $0 < \alpha_n = \dfrac{2}{K + 2\varepsilon}$, $\varepsilon > 0$, $K > 0$ is the Lipschitz constant in (41). We introduce the following notations

$$M_0 = \{\theta \in X / J(\theta) \leqslant J(\theta_0)\}, \quad X_{**} = \{\theta_{**} \in X / J(\theta_{**}) = \inf_{\theta \in X} J(\theta)\},$$

where $\theta_0 = (v_{10}, v_{20}, p_0, d_0, x_{10}, x_{20}, \lambda_0) \in X$ is the initial point of iteration process (42).

Theorem 6. *Suppose that the assumptions of Theorem 5 hold, functional $J(\theta)$, $\theta \in X$, is bounded from below, sequence $\{\theta_n\} \subset X$ is introduced by formula (42). Then*

$$1)\ J(\theta_n - J(\theta_{n+1}) \geqslant \varepsilon \|\theta_n - \theta_{n+1}\|^2, \quad n = 0, 1, 2, \ldots; \tag{43}$$
$$2)\ \lim_{n \to \infty} \|\theta_n - \theta_{n+1}\| = 0. \tag{44}$$

Proof. Since θ_{n+1} is the projection of point $\theta_n - \alpha_n J'(\theta_n)$, then

$$< \theta_{n+1} - \theta_n + \alpha_n J'(\theta_n), \theta_n - \theta_{n+1} >_H \geqslant 0,$$

$\forall\theta$, $\theta \in X$. Thanks to $J(\theta) \in C^{1,1}(X)$ this yields

$$J(\theta_n) - J(\theta_{n+1}) \geqslant \left(\frac{1}{\alpha_n} - \frac{K}{2}\right) \|\theta_n - \theta_{n+1}\|^2 \geqslant \varepsilon \|\theta_n - \theta_{n+1}\|^2.$$

Therefore, the scalar sequence $\{J(\theta_n)\}$ is strictly decreasing and inequality (43) holds. Identity (44) is implied by the lower boundedness of functional $J(\theta)$, $\theta \in X$. We observe that $J(\theta) > 0$, $\forall\theta$, $\theta \in X$. The proof is complete. □

Theorem 7. *Suppose that the assumptions of Theorem 5 hold, set M_0 is bounded and the inequality*

$$\begin{aligned} & < F_{0q}(t, q_1) - F_{0q}(t, q_2), q_1 - q_2 >_{\mathbb{R}^N} \geqslant 0, \quad \forall q_1,\ q_2 \in \mathbb{R}^N, \\ & N = m + m_2 + 2n + s + m_1 + r + 2(n + m_2). \end{aligned} \tag{45}$$

holds true. Then

1) set M_0 is weakly bicompact, $X_{**} \neq \emptyset$, $\emptyset$ is the empty set;

2) sequence $\{\theta_n\}$ is minimizing, that is,

$$\lim_{n\to\infty} J(\theta_n) = J_* = \inf_{\theta\in X} J(\theta);$$

3) sequence $\{\theta_n\} \subset M_0$ converges weakly to $\theta_{**} \in X_{**}$;

4) the following estimate for the rate of convergence

$$0 \leqslant J(\theta_n) - J_* \leqslant \frac{c_1}{n}, \quad c_1 = const > 0, \quad n = 1, 2, \dots,$$

holds true;

5) boundary value problem (1)–(6) is solvable if and only if

$$\lim_{n\to\infty} J(\theta_n) = J_* = \inf_{\theta\in X} J(\theta) = J(\theta_{**}) = 0;$$

6) if $J(\theta_{**}) = 0$, where $\theta_{**} = \theta_* = (v_1^*, v_2^*, p_*, d_*, x_0^*, x_1^*, \lambda_*) \in X_*$, then the solution to boundary value problem (1)–(6) is the function

$$x_*(t) = Py_*(t), \quad y_*(t) = z(t, v_1^*, v_2^*) + \lambda_2(t, \xi_0^*, \xi_1^*) + N_2(t)z(t_1; v_1^*, v_2^*), \quad t \in I.$$

7) if $J(\theta_{**}) > 0$, then boundary value problem (1)–(6) has no solution.

Proof. It follows from condition (5) that functional $J(\theta) \in C^{1,1}(X)$ is convex. The first statement of the theorem is implied by the fact that M_0 is a bounded convex closed set in a reflexive Banach space H and by the weak lower semi-continuity of functional $J(\theta)$ on a weakly bicompact set M_0. The second statement is due to the estimate $J(\theta_n) - J(\theta_{n+1}) \geqslant \varepsilon\|\theta_n - \theta_{n+1}\|^2$, $n = 0, 1, 2, \dots$. It follows that $J(\theta_{n+1}) < J(\theta_n)$, $\|\theta_n - \theta_{n+1}\| \to 0$ as $n \to \infty$, $\{\theta_n\} \subset M_0$. Then the convexity of functional $J(\theta)$, $\theta \in M_0$ yields that $\{\theta_n\}$ is minimizing. The third state is implied by the weak bicompactness of set M_0, $\{\theta_n\} \subset M_0$. Inequality $J(\theta_n) - J(\theta_{**}) \leqslant c_1\|\theta_n - \theta_{n+1}\|$ leads us to the estimate for the rate of convergence. Statements 5), 6) are yielded by Theorem 4. The proof is complete. □

We note that if $f(x, \lambda, t)$, $f_{0j}(x, x_0, x_1, \lambda, t)$, $j = \overline{1, m_2}$, $F(x, \lambda)$ are linear functions in variables (x, x_0, x_1, λ), then functional $J(\theta)$ is convex.

4. Conclusion

We developed a method for solving a boundary value problem with a parameter for ordinary differential equations under the presence of constraints on the state and integral constraints. The basis of the proposed approach is the immersion principle. The matter of this principle is that the original boundary value problem with a parameter under the presence of constraints on the state and integral constraints is replaced by an equivalent initial problem of optimal control. Such approach is possible thanks to finding the general solution to one class of first kind Fredholm integral equation. The existence of a solution to the boundary value problem with the parameter and constraints is reduced to constructing of a minimizing sequence and the lower bound of the functional is determined.

In the general case, optimization problem (33)–(36) can have infinitely many solutions $\{\theta_*\} \subset X$, for which $J(\{\theta_*\}) = 0$. Subject to the choice of the initial approximation, minimizing sequences converge to some element in set $\{\theta_*\}$. Let $\theta_* = (v_{1*}, x_0^*, x_1^*, \lambda_*)$, where $J(\theta_*) = 0$ is some solution. Here $x_0^* = x(t_0)$, $x_1^* = x(t_1)$, $(x_0^*, x_1^*) \in S_0 \times S_1 = S$, $\lambda_* \in \Lambda$, where x_0^* is the initial state of the system. Formulation of the problem involves conditions (7), (8) for the right hand side of differential equation (1) ensuring the uniqueness of the solution to the initial Cauchy problem. Therefore, differential equation (1) with initial state $x_0^* = x(t_0)$ has the unique solution for $t \in [t_0, t_1]$ as $\lambda = \lambda_* \in \Lambda$. Moreover, $x_1^* = x(t_1)$ and all constraints (2)–(6) hold true. No matter which solution is found by the iteration procedure, in the case $J(\theta_*) = 0$ we find the corresponding solution to boundary value problem (1)–(6).

The principal feature of the proposed method is that the solvability and constructing of the solution to the boundary value problem with the parameter is solved simultaneously by constructing minimizing sequences oriented for applying computer techniques. The solvability and construction of the solution to the boundary value problem is determined by solving optimization problem (33) – (36), where $\lim_{n\to\infty} J(\theta_n) = \inf_{\theta\in X} J(\theta) = 0$ gives solvability conditions, while the limiting points θ_* of sequence $\{\theta_m\}$ determine the solution to boundary value problem.

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