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ON DECAY OF SOLUTION TO LINEAR PARABOLIC EQUATION WITH DOUBLE DEGENERACY

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Abstract. We obtain the upper bound for the decay rate of the solution to the Dirichlet initial boundary value problem for a linear parabolic second order equation with a double degeneracy $\mu(x)u_t = (\rho(x)a_{ij}(t,x)u_{x_i})_{x_j}$ in an unbounded domain. For a wide class of revolution domains we prove a lower bound. We adduce the examples showing that the upper and lower bounds are in some sense sharp.

We prove the unique solvability of the problem in an unbounded domain by Galerkin's approximations method.

Keywords: parabolic equation with a double degeneracy, decay rate of a solution, upper bound, existence of a solution.

Mathematics Subject Classification: 35B30, 35B45, 35K10, 35K20, 35K65

1. Introduction

Let Ω be an unbounded domain in space $\mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$. In a cylindrical domain $D = \{t > 0\} \times \Omega$ we consider a linear second order equation

$$\mu(x)u_t = \sum_{i,j=1}^{n} (\rho(x)a_{ij}(t,x)u_{x_i})_{x_j}, \tag{1}$$

where weights $\mu(x) > 0$ and $\rho(x) > 0$ are measurable functions integrable over each bounded subset of Ω: $\mu, \rho \in L^1_{\text{loc}}(\overline{\Omega})$. For symmetric coefficients $a_{ij} = a_{ji}$ we impose the condition of the uniform ellipticity: there exist positive constants γ, γ_1 such that for each vector $y \in \mathbb{R}^n$ and almost each $(t,x) \in D$ the identities

$$\gamma|y|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(t,x)y_i y_j \leqslant \gamma_1|y|^2 \tag{2}$$

hold true.

On the lateral boundary of cylinder D we impose Dirichlet boundary condition:

$$u(t,x)\Big|_{\Gamma} = 0, \quad \Gamma = (0,\infty) \times \partial\Omega. \tag{3}$$

We shall deal with a generalized solution to problem (1), (3) subject to the initial condition

$$u(0,x) = \varphi(x) \in L_2(\Omega, \mu dx). \tag{4}$$

The present work is devoted to the studying the dependence of the decay rate as $t \to \infty$ of a solution to problem (1), (3), (4) on the geometry of unbounded domain Ω and behavior of weights μ, ρ as $x \to \infty$.

The first studies of the decay rate of the solution to a mixed problem of second order uniformly parabolic equation ($\mu = \rho \equiv 1$) on the geometry of the unbounded domain were made by A.K. Guschin

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in works [1, 2]. For a wide class of domains there was established the estimate for the solution to the second mixed problem:

$$|u(t,x)| \leqslant \frac{\|\varphi\|_{L_1(\Omega)}}{v(\sqrt{t})}, \qquad x \in \Omega,$$

where $v(r) = \text{mes}\{x \in \Omega : |x| < r\}$. The sharpness of this estimate was proved. In particular, for the solution of Cauchy problem this estimate becomes

$$|u(t,x)| \leqslant C \frac{\|\varphi\|_{L_1(\mathbb{R}^n)}}{(\sqrt{t})^n}.$$

More complete studies of the dependence of the behavior at infinity of solution to the second mixed problem on the geometry of the domain and on the initial function were made by A.V. Lezhnev in [3]. V.I. Ushakov [4] obtained the results close to that by A.K. Guschin for the third mixed problem in a non-cylindrical domain. Earlier in work [5] F.Kh. Mukminov proved the estimate for the decay rate of solution to the first mixed problem for a second order uniformly parabolic equation and its sharpness in the class of unboundedly monotonically increasing revolution domains was proved. In work [6] there were obtained sharp estimates for solution to a forth and sixth order parabolic equation with Rickyies condition on the lateral boundary of an unbounded parabolic domain.

We also mention work [7], where the dependence of the solution on the structure of nonlinearities in equations was studied.

A more complete survey of results related to the subject of our work can be found in [6]–[14].

We proceed to formulating our result. We introduce the functions

$$\lambda(r) = \inf_{g \in C_0^\infty(\Omega)} F_r(g), \quad F_r(g) = \frac{\int\limits_{\Omega[r]} \rho(x)|\nabla g|^2 dx}{\int\limits_{\Omega[r]} \mu g^2 dx}, \tag{5}$$

where $\Omega[r] = \{x \in \Omega \mid |x| < r\}$;

$$\widetilde{\lambda}(r) = \inf_{g \in C_0^\infty(\Omega)} \frac{\int\limits_{S_r} \rho(x)|\nabla g|^2 dS}{\int\limits_{S_r} \rho g^2 dS}, \tag{6}$$

where $S_r = \{x \in \Omega | \ |x| = r\}$. It is obvious that function $\lambda(r)$ is bounded on the interval $r > r_0$ if set $\Omega[r_0]$ is non-empty.

In the next statement we discuss a generalized solution to problem (see Section 2).

Theorem 1. *Let $u(t,x)$ be a solution to problem (1), (3), (4) with the initial function φ vanishing as $|x| > R_0$. Then there exists a number $\nu_1 > 0$ depending only on n, γ_1, R_0 and T and depending also on functions λ, $\widetilde{\lambda}$ such that for all $t > T$ the inequality*

$$\int\limits_{\Omega} \mu(x) u^2(t,x) dx \leqslant C \exp\left(-\nu_1 \int\limits_{R_0+1}^{r(t)} \sqrt{\widetilde{\lambda}(s)} ds \right) \int\limits_{\Omega} \mu(x) \varphi^2(x) dx \tag{7}$$

holds true, where $r = r(t)$ is an arbitrary continuous function satisfying the inequality

$$t\lambda(r) \geqslant \int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)} \ ds.$$

Constant C depends on γ, γ_1 and on function $\widetilde{\lambda}$.

It is known that in the case of a planar angle $\Omega = \{(r,\psi)| \ r > 0, 0 < \psi < \alpha\}$ as $\mu = \rho \equiv 1$, the decay rate of solution to problem (1), (3), (4) is power: $u(t,x) = O(t^{-(\pi/\alpha+1)})$ (see [5]). For such situations (i.e., when the decay of solution is power) estimate (7) provides non-adequate result since the exact value of constant ν_1 is not determined (i.e., exponent t is not determined precisely).

If inequality $\rho(x) \leqslant \mu(x)$ holds true for almost each $x \in \Omega$, we can obtain an estimate slightly weaker than (7) without employing function $\widetilde{\lambda}$ (see Theorem 2 in Section 3).

We note that the theorem remains true if we replace domains $\Omega[r]$, S_r by $\Omega(r) = \{x \in \Omega \mid x_1 < r\}$ and $S_r = \{(x_1, x') \in \Omega| \; x_1 = r\}$. At that we assume that domains $\Omega(r)$ are bounded for all $r > 0$.

Formally function $r(t) = R_0 + 1$ satisfies the inequality in theorem. But it is clear that inequality (7) becomes stronger if we choose the largest function $r(t) \geqslant R_0 + 1$ among admissible ones. In the case when function $\lambda(r)$ is continuous and positive at least at one point $r \geqslant R_0 + 1$, we define function $r(t)$ as the largest among the roots of the equation $t\lambda(r) = \int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)}ds$ (for sufficiently large t there exists at least one root). In the end of Section 3 we also provide a condition ensuring the continuity of function $\lambda(r)$. The same approach is applicable under the presence of the estimate $\lambda(r) \geqslant h(r)$ with replacing $\lambda(r)$ by a continuous function $h(r)$. In the simplest case $\rho = \mu = 1$ function $h(r)$ can be chosen by employing inequality (5.4) in [21, CH. II, Sect. 5], which in the case $\operatorname{mes}\Omega[r] \leqslant (1-\varepsilon)\operatorname{mes} B(r)$, where $B(r)$ is the ball of radius r, is written as

$$\int\limits_{\Omega[r]} u^2(t,x)dx \leqslant \beta\varepsilon^{-2}r^2 \int\limits_{\Omega[r]} |\nabla u|^2(t,x)dx, \quad \beta > 0. \tag{8}$$

It implies the inequality $\lambda(r) \geqslant \frac{\varepsilon^2}{\beta r^2}$. We also observe that inequality (8) applied to the cone with the vertex at point O and spherical basis S_r instead of $\Omega[r]$, gives the estimate $\widetilde{\lambda}(r) \geqslant \delta^2(r)/(\beta r^2)$, where $1 - \delta(r) = \operatorname{mes}_{n-1} S_r r^{1-n}/\omega_n$, ω_n is the measure on the unit sphere. In particular, when function $\delta(r)$ decays fast enough, the inequality $\int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)}ds < \infty$ is possible and then estimate (7) becomes meaningless.

In Section 4 we also provide examples of functions $r(t)$ for functions μ, ρ not being equal to 1. In Section 5 we provide Theorem 2 on the lower bound of the non-negative solution in the case when domain Ω is a revolution one. By the examples we show that inequality (7) of Theorem 1 is in some sense sharp.

2. Existence and uniqueness of solution

We introduce the following notations: $D_a^b = (a,b) \times \Omega$, $D^T = D_0^T$, $D = D_0^\infty$,

$$\|u\|^2_{D^T,\mu} = \int\limits_{D^T} \mu u^2 dxdt, \quad \|\nabla u\|_{D^T,\rho} = \int\limits_{D^T} \rho|\nabla u|^2 dxdt.$$

On the set of restrictions of functions in $C_0^\infty(D_{-1}^T)$ on D^T we define the norms

$$\|u\|^2_{H^{0,1}(D^T)} = \|u\|^2_{D^T,\mu} + \|\nabla u\|^2_{D^T,\rho}; \quad \|u\|^2_{H^{1,1}(D^T)} = \|u\|^2_{H^{0,1}(D^T)} + \|u_t\|^2_{D^T,\mu}.$$

The completions of these liner normed spaces are denoted by $\mathring{H}^{0,1}(D^T)$ and $\mathring{H}^{1,1}(D^T)$. For the uniqueness of the gradient of the functions in the introduced weighted space we impose the condition in work [20]:

$$\rho^{-1} \in L^1_{\mathrm{loc}}(\Omega).$$

We defined space $\mathring{H}^1(\Omega)$ as the completion of space $C_0^\infty(\Omega)$ by the norm $\|u\|^2_{H^1(\Omega)} = \int\limits_{\Omega}(\mu u^2 + \rho|\nabla u|^2)dx$.

A generalized solution to problem (1), (3), (4) in D^T is a function $u(t,x) \in \mathring{H}^{0,1}(D^T)$ satisfying the integral identity

$$\int\limits_{D^T}\Big(-\mu u v_t + \sum_{i,j=1}^{n} \rho a_{ij}(t,x) u_{x_i} v_{x_j}\Big)dxdt = \int\limits_{\Omega} \mu\varphi(x)v(0,x)dx \tag{9}$$

for each function $v(t,x) \in \mathring{H}^{1,1}(D^T)$.

Function $u(t,x)$ is a solution to problem (1), (3), (4) in D if for each $T > 0$ it is a solution to problem (1), (3), (4) in D^T.

The generalized solution of problem (1), (3), (4) in D^T exists and is unique. The existence can be proved by Galerkin method (see, for instance, [21]).

We choose a set of linearly independent functions $w_i(x) \in C_0^\infty(\Omega)$ such that their linear span is dense in $\overset{\circ}{H}{}^1(\Omega)$. Without loss of generality we can assume that these functions are orthonormalized in $L_2(\Omega, \mu dx)$.

We seek Galerkin approximations in the form

$$u^l(t,x) = \sum_{i=1}^{n} C_i^l(t) w_i(x). \tag{10}$$

We obtain the equations for the sought coefficients by the restriction

$$\int_\Omega (\mu(x) u_t^l w_s + \sum_{i,j=1}^{n} \rho(x) a_{ij}(t,x) u_{x_i}^l (w_s)_{x_j} dx = 0, \quad s = \overline{1,l}. \tag{11}$$

Thanks to the orthonormaliaty of functions w_i, conditions (11) lead us to the system of ordinary differential equations

$$(C_i^l)' + \sum_{j=1}^{n} b_{ij}(t) C_j^l = 0, \quad i = \overline{1,n}. \tag{12}$$

We choose the initial conditions for system of differential equations (12) as

$$C_i^l(0) = (\varphi, w_i). \tag{13}$$

Conditions (12), (13) determine the unique set of functions $C_i^l(t)$.

Let us prove the boundedness of set u^l of Galerkin approximations in space $\overset{\circ}{H}{}^{0,1}(D^T)$. We multiply identities (11) by C_s^l and sum they up. We obtain

$$\int_\Omega (\mu u_t^l u^l + \sum_{i,j=1}^{n} \rho a_{ij}(t,x) u_{x_i}^l u_{x_j}^l) dx dt = 0. \tag{14}$$

Integrating (14) over $t \in (0,T)$ and employing condition (2), we get

$$\frac{1}{2} \int_\Omega \mu(x) \left[(u^l(t,x))^2 - (u^l(0,x))^2 \right] dx + \gamma \int_{D^T} \rho(x) |\nabla u^l|^2 dx dt \leqslant 0. \tag{15}$$

It is obvious that

$$\|u^l(0,x)\|^2_{L_2(\Omega,\mu dx)} = \sum_{i=1}^{l} (\varphi, w_i)^2.$$

Then (15) can be rewritten as

$$\int_\Omega \mu(x) u^l(t,x))^2 dx + 2\gamma \int_{D^T} \rho(x) |\nabla u^l|^2 dx dt \leqslant \|\varphi\|^2_{D^T,\mu}. \tag{16}$$

It implies the boundedness of set u^l in subspace $\overset{\circ}{H}{}^{0,1}(D^T)$. This is why we can choose a subsequence weakly converging in this space to some function $u \in \overset{\circ}{H}{}^{0,1}(D^T)$. In order to avoid cumbersome superscripts, we assume that the sequence weakly converges itself.

We multiply (11) by function $d_s(t) \in C_0^\infty(-1,T)$ and integrate over $t \in (0,T)$. By denoting $v = d_s w_s$, integrating by parts and passing to the limit as $l \to \infty$, we obtain

$$\int_{D^T} \left(-\mu u (v)_t + \sum_{i,j=1}^{n} \rho a_{ij}(t,x) u_{x_i} (v)_{x_j} dx dt \right) = \int_\Omega \mu \varphi(x) v(0,x) dx. \tag{17}$$

We note that (17) is true not only for functions $v = d_s w_s$, but also for the sums of such functions. It remains to mention that by functions $v^m = \sum\limits_{s=1}^{m} d_s w_s$, we can approximate each function w in $C_0^\infty(D_{-1}^T)$ in the norm of space $\overset{\circ}{H}{}^{1,1}(D^T)$.

Let us show the uniqueness of solution to problem (1), (3), (4).

By $v_h(t,x)$ we denote the Steklov average of function $v(t,x)$:

$$v_h(t,x) = \frac{1}{h}\int_t^{t+h} v(\tau,x)d\tau,$$

which possesses the following properties:

1)$(v,u_{-h}) = (v_h,u)_{L_2(\mathbb{R}^{n+1},\mu dxdt)}$, where $(v,u)_{L_2(\mathbb{R}^{n+1},\mu dxdt)} = \int\limits_{\mathbb{R}^{n+1}} \mu vu\, dxdt$,

2) if $v \in \mathring{H}^{0,1}(D_0^T)$, then $(v_h)_{x_i} = (v_{x_i})_h$,

3) if $v, v_t \in L_2(\mathbb{R}^{n+1},\mu dxdt)$, then $(v_t)_h = (v_h)_t$,

4) if $v \in L_2(D^T,\mu dxdt)$, then for each $\delta > 0$ the convergence $v_h \to v$ is true in $L_2(D^{T-\delta},\mu dxdt)$ as $h \to 0$ $(h < \delta)$.

We substitute test function v_{-h} into integral identity (9), where v belongs to space $C_0^\infty(D_0^{T-\delta})$. It is possible since $v_{-h} \in C_0^\infty(D_0^T)$ as $0 < h < \delta$. Employing the properties of Steklov average, we get

$$\int\limits_{D^T} \left[\mu(u_h)_t v + \rho\sum_{i,j=1}^n (a_{ij}u_{x_i})_h v_{x_j}\right] dxdt = 0. \tag{18}$$

By passing to the limit we prove that the latter relation is true not only for $v \in C_0^\infty(D_0^{T-\delta})$, but also for $v \in \mathring{H}^{0,1}(D_0^{T-\delta})$.

We note that identities (18) are of the form

$$\int\limits_{D^T} \mu(u_h)_t v dxdt = l_h(v), \tag{19}$$

where $l_h(v)$ is a linear functional in space $\mathring{H}^{0,1}(D_0^{T-\delta})$.

Let us prove the uniform boundedness of linear functional $l_h(v)$ as $|h| < \delta_0$ in the unit ball of space $\mathring{H}^{0,1}(D_0^{T-\delta})$.

We consider $l_h(v)$ and in view of the uniform ellipticity we have

$$|l_h(v)| = \left|\int\limits_{D^{T-\delta}} \rho\sum_{i,j=1}^n (a_{ij}u_{x_i})_h v_{x_j} dxdt\right| \leqslant \int\limits_{D^{T-\delta}} \left(\frac{\gamma_1}{h}\int\limits_t^{t+h} \rho|\nabla u(\tau,x)|d\tau\right)|\nabla v(t,x)|dxdt$$

$$\leqslant \int\limits_{D^{T-\delta}} \gamma_1\rho\left(\frac{1}{h^2}\left(\int\limits_t^{t+h}|\nabla u(\tau,x)|d\tau\right)^2 + |\nabla v(t,x)|^2\right)dxdt.$$

Thus, we obtain that $|l_h(v)| \leqslant C$. The boundedness of linear functional $l_h(v)$ is proved.

We substitute function $v = (u_{h_1} - u_{h_2})\chi(t_1,t_2) \in \mathring{H}^{0,1}(D_0^{T-\delta})$ into identities $(19)_{h_1}$–$(19)_{h_2}$, where $\chi(t_1,t_2)$ is the characteristic function of the interval (t_1,t_2). We obtain

$$\left|\int\limits_{t_1}^{t_2}\int\limits_\Omega \mu((u_{h_1})_t - (u_{h_2})_t)(u_{h_1} - u_{h_2})dxdt\right| = |(l_{h_1} - l_{h_2})(\chi(u_{h_1} - u_{h_2}))| \leqslant C\|(u_{h_1} - u_{h_2})\|_{H^{0,1}} \leqslant \varepsilon.$$

The latter inequality for sufficiently small h_1, h_2 is implied by the convergence $u_h \to u$ in space $\mathring{H}^{0,1}(D_0^{T-\delta})$. After integration in t we obtain

$$\int\limits_\Omega \mu(u_{h_1} - u_{h_2})^2(t_1,x)dx \leqslant \int\limits_\Omega \mu(u_{h_1} - u_{h_2})^2(t_2,x)dx + 2\varepsilon.$$

We integrate this inequality in $t_2 \in [t_1, T-\delta]$:

$$(T-\delta-t_1)\int\limits_\Omega \mu(u_{h_1} - u_{h_2})^2(t_1,x)dx \leqslant \mu\|(u_{h_1} - u_{h_2})\|^2_{L_2(D^{T-\delta},\mu dx)} + 2\varepsilon(T-\delta-t_1).$$

Since $u_h \to u$ in $L_2(D^{T-\delta}, \mu dx)$, as $t_1 < T - 2\delta$ we have the inequality

$$\int\limits_{\Omega} \mu(u_{h_1} - u_{h_2})^2(t_1, x)dx \leqslant \frac{\varepsilon_1}{\delta} + 2\varepsilon.$$

It yields the uniform fundamentality in t_1 of the family of functions $u_h(t_1, x)$ in $L_2(\Omega, \mu dx)$. This is why $u_h(t,x) \rightrightarrows u(t,x)$ in $L_2(\Omega, \mu dx)$ as $h \to 0$ uniformly in $t \in [0, T-2\delta]$ and the limiting function is continuous in t in the norm of $L_2(\Omega, \mu dx)$. We substitute function $v = u_h\chi(0,t)$ into (18):

$$\int\limits_{D_0^t} (\mu(u_h)_t u_h + \rho \sum_{i,j=1}^{n} (a_{ij}u_{x_i})_h (u_h)_{x_j})\, dxdt = 0.$$

Integrating the first term in t and passing to the limit as $h \to 0$, we obtain

$$\frac{1}{2}\int\limits_{\Omega} \mu u^2(t,x)dx + \int\limits_{D_0^t} \rho \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}dxdt = \frac{1}{2}\int\limits_{\Omega} \mu u^2(0,x)dx. \tag{20}$$

If we prove that $u(0,x) = \varphi(x)$, the latter relation coincides with (9). In order to do it, we substitute a continuous test function $v(t,x) = \eta(\frac{t}{\varepsilon})\psi(x)$ into identity (9), where $\eta(t) = 1 - t$ as $t \in [0,1]$ and $\eta(t)$ is constant in other intervals $(-\infty, 0]$, $[1, \infty)$. Since $v_t = -\frac{1}{\varepsilon}\psi(x)$, identity (9) becomes

$$\int\limits_0^{\varepsilon}\int\limits_{\Omega} \frac{1}{\varepsilon}\mu(x)\psi(x)u(t,x)dtdx + l^{\varepsilon}(\psi) = \int\limits_{\Omega} \mu(x)\varphi(x)\psi(x)dx,$$

where linear functional $l^{\varepsilon}(\psi)$ tends to zero as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int\limits_{\Omega} \mu(x)\psi(x)u(0,x)dx = \int\limits_{\Omega} \mu(x)\varphi(x)\psi(x)dx$$

for each $\psi \in C_0^{\infty}(\Omega)$. It proves the intitial condition $u(0,x) = \varphi(x)$.

3. Upper bound for solution

We first establish two estimates characterising the decay of solution to problem (1), (3), (4) as $|x| \to \infty$.

Proposition 1. *Let $u(t,x)$ be a solution to problem (1), (3), (4) with initial function φ vanishing outside the ball of radius R_0. Suppose that the inequality*

$$\rho(x) \leqslant C\mu(x), \quad C > 0, \quad x \in \Omega, \tag{21}$$

holds true. Then for all $t > 0$, $r \geqslant R_0$ the inequality

$$\int\limits_{\Omega\setminus\Omega[r]} \mu u^2(t,x)dx \leqslant e\exp\left(-\widetilde{C}t^{-1}(r-R_0)^2\right)\int\limits_{\Omega} \mu(x)\varphi^2(x)dx \tag{22}$$

holds true, where $\widetilde{C}$ is a constant depending on γ and γ_1.

Proof. Let $\xi(\tau, r, \varrho)$ be a continuous nonnegative function vanishing as $\tau \leqslant r$ and being one as $\tau \geqslant r + \varrho$. In the remaining interval it is linear: $\frac{\partial\xi}{\partial\tau} = \frac{1}{\varrho}$. We substitute a test function $v = \eta(x; r, \varrho)u_h$, $\eta(x) = \xi^2(|x|, r, \varrho)$ into identity (18) to obtain

$$\int\limits_{D^T} \left[\frac{1}{2}\mu(u_h^2\eta)_t + \sum_{i,j=1}^{n} \rho(a_{ij}u_{x_i})_h(\eta u_h)_{x_j}\right]dxdt = 0. \tag{23}$$

We passing to the limit as $h \to 0$ in identity (23):

$$\int\limits_{\Omega} \mu(u^2(T,x) - \varphi^2(x))\eta dx + 2\int\limits_{D^T} \sum_{i,j=1}^{n} \rho a_{ij}u_{x_i}(\eta u)_{x_j}dxdt = 0.$$

By the condition $\operatorname{supp}\varphi\subset\Omega[R_0]$, for each $r\geqslant R_0$ and $\varrho>0$ we obtain easily the inequality

$$\int\limits_{\Omega}\mu\eta u^2(T,x)dx+2\int\limits_{D^T}\rho\sum_{i,j=1}^{n}\eta a_{ij}u_{x_i}u_{x_j}dxdt\leqslant -2\int\limits_{D^T}\rho\sum_{i,j=1}^{n}a_{ij}u_{x_i}u\frac{\partial\eta}{\partial x_j}dxdt \leqslant 2\int\limits_{D^T}\rho\gamma_1|u\nabla u\nabla\eta|dxdt. \tag{24}$$

Rewriting the latter, we have

$$\int\limits_{\Omega}\mu\eta u^2(T,x)dx+\int\limits_{D^T}\gamma\rho\eta|\nabla u|^2dxdt\leqslant 2\int\limits_{D^T}\rho\gamma_1|u\nabla u\nabla\eta|dxdt.$$

Employing the structure of function η, we obtain easily the inequality

$$\int\limits_{\Omega\backslash\Omega[r+\varrho]}\mu u^2(t,x)dx+\int\limits_0^t\int\limits_{\Omega\backslash\Omega[r+\varrho]}\rho\gamma|\nabla u|^2dxdt\leqslant\frac{C}{\varrho^2}\int\limits_0^t\int\limits_{\Omega[r+\varrho]\backslash\Omega[r]}\rho u^2dxdt.$$

We introduce the notation

$$H_r(t)=\int\limits_{\Omega\backslash\Omega[r]}\mu u^2(x,t)dx+\int\limits_0^t\int\limits_{\Omega\backslash\Omega[r]}\rho\gamma|\nabla u|^2dxdt$$

and employ condition (21) to establish that

$$H_{r+\varrho}(t)\leqslant\frac{C}{\varrho^2}\int\limits_0^t H_r(\tau)d\tau. \tag{25}$$

We apply inequality (25) inductively to sequence r_i, $i=0,1,2,\dots k$, $r_{i+1}=r_i+\varrho$, $r_0=R_0$. Since (20) implies the inequality

$$H_r(t)\leqslant A=\int\limits_{\Omega}\mu(x)\varphi^2(x)dx,\quad r>0,\quad t>0,$$

we get

$$H_{R_0+\varrho}(t)=\frac{ACt}{\varrho^2}. \tag{26}$$

Let us establish the inequality

$$H_{r_k}(t)\leqslant\frac{AC^kt^k}{\varrho^{2k}k!} \tag{27}$$

by induction in k.

Indeed,

$$H_{r_k+\varrho}(t)\leqslant\frac{C}{\varrho^2}\int\limits_0^t H_{r_k}(\tau)d\tau\leqslant\frac{C}{\varrho^2}\int\limits_0^t\frac{AC^k\tau^k}{\varrho^{2k}k!}d\tau=\frac{AC^{k+1}t^{k+1}}{\varrho^{2(k+1)}(k+1)!}$$

that completes the induction. Employing Stirling's inequality, by (27) we obtain easy that

$$H_{r_k}(t)\leqslant\frac{AC^ke^kt^k}{\sqrt{2\pi k}\varrho^{2k}k^k}\leqslant A\exp\left(-k\ln\frac{\varrho^2k}{Cet}\right). \tag{28}$$

We choose k equal to the integer part of the number $\frac{(r-R_0)^2}{Ce^2t}$. If $k=0$, then $\frac{(r-R_0)^2}{Ce^2t}<1$ and $H_r(t)\leqslant A=eAe^{-1}$ that implies inequality (22). If $k\geqslant 1$, then $k\geqslant\frac{(r-R_0)^2}{2Ce^2t}$. Now we let $\varrho=(r-R_0)/k$. Then $r_k=r$ and $\varrho^2k=\frac{(r-R_0)^2}{k}\geqslant Ce^2t$. Therefore, $\frac{\varrho^2k}{Cet}\geqslant 1$. Hence it follows from (28) that $H_r(t)=H_{r_k}(t)\leqslant Ae^{-k}$. It leads us to inequality (22). $\square$

Proposition 2. *Let $u(t,x)$ be the solution to problem (1), (3), (4) with an initial function φ vanishing outside a ball of radius R_0. Then for all $t>0$, $r \geqslant R_0+1$ the inequality*

$$\int\limits_{\Omega\backslash\Omega[r]} \mu u^2(t,x)dx \leqslant C\exp\left(-2\nu\int\limits_{R_0+1}^{r}\sqrt{\widetilde{\lambda}(s)}ds\right)\int\limits_{\Omega}\mu(x)\varphi^2(x)dx \tag{29}$$

holds true, where C, ν are constants depending on γ and γ_1, while C depends also on function $\widetilde{\lambda}$.

Proof. Let $\xi(\tau,r)$ be a continuous nonnegative function vanishing as $\tau \leqslant R_0$, being linear as $R_0 < \tau < R_0+1$ and equalling one as $\tau \geqslant r$. In the remaining interval it satisfies the condition $\frac{\partial\xi}{\partial\tau} = \nu\sqrt{\widetilde{\lambda}}\xi$, where number ν will be fixed later.

It is easy to see that $\xi_\tau = \xi(R_0+1,r)$ as $\tau\in(R_0,R_0+1)$, where

$$\xi(R_0+1,r) = \exp\left(-\nu\int\limits_{R_0+1}^{r}\sqrt{\widetilde{\lambda}(s)}ds\right).$$

We substitute a test function $v=\eta(x;r)u_h$, $\eta(x;r)=\xi^2(|x|,r)$, into identity (18) to obtain

$$\int\limits_{D^T}\left[\frac{1}{2}\mu(u_h^2\eta)_t + \sum_{i,j=1}^{n}\rho(a_{ij}u_{x_i})_h(\eta u_h)_{x_j}\right]dxdt = 0. \tag{30}$$

Passing to the limit as $h\to 0$ in identity (30), we get

$$\int\limits_{\Omega}\mu(u^2(T,x)-\varphi^2(x))\eta dx + 2\int\limits_{D^T}\sum_{i,j=1}^{n}\rho a_{ij}u_{x_i}(\eta u)_{x_j}dxdt = 0.$$

It implies easily the inequality

$$\int\limits_{\Omega}\mu\eta u^2(T,x)dx + 2\int\limits_{D^T}\rho\sum_{i,j=1}^{n}\eta a_{ij}u_{x_i}u_{x_j}dxdt \leqslant 2\int\limits_{D^T}\rho\gamma_1|u\nabla u\nabla\eta|dxdt. \tag{31}$$

Transforming the latter, we obtain

$$\int\limits_{\Omega}\xi^2\mu u^2dx + \int\limits_{D^T}\gamma\rho\xi^2|\nabla u|^2dxdt \leqslant \int\limits_{D^T}\rho\gamma_1\left(\varepsilon\xi^2|\nabla u|^2 + \frac{u^2\xi'^2}{\varepsilon}dx\right)dt.$$

Taking $\varepsilon = \frac{\gamma}{2\gamma_1}$, we obtain the inequality

$$\begin{aligned}\int\limits_{\Omega}\xi^2\mu u^2dx + \frac{\gamma}{2}\int\limits_{D^T}\rho\xi^2|\nabla u|^2dxdt \\ \leqslant \frac{2\gamma_1^2}{\gamma}\left(\int\limits_0^T\int\limits_{\Omega[r]\backslash\Omega[R_0+1]}\nu^2\rho u^2\xi^2\widetilde{\lambda}dxdt + \int\limits_0^T\int\limits_{\Omega[R_0+1]\backslash\Omega[R_0]}\rho u^2\xi^2(R_0+1)dxdt\right).\end{aligned} \tag{32}$$

Employing the definition of function ξ, we rewrite the latter terms:

$$\begin{aligned}\int\limits_{\Omega[r]\backslash\Omega[R_0+1]}\nu^2\rho u^2\xi^2\widetilde{\lambda}dx &= \int\limits_{R_0+1}^{r}\nu^2\xi^2(\tau)\widetilde{\lambda}(\tau)d\tau\int\limits_{S_\tau}\rho u^2dS \\ &\leqslant \int\limits_{R_0+1}^{r}\nu^2\xi^2(\tau)d\tau\int\limits_{S_\tau}\rho|\nabla u|^2dS = \nu^2\int\limits_{\Omega[r]\backslash\Omega[R_0+1]}\rho\xi^2|\nabla u|^2dx.\end{aligned} \tag{33}$$

In the same way,

$$\int\limits_{\Omega[R_0+1]\setminus\Omega[R_0]} \rho u^2 dx \leqslant \frac{1}{\inf\limits_{[R_0,R_0+1]} \widetilde{\lambda}(\tau)} \int\limits_{\Omega[R_0+1]\setminus\Omega[R_0]} \rho|\nabla u|^2 dx. \tag{34}$$

We let $\nu = \frac{\gamma}{2\gamma_1}$. Substituting (33) and (34) into (32) and estimating the right hand side in (34) by (20), we obtain inequality (29), $C = 2\gamma_1^2/(\gamma \inf\limits_{[R_0,R_0+1]} \widetilde{\lambda}(\tau))$. □

Theorem 2. *Let $u(t,x)$ be a solution to problem (1), (3), (4) with an initial function φ vanishing as $|x| > R_0$ and let inequality (21) hold true. Then there exists a constant $\nu_2 > 0$ depending only on n, γ_1, R_0 such that for all $t > 0$ the inequality*

$$\int\limits_\Omega \mu(x)u^2(t,x)dx \leqslant C\exp\left(-\nu_2 t\lambda(r(t))\right) \int\limits_\Omega \mu(x)\varphi^2(x)dx \tag{35}$$

holds true, where $r = r(t)$ is an arbitrary function satisfying the inequality $t\lambda(r) \leqslant t^{-1}(r-R_0)^2$. Constant C depends only on γ, γ_1 and n.

Proof of Theorems 1, 2. Let $T > 0$ be an arbitrary number. We introduce the notation

$$\varepsilon = \sup_{t\in[0,T]} \int\limits_{\Omega\setminus\Omega[r]} \mu u^2(t,x)dx.$$

The inequality

$$\int\limits_\Omega \mu u^2(t,x)dx \leqslant \varepsilon + \int\limits_{\Omega[r]} \mu u^2(t,x)dx \tag{36}$$

holds true. Since for almost each $t \in (0,T)$ function $u(t,x)$ is an element of space $\mathring{H}^1(\Omega)$, by (5) we obtain

$$\int\limits_\Omega \mu u^2(t,x)dx \leqslant \varepsilon + \lambda^{-1}(r) \int\limits_\Omega \rho|\nabla u|^2 dx. \tag{37}$$

By means of relation

$$\frac{d}{dt}\int\limits_\Omega \mu u^2(t,x)dx \leqslant -\gamma \int\limits_\Omega \rho|\nabla u|^2 dx$$

implied by (20), for the function $E(t) = \int\limits_\Omega \mu u^2(t,x)dx$ we get the inequality

$$\gamma(E(t)-\varepsilon)\lambda(r) \leqslant -\frac{d}{dt}E(t).$$

Solving this inequality, we find

$$E(T) - \varepsilon \leqslant e^{-T\lambda(r)\gamma}E(0). \tag{38}$$

To prove Theorem 2, we make use of estimate (22):

$$\varepsilon \leqslant e\exp\left(-\widetilde{C}T^{-1}(r-R_0)^2\right) \int\limits_\Omega \mu(x)\varphi^2(x)dx.$$

Then

$$E(T) \leqslant E(0)\left(e\exp\left(-\widetilde{C}T^{-1}(r-R_0)^2\right) + e^{-T\lambda(r)\gamma}\right). \tag{39}$$

The latter inequality is valid for all $r \geqslant R_0$. It is natural to find the infimum of the right hand side in r. But since we can find constructively the point of the infimum, we can take value $r(T) > R_0$ (as small as possible) to satisfy the inequality

$$T^{-1}(r-R_0)^2 \geqslant T\lambda(r).$$

The possibility of such choice for $r(T)$ follows from the boundedness of function $\lambda(r)$. Substituting $r = r(T)$ into (39), we arrive at the estimate in Theorem 2.

To prove Theorem 1, we employ estimate (29):

$$\varepsilon \leqslant C \exp\left(-2\nu \int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)}ds\right) \int\limits_{\Omega} \mu\varphi^2(x)dx.$$

We choose $r = r(T)$ (as large as possible) so that

$$T\lambda(r) \geqslant \int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)}ds.$$

Then (38) implies inequality (7) of Theorem 1. □

Let us show that function $\lambda(r)$ is continuous in a rather general situation. We call domain Ω regular if there exists a family of diffeomorphisms $\varphi_{r_1,r_2} : \Omega[r_1] \to \Omega[r_2]$, $0 < r_1 < r_2$, such that $\varphi_{r_1,r_2}(x) \to id$ in $C^1(\Omega[r_1])$ both as $r_1 \to r_2$ and as $r_2 \to r_1$.

Let us show for a regular domain function $\lambda(r)$ is continuous/ For each $\varepsilon > 0$, $r > 0$ there exists a function $g_r \in C_0^1(\Omega)$ (depending on ε) such that $F_r(g_r) < \lambda(r) + \varepsilon$. It is obvious that $\lambda(r_1) \leqslant F_{r_1}(g_{r_2})$. This is why

$$\lim\sup_{r_1 \to r_2} \lambda(r_1) \leqslant \lambda(r_2) + \varepsilon.$$

Then $F_{r_2}(g_{r_1}(\varphi_{r_1,r_2}(x))) \geqslant \lambda(r_2)$. Therefore,

$$\begin{aligned}\lambda(r_2) &\leqslant \lim\inf_{r_1\to r_2} F_{r_2}(g_{r_1}(\varphi_{r_1,r_2}(x))) = \lim\inf_{r_1\to r_2} F_{r_1}(g_{r_1}(\varphi_{r_1,r_2}(x))) \\ &= \lim\inf_{r_1\to r_2} F_{r_1}(g_{r_1}(x)) \leqslant \varepsilon + \lim\inf_{r_1\to r_2} \lambda(r_1).\end{aligned}$$

Thanks to the arbitrariness of $\varepsilon > 0$, the obtained relations yield the left continuity of function $\lambda(r)$. The right continuity can be proven in the same way.

4. Examples

We restrict ourselves by constructing examples in the case $n = 2$, while similar examples can be easily adapted for a multi-dimensional situation for the revolution domain

$$\Omega_f = \{(x_1, x')|\ x_1 > 0; |x'| < f(x_1)\}$$

defined by a positive continuous function $f(x_1)$, $f(x_1) \geqslant 1$, $x_1 > 0$. We shall obtain some estimates for functions λ, $\widetilde{\lambda}$ in the case of planar domain Ω_f.

For the simplicity we shall refer to the version of Theorem 1, when domains $\Omega[r]$, S_r are replaced by $\Omega(r) = \{x \in \Omega \mid x_1 < r\}$ and $S_r = \{(x_1, x') \in \Omega|\ x_1 = r\}$.

Let us establish an analogue of Steklov-Fridrichs inequality with weights. Let $g(s) \in C^1[0, r]$ and $g(0) = 0$. Squaring the identity

$$g(s) = g(s) - g(0) = \int\limits_0^s g'(t)dt,$$

it is easy to obtain

$$g^2(s) \leqslant \int\limits_0^r \rho^{-1}(t)dt \int\limits_0^r \rho(t)(g'(t))^2 dt.$$

Then we multiply by $\mu(s)$ and integrate in s:

$$\int\limits_0^r \mu(s)g^2(s)ds \leqslant \int\limits_0^r \mu(s)ds \int\limits_0^r \rho^{-1}(t)dt \int\limits_0^r \rho(t)(g'(t))^2 dt.$$

Assume that $g(x_1,x_2)\in C_0^\infty(\Omega)$. Then we have

$$\int\limits_0^{f(x_1)} \mu(x)g^2(x)dx_2 \leqslant \int\limits_0^{f(x_1)} \mu(x)dx_2 \int\limits_0^{f(x_1)} \rho^{-1}(x)dx_2 \int\limits_0^{f(x_1)} \rho(x)(g'_{x_2}(x))^2dx_2. \tag{40}$$

We introduce the notation $\Lambda(r)=\sup\limits_{0\leqslant x_1\leqslant r} M(x_1)$, where

$$M(x_1)=\int\limits_0^{f(x_1)} \mu(x)dx_2 \int\limits_0^{f(x_1)} \rho^{-1}(x)dx_2.$$

Then

$$\int\limits_0^{f(x_1)} \mu(x)g^2(x)dx_2 \leqslant \Lambda(r) \int\limits_0^{f(x_1)} \rho(x)(g'_{x_2}(x))^2dx_2. \tag{41}$$

or, integrating in x_1, we obtain

$$\int\limits_{\Omega[r]} \mu(x)g^2(x)dx \leqslant \Lambda(r) \int\limits_{\Omega[r]} \rho(x)(g'_{x_2}(x))^2dx. \tag{42}$$

As $\mu(x)$ and $\rho(x)$ we consider the functions

$$\rho(x_1,x_2)=\begin{cases} \widetilde{\rho}(x_1)(f(x_1)-|x_2|)^\alpha, & |x_2|\in[f(x_1)-1,f(x_1)]|, \\ \widetilde{\rho}(x_1), & |x_2|<f(x_1)-1, \end{cases}$$

$$\mu(x_1,x_2)=\begin{cases} \widetilde{\mu}(x_1)(f(x_1)-|x_2|)^\beta, & |x_2|\in[f(x_1)-1,f(x_1)]|, \\ \widetilde{\mu}(x_1), & |x_2|<f(x_1)-1, \end{cases}$$

where $|\alpha|<1$, $\beta>-1$. We shall define function $\widetilde{\mu}(x_1)$, $\widetilde{\rho}(x_1)$ later. For simplicity, we shall assume that $f(r)\geqslant \frac{|\alpha|}{1-|\alpha|}$ and $f(r)\geqslant \frac{-\beta}{1+\beta}$ as $r\geqslant R_0$.

Calculating $M(x_1)$ as $\mu=\rho$, by (41) we find that

$$\widetilde{\lambda}(r)\geqslant \left[\left(f(r)-\frac{\alpha}{1+\alpha}\right)\left(f(r)+\frac{\alpha}{1-\alpha}\right)\right]^{-1} \geqslant \frac{1}{2f^2(r)} \tag{43}$$

as $r\geqslant R_0$. Substituting this estimate into (29), we obtain

$$\int\limits_{\Omega\setminus\Omega(r)} \mu u^2(t,x)dx \leqslant C\exp\left(-2\nu\int\limits_{R_0+1}^{r}\frac{ds}{f(s)}\right)\int\limits_{\Omega}\mu(x)\varphi^2(x)dx.$$

It is easy to see that

$$\begin{aligned}\Lambda(r)&=\sup_{0\leqslant x_1\leqslant r}\frac{\widetilde{\mu}(x_1)}{\widetilde{\rho}(x_1)}\left(f(x_1)-\frac{\beta}{1+\beta}\right)\left(f(x_1)+\frac{\alpha}{1-\alpha}\right)\\ &\leqslant \max\left(\sup_{0\leqslant x_1\leqslant r} 4\frac{\widetilde{\mu}(x_1)}{\widetilde{\rho}(x_1)}f^2(x_1), \sup_{0\leqslant x_1\leqslant R_0} 4\frac{\widetilde{\mu}(x_1)}{\widetilde{\rho}(x_1)}\frac{|\beta|}{1+\beta}\frac{|\alpha|}{1-\alpha}\right).\end{aligned}$$

For the sake of simplicity we assume that function $\frac{\widetilde{\mu}(x_1)}{\widetilde{\rho}(x_1)}f^2(x_1)$ increases and

$$\frac{\widetilde{\mu}(R_0)}{\widetilde{\rho}(R_0)}f^2(R_0)\geqslant \sup_{0\leqslant x_1\leqslant R_0}\frac{\widetilde{\mu}(x_1)}{\widetilde{\rho}(x_1)}\frac{|\beta|}{1+\beta}\frac{|\alpha|}{1-\alpha}.$$

Due to (42) we have $\lambda(r)\geqslant\Lambda^{-1}(r)$ and this is why

$$\lambda(r)\geqslant\frac{\widetilde{\rho}(r)}{4\widetilde{\mu}(r)f^2(r)}. \tag{44}$$

Roughing a little bit the estimate in Theorem 1 (see its proof), we can choose function $r(t)$ satisfying the inequality $\frac{t\widetilde{\rho}(r)}{\widetilde{\mu}(r)f^2(r)} \geqslant \int\limits_{R_0+1}^{r} \frac{ds}{f(s)}$. Then estimate (7) becomes

$$\int\limits_{\Omega} \mu(x)u^2(t,x)dx \leqslant C\exp\left(-\nu_2 \int\limits_{R_0+1}^{r(t)} \frac{ds}{f(s)}\right) \int\limits_{\Omega} \mu(x)\varphi^2(x)dx. \tag{45}$$

In particular, if $f(s) = s^p$, $p \in (0,1)$, then

$$\int\limits_{R_0+1}^{r(t)} \frac{ds}{f(s)} \leqslant \frac{r^{1-p}}{1-p}.$$

Suppose for simplicity that

$$\frac{\widetilde{\rho}}{\widetilde{\mu}} = \frac{r^q}{1-p}, \quad q < 1-p,$$

then the inequality determining $r(t)$ casts into the form $t \geqslant r^{1-p-q}$ and we can choose $r(t) = t^{(1-p-q)^{-1}}$. In this case estimate (45) becomes

$$\int\limits_{\Omega} \mu(x)u^2(t,x)dx \leqslant C_1\exp\left(-C_2 t^{\frac{1-p}{1-p-q}}\right) \int\limits_{\Omega} \mu(x)\varphi^2(x)dx.$$

We observe that in the multi-dimensional case function $f(s) = s^p$ generates the revolution paraboloid and all the above arguments remain true with appropriately changed constants.

In the case $f(s) = s$ we have the interior of an angle in the plane (or of a cone in the multidimensional case). Then

$$\int\limits_{R_0+1}^{r(t)} \frac{ds}{f(s)} \leqslant \ln r.$$

As an example we choose functions $\widetilde{\rho}$, $\widetilde{\mu}$ so that $\frac{\widetilde{\rho}}{\widetilde{\mu}} = \frac{\ln r}{r^q}$, $q > 0$. Then the inequality determining $r(t)$ becomes $t \geqslant r^q$. We choose $r(t) = t^{1/q}$. Then estimate (45) casts into the form

$$\int\limits_{\Omega} \mu(x)u^2(t,x)dx \leqslant C_3\exp\left(-C_4 \ln t\right) \int\limits_{\Omega} \mu(x)\varphi^2(x)dx.$$

5. Lower bound

We recall Harnack inequality established by J.A. Moser for a uniformly parabolic equation [23]

$$u_t = \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j}. \tag{46}$$

We formulate it in a convenient for us form: for a nonnegative in a cylinder $Q = (0, 9C_1\rho^2] \times B(2\rho, \mathbf{w}) \subset \mathbb{R}^{n+1}$, $C_1 > 1$ solution to equation (46) the inequality

$$\max_{Q^-} u(\tau,x) \leqslant H \min_{Q^+} u(\tau,x),$$

holds true, where $Q^- = [\rho^2, 2\rho^2] \times B(\rho,\mathbf{w})$, $Q^+ = [8C_1\rho^2, 9C_1\rho^2] \times B(\rho,\mathbf{w})$, $B(\rho,\mathbf{w})$ is the ball of radius ρ centered at point $\mathbf{w} \in \Omega$, and constant $H \geqslant 1$ depends only on n, C_1 and the parabolicity constants of the equation.

We recall the notion of A_2-weight introduced by Muckenhoupt. This is a measurable function $\vartheta(x) : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the inequality

$$\int\limits_K \vartheta(x)dx \times \int\limits_K \frac{1}{\vartheta(x)}dx < C_0|K|^2$$

for each cube $K \subset \mathbb{R}^n$. It was proven in work [24] that if the identity $\rho = \mu = \vartheta$ holds in Q, where ϑ is some A_2-weight, then for each nonnegative in Q solution of equation (1) Harnack inequality holds true. At that, constant H depends only on C_0, C_1, n, γ and γ_1. Let us show that we can get rid of the assumption $\rho = \mu$ if $\mu = \vartheta$ and the inequalities

$$C_1^{-1} \leqslant \frac{\rho(x)\mu(\mathbf{w})}{\mu(x)\rho(\mathbf{w})} \leqslant C_1, \quad x \in B(2\rho, \mathbf{w}), \tag{47}$$

hold true. By the change $\tau = \frac{\rho(\mathbf{w})}{\mu(\mathbf{w})} t$ we obtain the equation

$$\operatorname{div}(\rho(x)a(t,x)\nabla u) = \mu(x)u_t = \frac{\mu(x)\rho(\mathbf{w})}{\mu(\mathbf{w})} u_\tau$$

or as $x \in B(2\rho, \mathbf{w})$:

$$\operatorname{div}(\vartheta(x)\frac{\rho(x)\mu(\mathbf{w})}{\mu(x)\rho(\mathbf{w})} a(\tau,x)\nabla u) = \vartheta(x)u_\tau.$$

The latter equation in Q is of the form (1) with $\rho = \mu = \vartheta$ and $\widetilde{a} = \frac{\rho(x)\mu(\mathbf{w})}{\mu(x)\rho(\mathbf{w})} a$. If variables $(\tau, x) \in Q$, then $(t,x) \in \widetilde{Q} = (0, 9C_1\rho^2 \frac{\mu(\mathbf{w})}{\rho(\mathbf{w})}] \times B(2\rho, \mathbf{w})$. Then $Q^- \to \widetilde{Q}^-$ $Q^+ \to \widetilde{Q}^+$ change obviously. For these new cylinders, Harnack inequality is still true.

Thus, if in the neighbourhood of each point $\mathbf{w} \in \Omega$ the inequality (47) holds true and in this neighborhood function $\mu(x)$ coincides with some weight ϑ (depending on point $\mathbf{w}$), then a non-negative solution to equation (1) is either positive everywhere in Ω or vanishes identically. This can be proven by the standard technique if the radius of the neighbourhood depends continuously on the point.

In what follows we shall consider a positive solution to equation (1).

Theorem 3. *Assume that $s > pf(s)$, $p \in (0,1)$ as $s \geqslant z_0$, Ω_f is a revolution domain and weight $\mu(x)$ coincides with some A_2-weight ϑ in $\Omega_{pf} \cap \{x_1 > z_0\}$. Assume that the inequalities*

$$\frac{f(x_1')}{f(x_1'')} \leqslant 2, \qquad \frac{\rho(x')\mu(x'')}{\mu(x')\rho(x'')} \leqslant C_1 \tag{48}$$

hold true for all $x', x'' \in \Omega_{pf}$ such that $x_1', x_1'' \in [s - pf(s), s + pf(s)]$ and all $s \geqslant z_0$. Then for a positive solution of equation (1) the inequality

$$\min_{x \in B(r', \mathbf{w})} u(t,x) \geqslant u(t_1, (z_0, 0)) \exp\left(-C_2 \int\limits_{z_0}^{\widetilde{r}(t)} \frac{ds}{f(s)}\right)$$

holds true, where $B(2r', \mathbf{w})$ is some ball inscribed in $\Omega_{pf} \cap \{z_0 < x_1 < \widetilde{r}(t)\}$, $t_1 > 0$ is some fixed number, $\widetilde{r}(t)$, $t \geqslant t_1$, is introduced as the smallest r satisfying the inequality

$$\int\limits_{z_0}^{r} \frac{ds}{f(s)} \geqslant tL(r), \quad L(r) = \inf_{[z_0, r]} \frac{4\rho(z,0)}{\mu(z,0)pf^2(z)},$$

while constant C_2 depends only on p, C_0, C_1, n, γ, γ_1.

Proof. Let $y_0 = z_0$ and $r \geqslant z_0$ be an arbitrary number. We construct a sequence of balls with radii r_i, $i = 1, 2, \ldots$, and touching points $\mathbf{v}_i = (y_{i-1} + 2r_i, 0)$ such that the double ball $B(2r_i, \mathbf{w}_i)$, where $\mathbf{w}_i = (z_i, 0)$, $z_i = y_{i-1} + r_i$, touches the set $\partial\Omega_{pf}$ from inside. We note that $r_{i+1} \leqslant 3r_i$ since otherwise $B(2r_i, \mathbf{w}_i) \subset B(2r_{i+1}, \mathbf{w}_{i+1})$, i.e.,ball $B(2r_i, \mathbf{w}_i)$ does not touch the boundary of Ω_{pf}.

We denote $\mu_i = \mu(\mathbf{w}_i)$, $\rho_i = \rho(\mathbf{w}_i)$, $t_1 = r_1^2 \frac{\mu_1}{\rho_1}$; $t_{i+1} = t_i + (9C_1 - 1)\frac{\mu_i}{\rho_i} r_i^2$.

If for some i the inequality $r_i \leqslant r_{i+1}$ holds true, then as $s = z_{i+1}$ we have $s - z_i \leqslant 2r_{i+1} \leqslant pf(s)$ and by (48) we obtain the inequality

$$\frac{\mu_{i+1}\rho_i}{\rho_{i+1}\mu_i} \leqslant C_1. \tag{49}$$

If $r_i > r_{i+1}$, then letting $s = z_i$, $z_{i+1} - s < 2r_i < pf(s)$, by (48) we again obtain (49). Moreover, as $s = z_i$, by (48) we get also an analogue of inequality (47):

$$C_1^{-1} \leqslant \frac{\rho(x)\mu(w_i)}{\mu(x)\rho(w_i)} \leqslant C_1, \ x \in B(2\rho_i, w_i)$$

and inequality

$$\frac{f(x_1')}{f(x_1'')} \leqslant 2, \ \forall x', x'' \in [s - 2r_i, s + 2r_i]. \tag{50}$$

Consider the cylinders

$$\widetilde{Q}_i = \left[t_i - \frac{\mu_i}{\rho_i} r_i^2, t_i + (9C_1 - 1)\frac{\mu_i}{\rho_i} r_i^2\right] \times B(2r_i, \mathbf{w}_i),$$
$$\widetilde{Q}_i^- = \left[t_i, t_i + \frac{\mu_i}{\rho_i} r_i^2\right] \times B(r_i, \mathbf{w}_i),$$
$$\widetilde{Q}_i^+ = \left[t_i + (8C_1 - 1)\frac{\mu_i}{\rho_i} r_i^2, t_i + (9C_1 - 1)\frac{\mu_i}{\rho_i} r_i^2\right] \times B(r_i, \mathbf{w}_i).$$

Let us show that if $t_{i+1} \leqslant T$, then $\widetilde{Q}_i \subset (0, T] \times \Omega_{pf}$. It is sufficient to establish that $t_i \geqslant \frac{\mu_i}{\rho_i} r_i^2$. The first step of induction is made. Then due to (49)

$$t_{i+1} = t_i + (9C_1 - 1)\frac{\mu_i}{\rho_i} r_i^2 \geqslant 9C_1 \frac{\mu_i}{\rho_i} r_i^2 \geqslant \frac{\mu_{i+1}}{\rho_{i+1}} r_{i+1}^2$$

that completes the induction.

Let k be the first index such that $y_{k+1} \geqslant r$ or $t_{k+1} \geqslant T$. Then by Harnack inequality

$$u(t_1, (y_0, 0)) \leqslant Hu(t_2, \mathbf{v}_1) \leqslant \ldots \leqslant H^k u(t_{k+1}, \mathbf{v}_k).$$

It yields $u(t_{k+1}, \mathbf{v}_k) \geqslant H^{-k} C_3$. Let us estimate number k from above. Let s_i be the abscissa for one of the points, where the ball $B(2r_i, \mathbf{w}_i)$ touches the boundary of domain Ω_{pf}. It is clear that $|z_i - s_i| \leqslant 2r_i$, $pf(s_i) \leqslant 2r_i$ and this is why due to (50), $f(s)/2 \leqslant f(s_i)$ as $s \in [y_{i-1}, y_i]$, and $r_i \geqslant pf(z_i)/4$. Then

$$k = \sum_{i=1}^{k} \frac{y_i - y_{i-1}}{2r_i} \leqslant \sum_{i=1}^{k} \frac{y_i - y_{i-1}}{pf(s_i)} \leqslant \sum_{i=1}^{k} \int_{y_{i-1}}^{y_i} \frac{2ds}{pf(s)} \leqslant \int_{y_0}^{r} \frac{2ds}{pf(s)}.$$

Let $\frac{\mu_m}{\rho_m} r_m^2 = \max\limits_{j \leqslant k} \frac{\mu_j}{\rho_j} r_j^2 \geqslant \max\limits_{z \in [z_0, r]} \frac{\mu(z,0)}{64 C_1 \rho(z,0)} (pf(z))^2$. The latter inequality follows from (48). For the indices $i = m+1, m+2, \ldots$ we replace the balls $B(2r_i, \mathbf{w}_i)$ by the balls $B(2r_m, \mathbf{w}_m)$. Cylinders $\widetilde{Q}_i, i = m+1, m+2, \ldots$ change appropriately. Since each cylinder increases t_i by $(9C_i - 1)\frac{\mu_m}{\rho_m} r_m^2$, then to reach value t we need at most

$$N = \left[\frac{t\rho_m}{(9C_1 - 1)\mu_m r_m^2}\right] \leqslant 2tL(r)/p$$

cylinders. Thus, we obtain the estimate

$$\min_{x \in B(r_m, \mathbf{w}_m)} u(t, x) \geqslant H^{-(k+N)} C_3 \geqslant \exp\left(-\left(\int_{z_0}^{r} \frac{2ds}{pf(s)} + 2tL(r)/p\right) \ln H\right), \tag{51}$$

that implies the statement of the theorem. □

We apply inequality (51) to the example in Section 4 to obtain

$$L(r) = \inf_{[z_0, r]} \frac{4\widetilde{\rho}(z)}{\widetilde{\mu}(z) p f^2(z)} = \frac{4\widetilde{\rho}(r)}{\widetilde{\mu}(r) p f^2(r)} \leqslant 16\lambda(r)/p.$$

Employing also inequality (43), we have

$$\int_{\Omega(r)} \mu(x) u^2(t, x) dx \geqslant \pi r_m^2 \min_{x \in B(r_m, v_m)} \mu(x) C_3 \exp\left(-\frac{8}{p^2} \ln H \left(\int_{z_0}^{r} p\sqrt{\widetilde{\lambda}}(s) ds\right) + 8t\lambda(r)\right).$$

Now the choice of $r = r(t)$ as in Introduction (under the assumption f continuity of function $\lambda(r)$):

$$t\lambda(r) = \int\limits_{R_0+1}^{r} \sqrt{\widetilde{\lambda}(s)}ds,$$

justifies in some sense the sharpness of upper estimate (7) if the factor at the exponential in the latter inequality is not too small.

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