ISSN 2304-0122 Ufa Mathematical Journal. Vol. 8. No 1 (2015). P. 1-12.

doi:10.13108/2016-8-1-1 UDC 517.538.2 + 517.984.26 + 517.547

SOME PROPERTIES OF PRINCIPAL SUBMODULES IN THE MODULE OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND POLYNOMIAL GROWTH ON THE REAL AXIS

N.F. ABUZYAROVA

Abstract. In the work we consider a topological module of entire functions $\mathcal{P}(a;b)$, which is the isomorphic image of Fourier-Laplace transform of Schwarz space formed by distributions with compact supports in a finite or infinite segment $(a;b) \subset \mathbb{R}$. We study the conditions ensuring that the principal submodule of module $\mathcal{P}(a;b)$ can be uniquely recovered by the zeroes of a generating function.

Keywords: entire functions, subharmonic functions, Fourier-Laplace transform, principal submodules, local description of submodules, invariant subspaces, spectral synthesis.

Mathematics Subject Classification: 30D15, 30H99, 42A38, 47E05

1. Introduction

Let $[a_1;b_1] \Subset [a_2;b_2] \Subset \dots$ be a sequence of segments exhausting a finite or infinite interval $(a;b)$ in the real axis, P_k be a Banach space consisting of entire functions φ having a finite norm

$$\|\varphi\|_k = \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{(1+|z|)^k \exp(b_k y^+ - a_k y^-)}, \quad y^\pm = \max\{0, \pm y\}, \quad z = x + \mathrm{i}y. \tag{1.1}$$

We denote $\mathcal{P}(a;b)$ the inductive limit of sequence $\{P_k\}$. The multiplication by independent variable z is continuous in this space and this is why $\mathcal{P}(a;b)$ is a topological module over ring of polynomials $\mathbb{C}[z]$. Each of the embedding $P_k \subset P_{k+1}$ is completely continuous and therefore, $\mathcal{P}(a;b)$ is a locally-convex space of type (LN^*) (see [1]). It is known (see, for instance, [2, Ch. I, Lect. 16, Thms. 1, 2]) that each element of space $\mathcal{P}(a;b)$ is a function of completely regular growth with order 1; its indicator diagram is a segment of the imaginary axis: $[\mathrm{i}c_\varphi; \mathrm{i}d_\varphi] \subset (\mathrm{i}a; \mathrm{i}b)$.

In the present work we study principal submodules of module $\mathcal{P}(a;b)$. We recall that a *principal submodule $\mathcal{J}_\varphi$ generated by a function $\varphi \in \mathcal{P}(a;b)$* is the closure of the set $\{p\varphi : \ p \in \mathbb{C}[z]\}$ in $\mathcal{P}(a;b)$.

For the sake of brevity, if else is not said, we shall say *"submodule"* meaning a closed submodule.

The submodules of module $\mathcal{P}(a;b)$ are in the duality relation with closed differentiation-invariant subspaces of space $C^\infty(a;b)$ (see [3], [4]). Namely, Fourier-Laplace transform $\mathcal{F}$ acting in the strongly dual space $(C^\infty(a;b))'$ by the rule

$$\mathcal{F}(S)(z) = (S, e^{-itz}), \quad S \in (C^\infty(a;b))'$$

is a linear topological isomorphism of spaces $(C^\infty(a;b))'$ and $\mathcal{P}(a;b)$ [5, Thm. 7.3.1]. At that, between the set $\{\mathcal{J}\}$ of the closed submodules of module $\mathcal{P}(a;b)$ and the set $\{W\}$ of

N.F. Abuzyarova, Some properties of principal submodules in the module of entire functions of exponential type and polynomial growth on the real axis.

The work is supported by grant no. 01201456408 of the Ministry of Education and Science of Russian Federation.

Submitted June 2, 2015.

closed differentiation-invariant subspaces of space $C^\infty(a;b)$ there is a one-to-one correspondence described by the rule $\mathcal{J} \longleftrightarrow W$ if and only if $\mathcal{J} = \mathcal{F}(W^0)$, where closed subspace $W^0 \subset (C^\infty(a;b))'$ consists of all distributions $S \in (C^\infty(a;b))'$ annulating W. The aim of the spectral synthesis for closed differentiation-invariant subspaces $W \subset C^\infty(a;b)$ was first considered in work [6] (for an arbitrary interval $(a;b) \subset \mathbb{R}$). This problem is dual to the problem on a (weak) localizability of submodules in $\mathcal{P}(a;b)$.

Let us recall the series of notions characterizing the property of submodules (see [3], [4], [7], [8]). For a submodule $\mathcal{J} \subset \mathcal{P}(a;b)$ we let $c_{\mathcal{J}} = \inf\limits_{\varphi \in \mathcal{J}} c_\varphi$, $d_{\mathcal{J}} = \sup\limits_{\varphi \in \mathcal{J}} d_\varphi$. The set $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ is called *indicator segment* of submodule $\mathcal{J}$.

The *divisor of a function* $\varphi \in \mathcal{P}(a;b)$ is defined by the formula

$$n_\varphi(\lambda) = \begin{cases} 0, & \text{if} \quad \varphi(\lambda) \neq 0, \\ m, & \text{if} \quad \lambda \text{ is a zero of } \ \varphi \text{ of multiplicity } \quad m, \end{cases}$$

for all $\lambda \in \mathbb{C}$, and the *divisor of submodule* $\mathcal{J} \subset \mathcal{P}(a;b)$ is introduced by the formula $n_{\mathcal{J}}(\lambda) = \min\limits_{\varphi \in \mathcal{J}} n_\varphi(\lambda)$.

A submodule $\mathcal{J}$ is *weakly localizable* if it contains *all* functions $\varphi \in \mathcal{P}(a;b)$ satisfying the conditions: 1) $n_\varphi(z) \geqslant n_{\mathcal{J}}(z)$, $z \in \mathbb{C}$; 2) the indicator diagram of function φ is contained in the set $\mathrm{i}[c_{\mathcal{J}}; d_{\mathcal{J}}]$. If $c_{\mathcal{J}} = a$ and $d_{\mathcal{J}} = b$, the weak localizability of $\mathcal{J}$ means that this submodule is *ample*.

Submodule $\mathcal{J}$ is called *stable at a point* $\lambda \in \mathbb{C}$ if the conditions $\varphi \in \mathcal{J}$ and $n_\varphi(\lambda) > n_{\mathcal{J}}(\lambda)$ imply $\varphi/(z-\lambda) \in \mathcal{J}$. Submodule $\mathcal{J}$ is *stable* if it is stable at each point $\lambda \in \mathbb{C}$.

It is clear that the *stability of submodule $\mathcal{J}$ is the necessary condition of its localizability.*

It follows from the results of [9, Sect. 4] that a principal submodule $\mathcal{P}(a;b)$ is always stable. It is can be also checked straightforwardly by employing the definition of stability and the description of the topology in $\mathcal{P}(a;b)$. By the duality principle [4, Prop. 1], the indicator segment of a principle submodule is $[c_\varphi; d_\varphi]$.

Given a function $\varphi \in \mathcal{P}(a;b)$, we denote by $\mathcal{J}(\varphi)$ a weakly localizable submodule with the divisor, which is equal to divisor n_φ of function φ, and with indicator segment $[c_\varphi; d_\varphi]$. In other words, submodule $\mathcal{J}(\varphi)$ consists of *all* functions $\psi \in \mathcal{P}(a;b)$ divisible by φ and having indicator $h_\psi = h_\varphi$.

Submodules $\mathcal{J}_\varphi$ and $\mathcal{J}(\varphi)$ have the same divisor, which is equal to n_φ, and the same indicator segment $[c_\varphi; d_\varphi]$. This is why the inclusion

$$\mathcal{J}_\varphi \subset \mathcal{J}(\varphi)$$

holds true. The identity

$$\mathcal{J}_\varphi = \mathcal{J}(\varphi) \tag{1.2}$$

is equivalent to the weak localizability of principal submodule $\mathcal{J}_\varphi$. As an example provided in work [10] shows, this identity is not always true.

There are two options to satisfy identity (1.2).

(I) Submodule $\mathcal{J}(\varphi)$, and therefore, principal submodule $\mathcal{J}_\varphi$, contain only functions $p\varphi$, $p \in \mathbb{C}[z]$. In other words, the generator φ is such that the set of entire functions of minimal type at order 1, which can be represented as Φ/φ, $\Phi \in \mathcal{P}(a;b)$, coincides with the set of polynomials $\mathbb{C}[z]$.

(II) The set $\mathcal{J}(\varphi) \setminus \{p\varphi : \quad p \in \mathbb{C}[z]\}$ is non-empty and for each function $\Phi \in \mathcal{J}(\varphi)$ there exists a generalized sequence of polynomials p_α such that $p_\alpha\varphi \to \Phi$ in the sense of the topology in space $\mathcal{P}(a;b)$.

The sufficient condition ensuring the first of the above options is the *invertibility* of function φ: function $\varphi \in \mathcal{P}(-\infty;+\infty)$ is called *invertible* (see [11]) if for each same function Φ the implication holds true: the condition "$\Phi \in \mathcal{P}(-\infty;+\infty)$, Φ/φ is an entire function" implies that $\Phi/\varphi \in \mathcal{P}(-\infty;+\infty)$, i.e., the principal ideal $\mathcal{I}_\varphi$ generated by this function in algebra $\mathcal{P}(-\infty;\infty)$ is closed.

Indeed, it is easy to see that if $\varphi \in \mathcal{P}(a;b)$ is invertible, then

$$\mathcal{J}(\varphi) = \mathcal{J}_\varphi = \{p\varphi : \quad p \in \mathbb{C}[z]\}. \tag{1.3}$$

The invertibility of the generating functions turns out to be not the necessary condition for (1.3). In what follows, in the second section, we shall construct an example of a non-invertible function $\varphi \in \mathcal{P}(a;b)$ satisfying relations (1.3).

Passing to case (II), we reproduce the above mentioned example in work [10]. Let $(a;b) = (-2\pi;2\pi)$ and

$$\varphi_0(z) = \frac{\sin \pi z}{U(z)V(z)}, \quad \text{where} \quad U(z) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}}, \quad V(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{2^{2n}+1}\right). \tag{1.4}$$

Theorem 1.2 in work [10] states (although in dual terms of admissibility of the spectral synthesis in the weak sense) that principal submodule $\mathcal{J}_{\varphi_0}$ is not weakly localizable in $\mathcal{P}(-2\pi;2\pi)$.

In the third section of the present work we obtain some necessary conditions for the weak localizability of principal submodule $\mathcal{J}_\varphi$ in $\mathcal{P}(a;b)$ in the case, when the set

$$\mathcal{J}(\varphi) \setminus \{p\varphi, \ p \in \mathbb{C}[z]\}$$

is non-empty. We also prove the following statement involving the above cited result [10, Thm. 1.2] as a particular case.

Theorem 3. *Suppose that the generator of submodule $\mathcal{J}_\varphi$ is of the form*

$$\varphi = \frac{\Phi}{\omega},$$

where $\Phi = e^{i\gamma z}S \in \mathcal{P}(a;b)$, S is a sine-like function, $\gamma \in \mathbb{R}$, ω is an entire function of the minimal type at order 1.

If the orders of function ω on the rays $\arg z = 0$ and $\arg z = \pi$ determined by the identities

$$\rho_0 = \limsup_{r\to+\infty} \frac{\ln\ln|f(r)|}{\ln r}, \quad \rho_\pi = \limsup_{r\to+\infty} \frac{\ln\ln|f(-r)|}{\ln r}, \quad \textit{respectively,}$$

satisfy one of the relations

$$\rho_0 < 1/4 < 1/2 \leqslant \rho_\pi \quad \textit{or} \quad \rho_\pi < 1/4 < 1/2 \leqslant \rho_0, \tag{1.5}$$

then submodule $\mathcal{J}_\varphi$ is not weakly localizable.

2. Example of non-invertible function satisfying relations (1.3)

Suppose that ends a and b of an interval satisfy the conditions

$$a < -\pi, \quad \pi < b.$$

We let

$$\varphi(z) = \frac{s(z)}{s_1(z)} + \frac{\pi z s(z)}{s_0(z)},$$

where

$$s(z) = \frac{\sin \pi z}{\pi z}, \quad s_1(z) = s(\sqrt{z}) = \frac{\sin \pi\sqrt{z}}{\pi\sqrt{z}}, \quad s_0(z) = \prod_{k=1}^{\infty}\left(1 + \frac{z}{2^{2k}}\right).$$

It is well-known that function s satisfy the estimates

$$|s(z)| \leqslant \frac{c_0 e^{\pi|\mathrm{Im}\, z|}}{\pi(1+|z|)}, \quad z \in \mathbb{C}, \tag{2.1}$$

$$|s(z)| \geqslant \frac{m_d e^{\pi|\mathrm{Im}\, z|}}{\pi|z|}, \quad |z-k| \geqslant d, \quad k \in \mathbb{Z}, \tag{2.2}$$

where c_0 is an absolute constant, $d \in (0;1/2)$ is an arbitrary number, m_d is a positive number depending on d. It follows from (2.1) that entire function s_1 obeys the upper bound:

$$|s_1(z)| \leqslant \frac{c_0 e^{\pi\sqrt{|z|}|\sin(\theta/2)|}}{\pi(1+\sqrt{|z|})}, \quad z = re^{i\theta}, \quad -\pi < \theta \leqslant \pi, \quad r > 0. \tag{2.3}$$

We summarize other auxiliary estimates as lemmata.

Lemma 1. *Let a number $d_0 \in (0;1/2)$ be small enough so that $\left|\frac{\sin \pi\xi}{\pi\xi} - 1\right| \leqslant 1/2$ as $\pi|\xi| \leqslant d_0$. Then there exists a constant $c_{d_0} > 0$ such that*

$$|s_1(z)| \geqslant \frac{c_{d_0} e^{\pi\sqrt{|z|}|\sin(\theta/2)|}}{1+|z|}, \quad z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{N}} \left\{z : \, |z-k^2| < 3d_0\right\}. \tag{2.4}$$

Proof. We first observe that for each z satisfying inequalities

$$\frac{d_0}{|k|} \leqslant |z-k| \leqslant d_0, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{2.5}$$

the estimate

$$|s(z)| \geqslant \frac{d_0}{4|z|^2} \tag{2.6}$$

holds true.

Employing standard methods, by inequalities (2.2) and (2.6) we obtain the estimate

$$|s(z)| \geqslant \frac{c_{d_0} e^{\pi|\mathrm{Im}\, z|}}{1+|z|^2} \quad z \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}\setminus\{0\}} \left\{z : \, |z-k| < \frac{d_0}{|k|}\right\}, \tag{2.7}$$

where c_{d_0} is a positive constant depending on d_0. Then the statement of the lemma is implied by (2.7). □

Lemma 2. *For each $\theta \in (-\pi;\pi)$ the asymptotic identity*

$$\ln s_0(re^{i\theta}) = \frac{(\ln r)^2}{\ln 8} + \frac{i\theta \ln r}{\ln 4} + o(\ln r), \quad r \to \infty, \tag{2.8}$$

holds true. There exist a number $\delta > 0$ and a set $E_0 \subset (-\infty;0)$ of zero relative measure such that for each $x \in (-\infty;0) \setminus E_0$ the identity

$$\ln|s_0(x)| \geqslant \delta \left(\ln(|x|+1)\right)^2 \tag{2.9}$$

holds true.

Proof. Counting function $n(r)$ of zeroes of function s_0 satisfies the asymptotic relation

$$n(r) = \frac{\ln r}{\ln 4} + o(\ln r), \quad r \to \infty. \tag{2.10}$$

Hence, by Theorem 1 in work [12], function s_0 has a *strong regular growth* and it satisfies asymptotic relation (2.8).

By (2.10), function s_0 satisfies the assumptions of Theorem 3.6.1 [13]. This theorem states that

$$\frac{\min\limits_{|z|=r}|s_0(z)|}{\max\limits_{|z|=r}|s_0(z)|} \to 1, \tag{2.11}$$

as $r \to +\infty$ outside some set E_0 of zero relative measure.

By (2.11) we obtain that for some number $\delta > 0$ inequality (2.9) holds true everywhere on the half-line $(-\infty; 0)$ except set E_0. □

Theorem 1. *Function φ is contained in $\mathcal{P}(a;b)$ and is not invertible. Submodules $\mathcal{J}_\varphi$ and $\mathcal{J}(\varphi)$ satisfy relations (1.3).*

Proof. We consider the function $\varphi_1 = s/s_1$. This function satisfies the following estimate on the real axis:

$$|\varphi_1(x)| \leqslant \frac{c_0}{\pi c_{d_0} e^{\pi\sqrt{|x|}}}, \quad x \leqslant 0, \tag{2.12}$$

$$|\varphi_1(x)| \leqslant \frac{c_0 e^{3d_0\pi}}{\pi c_{d_0}}, \quad x > 0. \tag{2.13}$$

The former of these estimates is a direct implication of estimates (2.1) and (2.4), while the other, (2.13), can be obtained from the same estimates in the standard way by employing maximum principle for analytic functions. In their turn, estimates (2.12) and (2.13) imply that function φ_1 is bounded on the real axis. Taking into consideration that it has type π at order 1, we conclude that

$$\varphi_1 \in \mathcal{P}(a;b). \tag{2.14}$$

Let us show that function $\varphi_2 = (\pi z s)/s_0$ is also contained in $\mathcal{P}(a;b)$. Both this function and function φ_1 have type π at order 1.

It follows from the proof of Lemma 2 that for each $\varepsilon \in (0;1/2)$ there exists $\delta > 0$ such that outside the union of the rings

$$A_j = \{(1-\varepsilon)4^j \leqslant |z| \leqslant (1+\varepsilon)4^j\}, \quad j = 1, 2, \ldots,$$

the inequality

$$\ln|s_0(z)| \geqslant \delta\left(\ln\left(|z|+1\right)\right)^2. \tag{2.15}$$

holds true. Hence, for all real

$$x \notin \bigcup_{j=1}^{\infty}(-(1+\varepsilon)4^j; -(1-\varepsilon)4^j)$$

the inequality

$$\ln|s_0(x)| \geqslant \delta\left(\ln\left(|x|+1\right)\right)^2 \tag{2.16}$$

holds true.

In order to estimate function φ_2 in the intervals

$$(-(1+\varepsilon)4^j; -(1-\varepsilon)4^j), \quad j \in \mathbb{N}, \tag{2.17}$$

we observe that by (2.15), on the boundary of ring A_j the inequality

$$\ln|\varphi_2(z)| \leqslant \ln\left|\frac{\sin\pi z}{1 - z^2/4^{2j}}\right| + 2\ln(2+\varepsilon) - \delta\left(\ln((1-\varepsilon)4^j + 1)\right)^2$$

holds true. Since the right hand of this inequality is a function harmonic in ring A_j, this inequality is true for each $z \in A_j$. Therefore, there exist positive numbers $\tilde{\delta} > \delta$ and $\tilde{c} > 1$ depending on δ and ε and independent of j such that in intervals (2.17) the estimate

$$|\varphi_2(x)| \leqslant \frac{\tilde{c}}{e^{\tilde{\delta}(\ln(|x|+1))^2}}, \quad x \in (-(1+\varepsilon)4^j; -(1-\varepsilon)4^j), \quad j \in \mathbb{N},$$

holds true. In view of (2.16) it implies that on the real axis the inequality

$$|\varphi_2(x)| \leqslant \frac{\tilde{c}}{e^{\tilde{\delta}(\ln(|x|+1))^2}} \tag{2.18}$$

holds true. Applying Paley-Wiener-Schwarz theorem [5, Thm. 7.3.1], we conclude that

$$\varphi_2 \in \mathcal{F}(C_0^\infty(a;b)) \subset \mathcal{P}(a;b). \tag{2.19}$$

It follows from (2.14) and (2.19) that function φ belongs to space $\mathcal{P}(a;b)$.

In order to prove the non-invertibility of function φ, we shall make use of the analyticity criterion by L. Ehrenpreis [14, Thm. I]:
function $\varphi \in \mathcal{P}(a;b)$ is invertible if and only if there exists a positive number a with the property: for each $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that

$$|x-y| \leqslant a\ln(1+|x|), \quad \varphi(y) \geqslant (a+|y|)^{-a}.$$

By (2.12) and (2.18), there exists a positive number c_1 such that on the ray $(-\infty;0)$ function φ satisfies the estimate

$$\ln|\varphi(x)| \leqslant -\tilde{\delta}\,(\ln(|x|+1))^2 + c_1.$$

Comparing this estimate and the invertibility criterion by L. Ehrenpreis, we conclude that function φ is not invertible.

Let us prove the latter of the statements formulated for function φ, which is the identity

$$\mathcal{J}(\varphi) = \{p\varphi : \ \ p \in \mathbb{C}[z]\}. \tag{2.20}$$

It follows from estimates (2.2), (2.4) and relation (2.8) that for each positive θ_0 there exists a constant $a_0 = a_0(\theta_0)$ such that outside the angles $\{z : \ |\arg z| < \theta_0\}$, $\{z : \ |\pi - \arg z| < \theta_0\}$ function φ admits the lower bound:

$$|\varphi(z)| \geqslant |s(z)| \left(\frac{\pi|z|}{|s_0(z)|} - \frac{1}{|s_1(z)|}\right) \geqslant \frac{a_0 e^{\pi|\operatorname{Im} z|}}{\exp\left((\ln(|z|+1))^2/\ln 8\right)}. \tag{2.21}$$

Let Φ be an arbitrary function in submodule $\mathcal{J}(\varphi)$. For some $C_0 > 0$ and $k \in \mathbb{N}$ we have

$$|\Phi(z)| \leqslant C_0(1+|z|)^k e^{\pi|\operatorname{Im} z|}, \quad z \in \mathbb{C}. \tag{2.22}$$

Employing this relation, estimate (2.21) and Phragmén-Lindelöf principle, it is easy to get that function $\omega = \Phi/\varphi$ satisfies the estimate

$$|\omega(z)| \leqslant C e^{k\ln(|z|+1)+(\ln(|z|+1))^2} \tag{2.23}$$

in the whole complex plane, where $C > 0$ is some constant. In particular, this estimate means that ω is an entire function of zero order.

Let us estimate function ω on the ray $(3d_0;+\infty)$. In order to do it, we observe that by (2.2), (2.3), (2.8), everywhere in the half-strip $\{z = x+iy : \ x > 3d_0, \ |y| \leqslant d_0\}$ but outside the circles $|z-k| < 3d_0$, $k \in \mathbb{N}$, the estimate

$$|\varphi(z)| \geqslant |s(z)| \left(\frac{1}{|s_1(z)|} - \frac{\pi|z|}{|s_0(z)|}\right) \geqslant \frac{b_0}{1+|z|} \tag{2.24}$$

holds true for some constant $b_0 > 0$.

Taking into consideration estimate (2.22) for function Φ, by (2.24) we obtain that for all positive x the inequality
$$|\omega(x)| \leqslant (C_0/b_0)(1+x)^{k+1} \tag{2.25}$$
holds true. Estimates (2.23) and (2.25) and Phragmén-Lindelöf principle imply that ω is a polynomial. Since this fact holds true for each entire function ω of the form Φ/φ, $\Phi \in \mathcal{J}(\varphi)$, we conclude that desired relation (2.20) holds true for the submodules. □

3. Necessary conditions of weak localizability of principle submodule

We denote by $\mathcal{P}_0(a;b) \subset \mathcal{P}(a;b)$ the image of the space of compactly supported infinitely differentiable functions $C_0^\infty(a;b) \subset (C^\infty(a;b))'$ under the transform $\mathcal{F}$.

We consider function $\varphi \in \mathcal{P}(a;b)$, for which submodule $\mathcal{J}_\varphi$ contains the elements of the form
$$\Phi = \omega\varphi, \quad \omega \quad \text{is an entire function not being a polynomial.} \tag{3.1}$$
In this section we obtained some conditions necessary for the weak localizability of principal submodule $\mathcal{J}_\varphi$

Theorem 2. *Principal submodule $\mathcal{J}_\varphi$ contains function Φ of the form (3.1) if and only if $\varphi \in \mathcal{P}_0(a;b)$.*

Proof. 1. *Necessity.* Let us prove the equivalent implication: the condition
$$\varphi \notin \mathcal{P}_0(a;b) \tag{3.2}$$
implies the identity
$$\mathcal{J}_\varphi = \{p\varphi : \ p \in \mathbb{C}[z]\}. \tag{3.3}$$
In accordance with the aforementioned Paley-Wiener-Schwarz theorem [5, Thm. 7.3.1], it follows from (3.2) that there exist a natural number k_0 and a real sequence
$$x_n, \quad n = 1, 2, \ldots, \quad |x_n| \to \infty,$$
for which
$$|\varphi(x_n)| \geqslant |x_n|^{-k_0}, \quad n = 1, 2, \ldots \tag{3.4}$$
On the other hand, the inclusion $\varphi \in \mathcal{P}(a;b)$ means that for some $C > 0$ and $m_0 \in \mathbb{N}\bigcup\{0\}$ the estimate
$$|\varphi(z)| \leqslant C(1+|z|)^{m_0} e^{b_{m_0}y^+ - a_{m_0}y^-} \tag{3.5}$$
holds true everywhere in $\mathbb{C}$, where $y^\pm = \max\{0, \pm y\}$, $z = x + \mathrm{i}y$, $a < a_{m_0} < b_{m_0} < b$. Estimates (3.4) and (3.5) imply that for each natural j, the closure of the set (possibly, an empty one)
$$P_j \bigcap \{p\varphi : \ p \in \mathbb{C}[z]\} \tag{3.6}$$
in Banach space P_j is contained in the set (possibly, an empty one)
$$P_j \bigcap \{p\varphi : \ p \in \mathbb{C}[z], \ \deg p \leqslant j + k_0 - m_0\},$$
which is, in its turn, a subset of set (3.6). Therefore, set (3.6) is closed for each $j \in \mathbb{N}$. In accordance with the criterion of the closedness in a space of type (LN^*) [1, Thm. 1], the set $\{p\varphi : \ p \in \mathbb{C}[z]\}$ is closed in $\mathcal{P}(a;b)$ and therefore, (3.3) holds true.

2. *Sufficiency.* Let $\varphi = \mathcal{F}(s)$, $s \in C_0^\infty(a;b)$, $[a_0;b_0]$ be the closure the convex hull of the support of function s, $[a_0;b_0] \Subset (a;b)$ and let $\varphi \in P_{k_1}$.

By Paley-Wiener-Schwarz theorem, there exist positive constants C_n, $n = 1, 2, \ldots$, such that the estimates
$$|\varphi(z)| \leqslant \frac{C_n}{(1+|z|)^n} e^{b_0 y^+ - a_0 y^-}, \quad z = x + iy \in \mathbb{C}, \quad n \in \mathbb{N}, \tag{3.7}$$
hold true.

We let

$$f(r) = \sup_{n\in\mathbb{N}}(n\ln(1+r) - \ln C_n),$$

and consider a subharmonic in $\mathbb{C}$ function $v(z) = f(|z|)$. According to Theorem 5 in work [15], there exists an entire function ω such that outside the set of circles with a finite sum of radii, the inequality

$$|\ln|\omega(z)| - v(z)| \leqslant m_0 \ln(1+|z|)$$

holds true for some natural number m_0. In particular, $\omega \notin \mathbb{C}[z]$. Therefore, $\Phi = \omega\varphi$ is an entire function of the form (3.1) belonging to submodule $\mathcal{J}(\varphi)$.

Let us show that $\Phi \in \mathcal{J}_\varphi$, in other words, that function Φ can be approximated in the topology of space $\mathcal{P}(a;b)$ by functions of the form $p\varphi$, where p is a polynomial.

The possibility of such approximation is implied by the following statement.

Lemma 3. *There exists a sequence of polynomials p_j converging to function ω on the real axis in the weighted norm $\|\cdot\|_V$ determined by the formula*

$$\|f\|_V = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{V(x)}, \tag{3.8}$$

where $V(x) = C_1(1+|x|)^{m_0+3}e^{v(x)}$ and constant C_1 comes from inequalities (3.7).

Proof of Lemma 3. In monograph [16, Ch. VI], as weight V, there was chosen an *even* weight function W defined on the real axis and satisfying the conditions

1) $W(x) \geqslant 1$, $x \in \mathbb{R}$,
for each natural n, the fraction $x^n/W(x)$ tends to zero as $x \to \pm\infty$,
$\ln W(x)$ is a convex function of $t = \ln|x|$;
2) for each $\delta > 1$ there exists a constant $C_\delta > 0$ such that

$$x^2 W(x) \leqslant C_\delta(\delta x), \quad x \in \mathbb{R}.$$

It follows from de Branges theorem [16, VI.H.1] and the theorems proved by P. Koosis in the same work [16, VI.H.2] that for weight W satisfying Conditions 1) and 2) each entire function ω of minimal type at order 1 growing over the real axis slower than W:

$$\frac{|\omega(x)|}{W(x)} \to 0, \quad x \to \pm\infty,$$

is approximated by polynomials in the norm $\|\omega\|_W = \sup\limits_{x\in\mathbb{R}} \frac{|\omega(x)|}{W(x)}$.

The function $\tilde{V}(x) = C_1(1+|x|)^{m_0+1}e^{v(x)}$ satisfies Conditions 1) and, generally speaking, does not satisfy Condition 2). However, tracking the proof by P. Koosis in [16, VI.H.2], we see that it is possible to approximate function ω by polynomials on the real axis in the norm $\|\cdot\|_V$, $V = (1+|x|)^2\tilde{V}$. □

The definition of function V yields that there exists a constant $C_0 > 0$ such that

$$|p_j(x)\varphi(x)| \leqslant C_0(1+|x|)^{m_0+3}, \quad j = 1, 2, \dots$$

on the real axis. Employing Phragmén-Lindelöf principle, we obtain on the complex plane

$$|p_j(z)\varphi(z)| \leqslant \tilde{C}_0(1+|z|)^{m_0+3}e^{b_0 y^+ - a_0 y^-}, \quad j = 1, 2, \dots$$

Taking into consideration that space $\mathcal{P}(a;b)$ belongs to the class of locally-convex spaces of type (LN^*) and employing the properties of such spaces, by the above estimates we get that there exists a subsequence in this sequence converging to function Φ in $\mathcal{P}(a;b)$. □

Remark 1. *Function $\varphi_1 = (\sin \pi z)/(\sqrt{z} \sin \pi \sqrt{z})$ considered in Section 1 does not belong to class $\mathcal{P}_0(a;b)$, but the set*

$$\mathcal{J}(\varphi_1) \setminus \{p\varphi : \ \mathbb{C}[z]\}$$

contains the function $\frac{\sin \pi\sqrt{z}}{\sqrt{z}}$ and therefore, it is non-empty. Thus, as opposed to principal submodule $\mathcal{J}_\varphi$, submodule $\mathcal{J}(\varphi)$ can contain functions $\omega\varphi$, $\omega \notin \mathbb{C}[z]$ also when generating function φ is not in class $\mathcal{P}_0(a;b)$. Nevertheless, the proven theorem implies that principal submodule $\mathcal{J}_\varphi$ with a generator $\varphi \notin \mathcal{P}_0(a;b)$ is weakly localizable if and only if relations (1.3) hold true.

Proof of Theorem 3. We begin with an auxiliary statement.

Lemma 4. *Under the assumptions of the theorem there exists a positive number d such that for each natural n function φ can be represented as the product of two entire functions $\varphi_{1,n}$ and $\varphi_{2,n}$ satisfying the condition: for all z outside the strip $|\operatorname{Im} z| < 3d$ the inequalities*

$$\big|\ln|\varphi_{1,n}(z)| - 2^{-n}\ln|\varphi(z)|\big| \leqslant \ln(1+|z|) + A_0 \tag{3.9}$$

hold true, where A_0 is a positive constant depending only on d, a, b.

Proof of Lemma 4. Since the zero set of function φ is a part of the zero set of a sine-like function, it is contained in some horizontal strip $|\operatorname{Im} z| < d/2$, see, for instance, [2, Ch. III, Lect. 22]).

We shall make use of the following theorem in work [17, Thm. 2]:
Let f be an entire function, whose zeroes are located in the strip $|\operatorname{Im} z| \leqslant d/2$, and there exists an entire function F divisible by function f and satisfying the conditions

$$\ln|F(z)| \leqslant H(z), \quad F(0) = 1, \tag{3.10}$$

where function H is Lipschitz:

$$|H(z') - H(z'')| \leqslant \sigma|z' - z''|, \quad z', z'' \in \mathbb{C}.$$

Then f is represented as the product of two entire functions f_1 and f_2, and for z, $|\operatorname{Im} z| \geqslant 3d$, and each $p \geqslant 1$ the relation

$$|\ln|f_1(z)| - \ln|f_2(z)|| \leqslant \frac{C_0}{p}\left(H(z) - \ln|F(z)|\right) + C_1 + \ln(1+|z|) + C_2 + C_3 e^p \tag{3.11}$$

holds true, where C_j are some constants depending on σ, d, $H(0)$.

We let $f = \varphi$, $F = \Phi$, $H(re^{i\theta}) = h_\Phi(\theta) r$, h_Φ is the indicator of function Φ, $\sigma = \max\limits_{\theta\in[0;2\pi]} |h_\Phi(\theta)|$, $p = 1$. Since by the properties of sine-like function [2] as $|\operatorname{Im} z| \geqslant 3d$ we have

$$|H(z) - \ln|F(z)|| = |h_\Phi(\arg z)|z| - \ln|\Phi(z)|| \leqslant C_4,$$

where constant C_4 depend only on function Φ, we obtain the representation of function φ as the product of two entire functions $\varphi_{1,1}$ and $\varphi_{2,1}$, and

$$|\ln|\varphi_{1,1}(z)| - \ln|\varphi_{2,1}(z)|| \leqslant \ln(1+|z|) + A_0, \quad |\operatorname{Im} z| \geqslant 3d, \tag{3.12}$$

constant A_0 depends only on function Φ.

By (3.12) and identity

$$\ln|\varphi| = \ln|\varphi_{1,1}| + \ln|\varphi_{2,1}|$$

we obtain the estimate

$$\left|\ln|\varphi_{1,1}(z)| - \frac{1}{2}\ln|\varphi(z)|\right| \leqslant \frac{1}{2}\ln(1+|z|) + \frac{A_0}{2}, \quad |\operatorname{Im} z| \geqslant 3d. \tag{3.13}$$

Applying the above cited theorem by R.S. Yulmukhametov to the function $f = \varphi_{1,1}$ with the same F, H, σ and p as above, we obtain the representation

$$\varphi_{1,1} = \varphi_{1,2}\varphi_{2,2},$$

where entire function $\varphi_{1,2}$ satisfies the estimate

$$\left|\ln|\varphi_{1,2}(z)| - \frac{1}{2}\ln|\varphi_{1,1}(z)|\right| \leqslant \frac{1}{2}\ln(1+|z|) + \frac{A_0}{2}, \quad |\operatorname{Im} z| \geqslant 3d.$$

This estimate and (3.13) yield

$$\left|\ln|\varphi_{1,2}(z)| - \frac{1}{2^2}\ln|\varphi(z)|\right| \leqslant \left(\frac{1}{2}+\frac{1}{2^2}\right)(\ln(1+|z|)+A_0), \quad |\operatorname{Im} z| \geqslant 3d.$$

Repeating this process, in n steps we obtain the representation of function φ as the product of two entire functions $\varphi_{1,n}$ and $\varphi_{2,n}$ and for all z outside the strip $|\operatorname{Im} z| < 3d$ desired estimate (3.9) holds true. □

Let us prove that under the assumption of the theorem function Φ can not belong to principal module $\mathcal{J}_\varphi$. We assume the opposite: there exists a generalized sequence of polynomials p_α such that $p_\alpha\varphi$ converges to Φ in space $\mathcal{P}(a;b)$. We fix a natural number n_0, for which function $\varphi\varphi_{1,n_0}$ lies in $\mathcal{P}(a;b)$. Employing the properties of space $\mathcal{P}(a;b)$, it is easy to prove the existence of a *countable* subsequence $p_{\alpha_k}\varphi\varphi_{1,n_0}$, $k=1,2,\dots$, converging to function $\Phi\varphi_{1,n_0}$ in one of the norms $\|\cdot\|_{m_0}$ (cf. (1.1)). In particular, this subsequence is bounded in the sense of this norm: for some constant $C>0$ and all natural numbers k we have

$$|p_{\alpha_k}(z)\varphi(z)\varphi_{1,n_0}(z)| \leqslant C(1+|z|)^{m_0}\exp(b_{m_0}y^+ - a_{m_0}y^-), \quad y=\operatorname{Im} z, \quad z\in\mathbb{C}.$$

By these inequalities, Lemma 4 and the properties of sine-like functions we obtain that on the line $\operatorname{Im} z = y_0$, $|y_0|\geqslant 3d$, the estimates

$$|p_{\alpha_k}(z)| \leqslant \tilde{C}(1+|z|)^{m_0+1}|\omega(z)|^{1+2^{-n_0}} \tag{3.14}$$

hold true, where $\tilde{C}$ is a positive constant depending only on d.

Assume that the former of relations (1.5) hold true and let us estimate $|p_{\alpha_k}(z)|$ on the half-line $z=x+iy_0$, $x>0$, $y_0\geqslant 3d$.

According to the remark after Theorem3 in [2, §14.2] and in view of the fact that function ω has the minimal type at order 1, for all $x\in\mathbb{R}$, $y_0>0$ we can write

$$\ln|\omega(x+iy_0)| = \frac{y_0}{\pi}\int_{-\infty}^{+\infty}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t + \sum_{j=1}^{\infty}\ln\left|\frac{x+iy_0-\lambda_j}{x+iy_0-\bar{\lambda}_j}\right|,$$

where $\{\lambda_j\}$ is the set of zeroes of function ω in the upper half-plane.

Let us estimate $\int\limits_{-\infty}^{+\infty}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t$ for positive x and y_0. We have

$$\int_{-\infty}^{+\infty}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t = \int_{-\infty}^{0}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t + \int_{0}^{2x}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t + \int_{2x}^{+\infty}\frac{\ln|\omega(t)|}{(t-x)^2+y_0^2}\mathrm{d}\,t = I_1+I_2+I_3. \tag{3.15}$$

For the first term I_1 we have the estimate

$$|I_1| \leqslant \int_{-\infty}^{+\infty}\frac{|\ln|\omega(t)||}{t^2+y_0^2}\mathrm{d}\,t < +\infty, \tag{3.16}$$

the finiteness of the integral is implied by the remark in [2, Sect. 14.2]. For each positive number $\varepsilon < 1/8-\rho_0/2$ there exist positive constants b_ε, c_ε such that

$$\ln|\omega(x)| \leqslant b_\varepsilon x^{\rho_0+\varepsilon} + c_\varepsilon$$

for all $x > 0$. This is why terms I_2 and I_3 can be estimated as follows:

$$I_2 \leqslant (2^{\rho_0+\varepsilon} b_\varepsilon x^{\rho_0+\varepsilon} + c_\varepsilon) \int_0^{2x} \frac{\mathrm{d}\,t}{(t-x)^2 + y_0^2} \leqslant \frac{\pi}{y_0} \left(2^{\rho_0+\varepsilon} b_\varepsilon x^{\rho_0+\varepsilon} + c_\varepsilon\right), \tag{3.17}$$

$$I_3 \leqslant (b_\varepsilon + c_\varepsilon) \left(\int_1^{+\infty} \frac{t^{\rho_0+\varepsilon}}{t^2/4 + y_0^2} \mathrm{d}\,t + y_0^{-2} \right) \leqslant (b_\varepsilon + c_\varepsilon) \left(4 \int_1^{+\infty} \frac{\mathrm{d}\,t}{t^{2-\rho_0-\varepsilon}} + y_0^{-2} \right). \tag{3.18}$$

It follows from relations (3.14)–(3.18) that on the half-line $z = x + iy_0$, $x > 0$, $y_0 \geqslant 3d$ the estimates

$$|p_{\alpha_k}(z)| \leqslant C'(1 + |z|)^{m_0+1} \exp(C''|z|^{\rho_0+\varepsilon}), \quad k = 1, 2, \ldots,$$

hold true, where C', C'' are positive constants depending on ε and y_0 and independent of x and k.

Employing Phragmén-Lindelöf principle, by these estimates it is easy to see that the inequalities

$$|p_{\alpha_k}(z)| \leqslant C \exp(|z|^{\rho_0+2\varepsilon}), \quad k = 1, 2, \ldots,$$

hold true in the complex plane and constant $C > 0$ depends on ε but is independent of k and z. In its turn, it implies that function ω (being equal to the limit of sequence p_{α_k}) should have order in the whole plane less than $1/4$ that is impossible by conditions (1.5). □

Remark 2. *The condition* $\max(\rho_0, \rho_\pi) \geqslant 1/2$ *is necessary for the strict inequality* $\min(\rho_0, \rho_\pi) < \max(\rho_0, \rho_\pi)$ *by Wiman theorem (see, for instance,* [18, Ch. 1, Sect. 18, Thm. 30]*).*

Remark 3. *Function $V(-z)$, where $V(z)$ is the function in the definition of φ_0 in (1.4), satisfies both relations, (2.8) and (2.9) of Lemma 2. Employing this fact and Lemma 1, it is easy to make sure that function φ_0 in work* [10] *cited in Introduction satisfies the assumptions of the proven theorem. Namely, $\varphi_0 = \frac{\sin \pi z}{\omega}$, where $\omega = UV$, and orders ρ_0 and ρ_π of function ω are equal to* 0 *and* 1/2*, respectively. Applying Theorem 3 gives the proof of the absence of the weak localizability for principal module $\mathcal{J}_\varphi$ in each module $\mathcal{P}(a; b)$, $a < -\pi$, $\pi < b$, and this proof is different in comparison with that given in* [10].

BIBLIOGRAPHY

1. J. Sebastian-e-Silva. "On some classes of locally convex spaces important in applications" // Matematika. Sbornik Perevodov. **1**, 60–77 (1957). (in Russian).
2. B.Y. Levin (in collaboration with Yu. Lyubarskii, M. Sodin, V. Tkachenko). *Lectures on entire functions.* (Rev. Edition). AMS, Providence, Rhode Island (1996).
3. N.F. Abuzyarova. *Spectral synthesis in the Schwartz space of infinitely differentiable functions* // Dokl. Akad. Nauk. **457**:5, 510-513 (2014). [Dokl. Math. **90**:1, 479-482 (2014).]
4. N.F. Abuzyarova. *Closed submodules in the module of entire functions of exponential type and polynomial growth on the real axis* // Ufimskij Matem. Zhurn. **6**:4, 3–18 (2014). [Ufa Math. J. **6**:4, 3–17 (2014).]
5. L. Hörmander. *The analysis of linear partial differential operators I: distribution theory and Fourier analysis.* Springer, Berlin, (1990).
6. A. Aleman, B. Korenblum. *Derivation-invariant subspaces of C^∞* // Comp. Meth. Funct. Theory. **8**:2, 493–512 (2008).
7. I.F. Krasičkov-Ternovskiĭ. *Local description of closed ideals and submodules of analytic functions of one variable. I* // Izvestia AN SSSR. Ser. Matem. **43**:1, 44–66 (1979). [Math. USSR-Izvestiya. **14**:1, 41-60 (1980).]
8. I.F. Krasičkov-Ternovskiĭ. *Invariant subspaces of analytic functions. I. Spectral analysis on convex regions* // Matem. Sbornik. **87(129)**:4, 459–489 (1972). [Math. USSR-Sbornik. **16**:4, 471–500 (1972).]

9. I.F. Krasičkov-Ternovskiĭ. *Local description of closed ideals and submodules of analytic functions of one variable. II* // Izvestia AN SSSR. Ser. Matem. **43**:2, 309–341 (1979). [Math. USSR-Izvestiya. **14**:2, 289–316 (1980).]
10. A. Aleman, A. Baranov, Yu. Belov. *Subspaces of C^∞ invariant under the differentiation* // J. Funct. Anal. **268**:8, 2421–2439 (2015).
11. C.A. Berenstein, B.A. Taylor. *A new look at interpolation theory for entire functions of one variable* // Adv. Math. **33**:2, 109–143 (1980).
12. N.V. Zabolotskii. *Strongly regular growth of entire functions of order zero* // Matem. Zametki. **63**:2, 196–208 (1998). [Math. Notes. **63**:2, 172–182 (1998).]
13. R.P. Boas, Jr. *Entire functions.* Acad. Press, New-York (1954).
14. L. Ehrenpreis. *Solution of some problems of division. Part. IV. Invertible and Elliptic Operators* // Amer. J. Math. **82**:3, 522–588 (1960).
15. R.S. Yulmukhametov. *Approximation of subharmonic functions* // Anal. Math. **11**:3, 257–282 (1985). (in Russian).
16. P. Koosis. *The logarithmic integral I.* Cambridge Univ. Press, Cambridge (1998).
17. R.S. Yulmukhametov. *Factoring an entire function into two "almost equal" functions* // Sibir. Matem. Zhurn. **38**:2, 463–473. [Siber. Math. J. **38**:2, 401–410].
18. B.Ya. Levin. *Distribution of the zeros of entire functions.* Fizmatgiz, Moscow (1956). [Mathematische Lehrbücher und Monographien. II. Abt. Band 14. Akademie-Verlag, Berlin (1962). (in German).]

Natalia Fairbakhovna Abuzyarova,
Bashkir State University,
Zaki Validi str. 32,
450074, Ufa, Russia
E-mail: abnatf@gmail.com