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ON THE SOLUTIONS OF POLYNOMIAL GROWTH FOR A MULTIDIMENSIONAL GENERALIZED CAUCHY-RIEMANN SYSTEM

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Abstract. For a multidimensional generalized Cauchy-Riemann system we study the Noether property in Hölder spaces of functions bounded on the whole plane. For the case of constant coefficients we consider the solutions defined on the whole plane or on the half-plane and having a polynomial growth at the infinity. For the two- and three-dimensional cases we find appropriate conditions for the coefficients ensuring that the solutions to the first problem is finite-dimensional or zero or infinite-dimensional, respectively.

Keywords: multi-dimensional generalized Cauchy-Riemann systems, power-growth solutions, Hölder space, Noether property.

Mathematics Subject Classification: 35J46, 35J56

1. Introduction

Let C_α be the Banach space of complex vector functions $w(z) = (w_1(z), \dots, w_n(z))$ bounded in the whole plane and Hölder uniformly continuous with exponent α. The norm in C_α is introduced by the formula

$$\|w\|_\alpha = \sup \|w(z)\| + \sup_{z_1 \neq z_2} |z_1 - z_2|^{-\alpha} \|w(z_1) - w(z_2)\|,$$

where $\|\cdot\|$ is the norm in C^n. Let C^1_α be the Banach space of complex vector functions $w \in C_\alpha$ such that $w_z, w_{\bar z} \in C_\alpha$. The norm in C^1_α is introduced by the formula

$$\|w\|_{\alpha,1} = \|w\|_\alpha + \|w_z\|_\alpha + \|w_{\bar z}\|_\alpha.$$

We consider the multi-dimensional generalized Cauchy-Riemann system

$$Lw \equiv w_{\bar z} + A(z)\bar w = 0, \tag{1}$$

where $A(z)$ is a matrix with the columns belonging to space C_α. For system (1), many properties of generalized one-dimensional Cauchy-Riemann system are not preserved [1], we mention the properties like Liouville theorem, the finite dimension of space formed by solutions of power growth, etc. For instance, as $n = 2$ and $A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$, system (1) has infinitely many linearly independent bounded in the whole plane solutions $w_a(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(\bar a z + a \bar z)} + \frac{1}{a} \begin{pmatrix} 2i \\ -1 \end{pmatrix} e^{-i(\bar a z + a\bar z)}$, where $a = \sqrt{3} e^{i\alpha}$.

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2. Solutions in the whole plane

As $n = 2$, in work [2] for particular cases of system (1) with a constant matrix A there were considered solutions in the whole plane with the at most power growth at infinity. In works [3, 4] for arbitrary n, the issue on nontrivial solvability of system (1) with a constant matrix A was considered in the class of functions growing at most as $|z|^N$ as $z \to \infty$. In the case of finite-dimensional space P_N of such solutions there was obtained the formula for the dimension of this space:

$$\dim P_N = 2n(N+1) - 2\sum_{k=0}^{N} \operatorname{rank} B_k, \tag{2}$$

where $B_{2j} = \bar{A}(A\bar{A})^j, B_{2j+1} = (A\bar{A})^{j+1}$, $j = 0, 1, \dots, [N/2]$. In [5] for weakly oscillating at infinity coefficients, i.e., satisfying the condition

$$\lim_{z\to\infty} \max_{|z-\zeta|\leqslant 1} \|A(z) - A(\zeta)\| = 0,$$

there were found necessary and sufficient condition of Noether property for operator $L : C^1_\alpha \to C_\alpha$. Under the weak oscillation of the coefficients at infinity, the sequence $\{A(z + h_k)\}$, where $h_k \to \infty$, contains a subsequence converging uniformly on each compact set, at that, the limiting matrix is constant and depends on the choice of sequence h_k. The set of all matrices constructed by all possible sequences $h_k \to \infty$ is denoted by $H(A)$.

The following theorem holds true.

Theorem 1. *Operator $L : C^1_\alpha \to C_\alpha$ is Noether if and only if for each $A_0 \in H(A)$ the matrix $A_0\overline{A}_0$ has no eigenvalues on the semi-axis $(-\infty, 0]$.*

In view of this theorem, it is important to find necessary and sufficient conditions for absence of eigenvalues of matrix $A_0\overline{A}_0$ on semi-axis $(-\infty, 0]$.

As $n = 2, 3$, the condition of absence of eigenvalues on semi-axis $(-\infty, 0]$ for matrix $A_0\overline{A}_0$ can be written in terms of the entries of matrix A_0. As $n = 2$, such conditions are provided in [4] without proof.

Theorem 2. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the matrix $A\overline{A}$ has a negative eigenvalue if and only if the following four conditions hold true simultaneously:*

$$\det A \neq 0, \quad |a| = |d|, \quad |a|^2 + b\overline{c} < 0, \quad ad\overline{bc} \geqslant 0. \tag{3}$$

Proof. Necessity. Suppose that matrix $A\overline{A}$ has a negative eigenvalue. The characteristic equation for matrix $A\overline{A}$ reads as

$$\lambda^2 - \alpha\lambda + \Delta^2 = 0, \tag{4}$$

where

$$\alpha = |a|^2 + |d|^2 + 2\mathrm{Re}\,(b\overline{c}), \Delta = |\det A|. \tag{5}$$

Let us show that

$$\alpha + 2\Delta \geqslant 0. \tag{6}$$

Indeed, we have

$$\begin{aligned} \alpha + 2\Delta &\geqslant |a|^2 + |d|^2 + 2\mathrm{Re}\,(b\overline{c}) + 2(|bc| - |ad|) \\ &= (|a| - |d|)^2 + 2(\mathrm{Re}\,(b\overline{c}) + |b\overline{c}|) \geqslant 0. \end{aligned} \tag{7}$$

Then $\Delta > 0$, otherwise inequality (6) follows that $\alpha \geqslant 0$ and the eigenvalues of matrix $A\overline{A}$ are non-negative, $\lambda_1 = 0$, $\lambda_2 = \alpha$.

If $\alpha \geqslant 0$, then equation (4) has no negative root. Thus, $\alpha < 0$ and $\alpha - 2\Delta < 0$. Since the discriminant of equation (4) $D = (\alpha - 2\Delta)(\alpha + 2\Delta)$ is non-negative, the latter inequality and (6) imply that $D = 0$, i.e.,

$$\alpha + 2\Delta = 0. \tag{8}$$

Then by inequality (7) we obtain:

$$|a| = |d|, \mathrm{Re}\,(b\overline{c}) + |b\overline{c}| = 0. \tag{9}$$

Hence, inequality $\alpha < 0$ means that $|a|^2 + b\overline{c} < 0$.

It remains to check inequality $ad\overline{bc} \geqslant 0$. Due to identities (9), we rewrite identity (8) as

$$|ad - bc| = |bc| - |ad|. \tag{10}$$

Squaring this identity and simplifying, we get

$$\mathrm{Re}\ (ad\overline{bc}) = |adbc|. \tag{11}$$

The latter identity is equivalent to the inequality $ad\overline{bc} \geqslant 0$. All the relations in (3) are checked and their necessity is proven.

Sufficiency. Assume that conditions (3) hold true. Let us show that the eigenvalues of the matrix $A\overline{A}$ are negative. The fourth condition in (3) is equivalent to the identity (11), which, in its turn, is equivalent to identity (10). Then in view of the second and third condition in (3) we have

$$\Delta = |ad - bc| = |bc| - |ad| = -(|a|^2 + b\overline{c}).$$

Therefore,

$$\alpha + 2\Delta = 2|a|^2 + 2b\overline{c} - 2(|a|^2 + b\overline{c}) = 0.$$

Thus, the discriminant of equation (4) $D = (\alpha - 2\Delta)(\alpha + 2\Delta)$ vanishes and this equation has the double root

$$\lambda_{1,2} = \frac{\alpha}{2} = -\Delta,$$

which is negative. The sufficiency of conditions (3) is proven. The proof is complete. □

For the matrix 3×3 the following theorem is true.

Theorem 3. *Let A be a constant 3×3 matrix, $k_1 = \mathrm{Sp}\,(A\overline{A})$, $k_2 = \frac{1}{2}[\mathrm{Sp}\,(A\overline{A})^2 - (\mathrm{Sp}\,(A\overline{A}))^2]$ and γ be a curve formed by the left branch of the parabola $y = -\frac{x^2}{3}$, $x < 0$, and the semi-axis $y = 0$, $x \geqslant 0$. Then the matrix $A\overline{A}$ has no eigenvalues on the semi-axis $(-\infty, 0]$ if and only if $\det A \neq 0$ and one of the following conditions holds true:*

a) the point (k_1, k_2) is located above curve γ and

$$\mu^3 - k_1\mu^2 - k_2\mu - |\det A|^2 < 0, \tag{12}$$

where $\mu = \frac{1}{3}(k_1 - \sqrt{k_1^2 + 3k_2})$;

b) the point (k_1, k_2) is located on curve γ or below.

We observe that the trace of matrices $(A\overline{A})^m$, $m = 0, 1, 2, \ldots$, is a real number.

Proof. Necessity. Suppose that matrix $A\overline{A}$ has no negative eigenvalues on semi-axis $(-\infty, 0]$. Let us show that $\Delta = |\det A| \neq 0$ and either condition a) or condition b) holds true. The characteristic equation for matrix $A\overline{A}$ reads as

$$p(\lambda) \equiv -\lambda^3 + k_1\lambda^2 + k_2\lambda + \Delta^2 = 0. \tag{13}$$

Let λ_1, λ_2, λ_3 be the roots of this equation. We can assume that either $\lambda_j > 0$, $j = 1, 2, 3$, or $\lambda_1 = \overline{\lambda_2} = \varepsilon + i\delta, \lambda_3 > 0$, and $\delta \neq 0$. It is obvious that $\Delta \neq 0$.

By D_+ we denote an open domain formed by the points (x, y) above curve γ, and D_- stands for the complement of this domain to the whole plane. Let K be a point (k_1, k_2). Two cases are possible: 1) $K \in D_+$; 2) $K \in D_-$. In the first case either $k_1^2 + 3k_2 > 0$ and $k_2 \leqslant 0$ or $k_1 \geqslant 0$ and $k_2 > 0$. Then as $\lambda < 0$, function $p(\lambda)$ has a minimum at the point $\lambda = \mu$ and since $p(\lambda) \to +\infty$ as $\lambda \to -\infty$, then $p(\mu) > 0$, i.e., inequality (12) holds true. Hence, in the first case condition a) is satisfied. In the second case condition b) holds true. The proof of the necessity is complete.

Sufficiency. Let $\det A \neq 0$ and one of conditions a) and b) is satisfied. Let us show that matrix $A\overline{A}$ has no eigenvalues on semi-axis $(-\infty, 0]$. It is obvious that $p(0) > 0$. If condition a) is satisfied, then $K \in D_+$ and $\min_{\lambda<0} p(\lambda) = p(\mu) > 0$. Hence, the roots of equation (13) do not lie on semi-axis $(-\infty, 0]$. If condition b) is satisfied, then for $k_1 \geqslant 0$ we have $p(\lambda) > 0$ $\forall \lambda \leqslant 0$, while for $k_1 < 0$ we have $p'(\lambda) < 0$ $\forall \lambda \in (-\infty, +\infty)$. Hence, $p(\lambda) > p(0) > 0$ $\forall \lambda \leqslant 0$. Therefore, equation (13) has no roots on semi-axis $(-\infty, 0]$. The proof is complete. □

Let $M = \sigma(A\overline{A}) \bigcap (-\infty, 0]$, $M_1 = \sigma(A\overline{A}) \bigcap (-\infty, 0)$, where $\sigma(A\overline{A})$ is the spectrum of matrix $A\overline{A}$. For the case $n = 3$ and a degenerate matrix A we have the following theorem on the structure of sets M and M_1.

Theorem 4. *Let $\delta = k_1^2 + 4k_2$. The following statements hold true:*
a) if $\delta < 0$, then $M = \{0\}$ and $M_1 = \emptyset$;
b) if $\delta = 0$, then
$M = \{0\}$ as $k_1 \geqslant 0$ and $M = \{\frac{k_1}{2}, 0\}$ as $k_1 < 0$;
$M_1 = \emptyset$ as $k_1 \geqslant 0$ and $M_1 = \{\frac{k_1}{2}\}$ as $k_1 < 0$;
c) if $\delta > 0$, then as $k_2 > 0$,
$M = \{0\}$, if $k_1 \geqslant 0$ and $M = \{\frac{k_1}{2}, 0\}$, if $k_1 < 0$;
$M_1 = \emptyset$, if $k_1 \geqslant 0$ and $M_1 = \{\frac{k_1}{2}\}$, if $k_1 < 0$;
as $k_2 = 0$,
$M = \{0\}$, if $k_1 \geqslant 0$ and $M = \{k_1, 0\}$, if $k_1 < 0$;
$M_1 = \emptyset$, if $k_1 \geqslant 0$ and $M_1 = \{k_1\}$, if $k_1 < 0$;
as $k_2 < 0$,
$M = \{0\}$, if $k_1 > 0$ and $M = \{\frac{1}{2}(k_1 \pm \sqrt{\delta}), 0\}$, if $k_1 < 0$;
$M_1 = \emptyset$, if $k_1 \geqslant 0$ and $M_1 = \{\frac{1}{2}(k_1 \pm \sqrt{\delta})\}$, if $k_1 < 0$.

3. Dimension of space P_N for $n = 2$ and $n = 3$

Let $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If conditions (3) are satisfied, space P_N is infinite-dimensional. If $\det A \neq 0$ and other conditions (3) are not simultaneously satisfied, space P_N is zero. If $\det A = 0$, space P_N is non-trivial and finite-dimensional. In this case as $A = 0$, by formula (2) we have $\dim P_N = 4(N + 1)$, which is in accordance with Liouville formula. If $A \neq 0$, without loss of generality we can assume that $A = \begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$, $|a| + |b| > 0$, λ is a complex number, and by (2) we obtain easily the formulae:

$\dim P_N = 4N + 2$ as $a + \overline{\lambda} b = 0$ and $\dim P_N = 2N + 2$ as $a + \overline{\lambda} b \neq 0$.

Let $n = 3$. If $A \neq 0$, then in view of Theorem 2 we obtain that in case a) space P_N is zero, while in case b) it is infinite-dimensional. If $\det A = 0$, space P_N is non-trivial and finite-dimensional. In this case, as $A = 0$, by formula (2) we have $\dim P_N = 4(N + 1)$, which is in agreement with Liouville theorem. Assume now that $\det A = 0$ and $A \neq 0$. Then $\dim P_N = 2(N + 1)$ in the following cases: 1) as $\delta < 0$; 2) as $\delta = 0$ and $k_1 > 0$; 3) as $\delta > 0$, $k_1 > 0$, and $k_2 < 0$. We have

$\dim P_N = 6N + 4 - 2[\mathrm{rank}\, A + \mathrm{rank}\, (\bar{A}A\bar{A})]$ as $\delta = 0, k_1 = 0$ and $\mathrm{rank}\, (A\bar{A}) = 1$,
$\dim P_N = 6N - 4 - 2[\mathrm{rank}\, (\bar{A}A\bar{A}) + \mathrm{rank}\, (\bar{A}(A\bar{A})^2]$ as $\delta = 0, k_1 = 0$ and $\mathrm{rank}\, (A\bar{A}) = 2$,
$\dim P_N = 4N + 6 - 2\, \mathrm{rank}\, A$ as $\delta > 0,\ k_1 > 0$ and $\mathrm{rank}\, (A\bar{A}) = 1$,
$\dim P_N = 4N + 2 - 2\, \mathrm{rank}\, (\bar{A}A\bar{A}$ as $\delta > 0,\ k_1 > 0$ and $\mathrm{rank}\, (A\bar{A}) = 2$.

Finally, space P_N is infinite-dimensional in the following cases: 1) as $\delta = 0, k_1 < 0$; 2) as $k_1 < 0$ and $k_2 = 0$; 3) as $k_2 > 0$; 4) as $\delta > 0,\ k_1 < 0$ and $k_2 < 0$.

4. Problem in half-plane

For the system with constant coefficients

$$w_{\bar{z}} + A\bar{w} = 0, \tag{14}$$

where A is a complex $n \times n$ matrix, we consider the following problem.

Problem P. Find a vector function $w(z)$ in the class $C^1(G) \cap C(\bar{G})$, $G = \{z : \mathrm{Im}\, z > 0\}$, solving system (14) in domain G and satisfying the condition

$$\|w(z)\| \leqslant K(1 + |z|)^N, z \in \overline{G}, \tag{15}$$

where K is a constant depending on $w(z)$, $N \in \{0, 1, \dots\}$.

If $w(z)$ solves problem P, then the vector function $U(\xi, y) = (\omega, v)^T$, where $\omega = F_x w$, $v = F_x \overline{w}$, F_x is the Fourier transform w.r.t. variable x, belongs to space $S' = S'(R)$ for each $y \geqslant 0$ and solves the system of ordinary differential equations

$$\frac{dU}{dy} + B(\xi)U = 0, \tag{16}$$

where $B(\xi) = \begin{pmatrix} \xi E_n & -2i\overline{A} \\ 2iA & -\xi E_n \end{pmatrix}$, E_n is the unit $n \times n$ matrix. We observe that vector functions ω and v are related by the following identity

$$v(\xi, y) = \overline{\omega(-\xi, y)}. \tag{17}$$

And vice versa, if vector function $U(\xi, y) = (\omega, v)^T$ solves system (16) for each $\xi \in R$, condition (17) is satisfied and $\omega \in S'\ \forall y \in (0, \infty)$, then the vector function $w = F_\xi^{-1}\omega$ is a solution to equation (14) in domain G. Let $S(\xi)$ be a matrix transforming matrix $B(\xi)$ to the quasi-diagonal form: $\Lambda(\xi) = S^{-1}(\xi)B(\xi)S(\xi)$. Assume that matrix $A\overline{A}$ has no eigenvalues on semi-axis $(-\infty, 0]$. Then matrix $B(\xi)$ is invertible for each $\xi \in R$ and has pure imaginary eigenvalues. Thus, matrix $\Lambda(\xi)$ can be represented in the block-diagonal form

$$\Lambda(\xi) = \begin{pmatrix} \Lambda_+(\xi) & O \\ O & \Lambda_-(\xi) \end{pmatrix},$$

where $\Lambda_+(\xi)$ ($\Lambda_-(\xi)$) is $n \times n$ matrix formed by Jordan blocks of matrix $B(\xi)$ associated with the eigenvalues with positive (negative) real part. Suppose that matrix $S(\xi)$ is represented in the block form:

$$S(\xi) = \begin{pmatrix} S_1(\xi) & S_2(\xi) \\ S_3(\xi) & S_4(\xi) \end{pmatrix},$$

where $S_j(\xi)$, $j = 1, 2, 3, 4$, are $n \times n$ matrices and let $e_k(\xi) \in C^n\ \forall \xi \in R$, $k = 1, 2$. Then the general solution to system (16) reads as

$$U(\xi, y) = \begin{pmatrix} S_1(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi) + & S_2(\xi)e^{-\Lambda_-(\xi)y}e_2(\xi) \\ S_3(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi) + & S_4(\xi)e^{-\Lambda_-(\xi)y}e_2(\xi) \end{pmatrix}.$$

The entries of matrix $S_k(\xi)$ and $S_k^{-1}(\xi)$ have at most power growth $|\xi| \to \infty$. Hence, in order to have $U \in S'$ for each $y > 0$, it is necessary to have $e_2(\xi) = 0$. Therefore,

$$U(\xi, y) = \begin{pmatrix} S_1(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi) \\ S_3(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi) \end{pmatrix},$$

i.e.,

$$\omega(\xi, y) = S_1(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi)$$

and

$$v(\xi, y) = S_3(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi).$$

To satisfy condition (17) and to find a solution to problem P, vector function $e_1(\xi) \in S'$ is to be chosen to satisfy the identity

$$S_3(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi) = \overline{S_1(-\xi)}e^{-\overline{\Lambda_+(\xi)}y}\overline{e_1(-\xi)}$$

and the vector function

$$w = F_\xi^{-1}[S_1(\xi)e^{-\Lambda_+(\xi)y}e_1(\xi)]$$

is to satisfy condition (15).

BIBLIOGRAPHY

1. I.N. Vekua. *Generalized analytic functions.* Nauka, Moscow (1988). (in Russian).
2. D. Safarov. *Dimension of the space of power-growth solutions of elliptic systems of a certain class* // Diff. Uravn. **15**:1, 112–115 (1979). [Diff. Equat. **15**:1, 77–80 (1979).]
3. S. Baizaev. *On slow growing solutions to one multi-dimensional elliptic system* // Dokl. AN TadzhSSR. **34**:6, 329–332 (1991). (in Russian).
4. S. Baizaev. *Elliptic systems with bounded coefficients on plane.* Novosibirsk State Univ., Novosibirsk (1999). (in Russian).
5. S. Baizaev, E. Mukhamadiev. *On Noether property and index of multi-dimensional elliptic operators in Hölder spaces* // Modern methods in the theory of boundary value problems. Proc. of Voronezh spring mathematical schools “Pontryagin’s readings”. P. 1. Voronezh, 75–84 (2000). (in Russian).

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