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GENERAL REGULARIZED TRACE FORMULAE FOR LOADED EQUATIONS

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Abstract. We consider regularized traces for differential operators with the coefficients at the powers of a spectral parameter being the values of an unknown function at prescribed points in its domain. Such differential operators are interpreted as polynomial operator pencils whose coefficients are unbounded fininte-dimensional operators. Basing on the theory of M.V. Keldysh, we construct general regularized trace formulae for such operator pencils. The obtained formulae develop a known result by V.A. Sadovnichii and V.A. Lyubishkin for relative finite-dimensional perturbations of self-adjoint operators.

Keywords: spectrum, operator pencil, regularized traces.

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1. Introduction

We consider the operator pencil

$$N_\lambda = A - Q_0 - \lambda Q_1 - \cdots - \lambda^{n-1} Q_{n-1} - \lambda^n E, \tag{1}$$

where A is an unbounded self-adjoint operator in a separable Hilbert space $\mathfrak{H}$ with a compact resolvent. Operators $Q_0, Q_1, \ldots, Q_{n-1}$ are assumed to be A-finite-dimensional, i.e., they read as $Q_j = P_j A$, where P_j are finite-dimensional bounded operators in $\mathfrak{H}$:

$$\forall h \in \mathfrak{H} \quad P_j h = \sum_{l=1}^{n_j} (h, \varphi_l^j) \psi_l^j, \tag{2}$$

where $\varphi_l^j, \psi_l^j \in \mathfrak{H}$, $j = 0, 1, \ldots, n-1$; $l = 1, \ldots, n_j$. We observe that if vectors φ_l^j do not belong to the domain of operator A, then Q_j is an unbounded operator in $\mathfrak{H}$.

The above operator pencils appear, for instance, while solving initial boundary value problems for loaded equations [1, 2]

$$\frac{\partial^2 u(t,x)}{\partial^2 t} = \frac{\partial^2 u(t,x)}{\partial^2 x} + \sum_{i=1}^{\nu} a_i(x) u(t, x_i) + \sum_{j=1}^{\mu} b_j(x) \frac{\partial u(t, z_j)}{\partial t}.$$

by the Fourier method.

As a regularized trace formula for pencil (1) we call the formula

$$\sum_{\nu} (\mu_\nu^s - \eta_\nu^s - c_\nu(s)) = F(s), \tag{3}$$

where μ_ν and η_ν are the eigenvalues of pencils N_λ and $A_{\lambda^n} \stackrel{def}{=} A - \lambda^n E$, respectively, s is the arc length, $c_\nu(s)$ and $F(s)$ are some quantities. In the left hand side of (3) the sum symbol denotes a summation, probably with some brackets, over all eigenvalues of pencils N_λ and A_{λ^n}, and the way of placing brackets depend on the behavior of spectrum of operator A.

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As $n = 1$ and $s = 1$, a regularized trace formula was obtained first in work [3] for a relative finite-dimensional perturbation of an unbounded self-adjoint operator under rather general assumptions for the sparseness of its spectrum. In work [4] there were obtained regularized traces as $n = 1$ and $s > 1$ for relative finite-dimensional perturbation and they are expressed as recurrent formulae. As $s > 1$, the construction of regularized trace formulae in the case of infinite-dimensional perturbations is a more complicated problem. By the methods of the perturbation theory for abstract operators with discrete spectra in Hilbert space, formulae (3) were obtained in [5] (see also [6]) under a condition for the sparseness of the spectrum of an unperturbed operator. An essential progress in this direction was made in [7], where the restrictions for the sparseness of the spectrum were omitted. The survey and detailed analysis of the results obtained in the theory of regularized traces of operators were provided in [8].

Paper [9] was likely the first work devoted to constructing regularized trace formulae by analytic methods for a loaded ordinary differential equation, which in some cases can be treated as operator pencil (1). In the present work we obtain regularized trace formulae (3) for operator pencils (1) and arbitrary $s \in \mathbb{N}$.

It is interesting to note that the history of regularized traces for polynomial operator pencils reproduce the history of operator traces. Works [10]–[14] are devoted to constructing the formulae for the sums of the inverses for the eigenvalues of polynomial operator pencils. The main approach in these works is the linearization method and the known Lidskii's theorem on trace of a nuclear operator [15].

2. Preliminaries

In what follows we assume that $\lambda = 0 \notin \sigma(A)$, i.e., $T = A^{-1}$ is a compact operator. From original pencil (1) we pass to the pencil $L_\lambda = N_\lambda A^{-1}$:

$$L_\lambda = E - P_0 - \lambda P_1 - \cdots - \lambda^{n-1} P_{n-1} - \lambda^n T. \tag{4}$$

A complex number μ is an eigenvalue of pencil L_λ if $L_\mu y = 0$ for some non-zero vector $y \in \mathfrak{H}$. It was shown in work [16] that the spectrum of pencil L_λ is formed by a discrete set of eigenvalues $\sigma(L_\lambda) = \{\mu_k\}_{k=1}^\infty$ with the only accumulation point at infinity.

Let $\{\lambda_\nu\}_{\nu=1}^\infty$ be the eigenvalues of pencil $T_\lambda = E - \lambda T$, i.e., $\sigma(T_\lambda) = \{\lambda_\nu\}_{\nu=1}^\infty$. We consider also the pencil $T_{\lambda^n} \stackrel{def}{=} E - \lambda^n T$ whose eigenvalues are denoted by η_k, i.e., $\sigma(T_{\lambda^n}) = \{\eta_k\}_{k=1}^\infty$. We index the eigenvalues in the order of ascending absolute values with the multiplicities taken into account. We have the following lemma [16]

Lemma 1 (M.V. Keldysh). *Let $E - L_\lambda$ be an analytic in $\mathfrak{D} \subseteq \mathbb{C}$ operator function with the values in the ideal $\mathfrak{S}_\infty$ of compact operators. Then the trace of the principal part of the operator $\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1}$ for the pole $\lambda = c$ is equal to $\frac{N}{\lambda - c}$, where N is an algebraic multiplicity of eigenvalue $\lambda = c$ of pencil L_λ.*

If we denote by $[\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1}]$ the principal part of operator $\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1}$, and by $\mathrm{Tr}(\bullet)$ we denote the trace, Lemma (1) implies the relation

$$\frac{1}{2\pi i} \oint_{\Gamma_c} \lambda^s Tr\left(\left[\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1}\right]\right) d\lambda = N c^s, \tag{5}$$

where Γ_c is a circle of a sufficiently small radius centered $\lambda = c$ passed counterclockwise.

3. Preliminary regularized trace formula

Suppose that the counting function of characteristic values of operator $T = A^{-1}$ satisfies the condition

$$\lim_{r \to \infty} \frac{N(r)}{r^\alpha} = \varepsilon < \infty \qquad \text{as} \quad 0 < \alpha \leqslant \frac{1}{n}, \tag{6}$$

where ε is a positive constant (i.e., $0 < \varepsilon \leqslant \infty$). We introduce the notations: $r_k = |\lambda_k^{1/n}|$, $d_k = r_{k+1} - r_k$. The proof of the next lemma was provided in [17].

Lemma 2. *Under condition (6) for function $N(\lambda)$, there exists a subsequence of the natural series $\{k_\nu\}_{\nu=1}^{\infty}$ such that $d_{k_\nu} = r_{k_\nu+1} - r_{k_\nu} \geqslant \varepsilon_0$ $\forall \nu \in \mathbb{N}$, where $\varepsilon_0 > 0$ is a constant.*

Corollary 1. *There exists an infinite system of expanding concentric circles $\{\Gamma_\nu\}_{\nu=1}^{\infty}$ centered at the origin and containing no spectrum of pencil T_{λ^n} such that the distance δ_ν from circle Γ_ν to spectrum $\sigma(T_{\lambda^n})$ satisfies the condition $\delta_\nu \geqslant \varepsilon_0/2$ $\forall \nu \in \mathbb{N}$.*

Доказательство. As Γ_ν we choose the circle of radius $\widetilde{R}_\nu = r_{k_\nu} + \frac{1}{2} d_{k_\nu}$ centered at the origin. Then Γ_ν contains no spectrum $\sigma(T_{\lambda^n})$, i.e., points in $\sigma(T_{\lambda^n})$ are located on circumferences of radii r_k, $k \in \mathbb{N}$, centered at the origin. Moreover, since the points in spectrum $\sigma(T_{\lambda^n})$ are located at the rays $\arg \lambda = \frac{k\pi}{n}$, $k = 0, 1, \ldots, 2n-1$, then $\delta_\nu \geqslant d_{k_\nu}/2$. Hence, in accordance with Lemma 2, $\delta_\nu \geqslant \varepsilon_0/2$ $\forall \nu \in \mathbb{N}$. □

Lemma 3. *Let P be a finite-dimensional operator in $\mathfrak{H}$: $P = \sum\limits_{l=1}^{t} (\bullet, \varphi_l)\psi_l$, $\psi_l \in \mathfrak{D}(T^{-1})$, $l = 1, 2, \ldots, t$. Then for $j = 0, 1, 2, \ldots, n-1$ and $R_\lambda = (E - \lambda^n T)^{-1}$ the relations $\|\lambda^j R_\lambda P\| \to 0$ hold true as $\lambda \in \Gamma_\nu$ and $\nu \to \infty$ uniformly in $\arg\lambda$. (see [17]).*

By means of Lemma 3 we can represent operator function L_λ^{-1} as a series.

Corollary 2. *For $\lambda \in \Gamma_\nu$ and sufficiently large ν the formula*

$$L_\lambda^{-1} = \sum_{k=0}^{\infty} \left\{ R_\lambda \sum_{j=0}^{n-1} \lambda^j P_j \right\}^k R_\lambda \tag{7}$$

holds true, where the series converges in the operator topology uniformly in $\arg\lambda$.

Multiplying the left and right hand sides of this identity by the left and right hand sides of the identity $\frac{\partial L_\lambda}{\partial \lambda} = -\sum_{j=0}^{n-1} j\lambda^{j-1} P_j - n\lambda^{n-1} T$, we obtain

$$\begin{aligned} \frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1} + n\lambda^{n-1} T R_\lambda &= -\sum_{j=1}^{n-1} j\lambda^{j-1} P_j R_\lambda \\ &- \sum_{j=1}^{n-1} j\lambda^{j-1} P_j \sum_{k=1}^{\infty} \left\{ R_\lambda \sum_{l=0}^{n-1} \lambda^l P_l \right\}^k R_\lambda - n\lambda^{n-1} T \sum_{k=1}^{\infty} \left\{ R_\lambda \sum_{l=0}^{n-1} \lambda^l P_l \right\}^k R_\lambda. \end{aligned} \tag{8}$$

Integrating identity (8) over the contour Γ_ν, $\nu \geqslant m_0$, by formula (5) with $s = 0$, in the left hand side we obtain:

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\Gamma_\nu} \operatorname{Tr}\left(\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1} + n\lambda^{n-1} T R_\lambda \right) d\lambda &= \frac{1}{2\pi i} \oint_{\Gamma_\nu} \operatorname{Tr}\left(\left[\frac{\partial L_\lambda}{\partial \lambda} L_\lambda^{-1} \right] \right) d\lambda + \\ + \frac{1}{2\pi i} \oint_{\Gamma_\nu} \operatorname{Tr}\left(\left[n\lambda^{n-1} T R_\lambda \right] \right) d\lambda &= M_\nu - N_\nu. \end{aligned} \tag{9}$$

Here M_ν and N_ν are numbers of the eigenvalues taken counting multiplicities of pencils L_λ and T_{λ^n} lying respectively inside contour Γ_ν.

Lemma 4. *The left hand side of identity (9) tends to zero as $\nu \to \infty$. Since N_ν and M_ν are natural numbers, there exists an index m_0 such that for $\nu \geqslant m_0$ we have $M_\nu = N_\nu$, i.e., starting from some index m_0, all circles Γ_ν contain the same number (counting multiplicities) of the eigenvalues of pencils L_λ and T_{λ^n}.*

The proof will be given later once we study function $F(s)$ defined by identity (11).

In what follows we assume that index m_0 is such that inequality $\nu \geqslant m_0$ ensures all the aforementioned conditions. It follows from Lemma 4 that as $\nu \geqslant m_0$, between contours Γ_{m+1} and Γ_m there is the same number (counting multiplicities) of the eigenvalues of pencils L_λ and T_{λ^n}, namely $N_{m+1} - N_m$ eigenvalues. Hence, multiplying (8) by $\lambda^s(2\pi i)^{-1}$, calculating the trace, integrating over contour Γ_ν, passing to the limit $\nu \to \infty$ by formula (5), in the left hand side we obtain:

$$\lim_{\nu\to\infty} \sum_{k=1}^{N_\nu} (\mu_k^s - \eta_k^s) = \left(\sum_{k=1}^{N_{m_0}} + \sum_{\nu=m_0}^{\infty} \sum_{k=N_\nu+1}^{N_{\nu+1}} \right) (\mu_k^s - \eta_k^s). \tag{10}$$

In order to get formula (3), we need to study the right hand side of (8) after the above described procedure, namely, the expression:

$$\begin{aligned} F(s) = -\lim_{\nu\to\infty} \Bigg\{ & \sum_{j=1}^{n-1} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s-1} \operatorname{Tr}(P_j R_\lambda) d\lambda \\ & + \sum_{j=1}^{n-1} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \operatorname{Tr} \left(\sum_{k=1}^{\infty} \lambda^{j+s-1} P_j \left(R_\lambda \sum_{l=0}^{n-1} \lambda^l P_l \right)^k R_\lambda \right) d\lambda \\ & + \frac{n}{2\pi i} \oint_{\Gamma_\nu} \operatorname{Tr} \left(\sum_{k=1}^{\infty} \lambda^{n+s-1} T \left(R_\lambda \sum_{l=0}^{n-1} \lambda^l P_l \right)^k R_\lambda \right) d\lambda \Bigg\}. \end{aligned} \tag{11}$$

We have

$$\left(\sum_{l=0}^{n-1} \lambda^l R_\lambda P_l \right)^k = \left(R_\lambda P_0 \right)^k + \sum_{m=1}^{k(n-1)} \lambda^m \sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_k=m \\ 0\leqslant\alpha_1,\ldots,\alpha_k\leqslant n-1}} R_\lambda P_{\alpha_1} \cdots R_\lambda P_{\alpha_k}.$$

Denoting the internal sum by $\sum\limits_{m} P_{\alpha_1\ldots\alpha_k}$, we arrive at

$$\left(\sum_{l=0}^{n-1} \lambda^l R_\lambda P_l \right)^k = \left(R_\lambda P_0 \right)^k + \sum_{m=1}^{k(n-1)} \lambda^m \sum_{m} P_{\alpha_1\ldots\alpha_k}. \tag{12}$$

Employing identity (12), by (11) we obtain:

$$\begin{aligned} F(s) = -\lim_{\nu\to\infty} \Bigg\{ & \sum_{j=1}^{n-1} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s-1} \operatorname{Tr}(R_\lambda P_j) d\lambda \\ & + \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s-1} \operatorname{Tr} \left(R_\lambda P_j \left(R_\lambda P_0 \right)^k \right) d\lambda \\ & + \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} \sum_{m=1}^{k(n-1)} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s+m-1} \operatorname{Tr} \left(R_\lambda P_j \sum_{m} P_{\alpha_1\ldots\alpha_k} \right) d\lambda \\ & + \sum_{k=1}^{\infty} \frac{n}{2\pi i} \oint_{\Gamma_\nu} \lambda^{n+s-1} \operatorname{Tr} \left(T R_\lambda \left(R_\lambda P_0 \right)^k \right) d\lambda \\ & + \sum_{k=1}^{\infty} \sum_{m=1}^{k(n-1)} \frac{n}{2\pi i} \oint_{\Gamma_\nu} \lambda^{m+n+s-1} \operatorname{Tr} \left(R_\lambda T \sum_{m} P_{\alpha_1\ldots\alpha_k} \right) d\lambda \Bigg\}. \end{aligned} \tag{13}$$

We denote by $J_l^\nu(s)$ $(l = 1, 2, \dots, 5)$ in the right hand side of (13), respectively. Our next step is to calculate $\lim\limits_{\nu\to\infty} J_l^\nu(s)$ $\forall s \in \mathbb{N} \cup \{0\}$ $l = 1, 2, \dots, 5$. In order to do it, we need some preliminary formulae to which the next section is devoted.

We note that the subsequent considerations imply easily the well-definiteness of the passage to the limit as $\nu \to \infty$ in infinite series in (13).

4. Auxiliary formulae

We recall that T is a compact self-adjoint operator in separable Hilbert space $\mathfrak{H}$ and its counting function satisfies condition (6). Let $\{e_j\}_{j=1}^\infty$ be the orthonormalized basis in space $\mathfrak{H}$ formed by the eigenvectors of operator T. In what follows by n we denote the orders of pencils (1), (4).

4.1. Calculations with P_λ^k. We employ the system of circles $\{\Gamma_\nu\}_{\nu=1}^\infty$ constructed in Corollary 1. Let $N \geqslant 0$ be an integer number and P be a finite-dimensional operator and $\psi_l \in \mathfrak{D}(T^{-(N+1)})$, $l = 1, \dots, t$. Employing then the identity $R_\lambda e_j = \lambda_j(\lambda_j - \lambda^n)^{-1} e_j$, we obtain

$$\operatorname{Tr}(R_\lambda P) = \sum_{l=1}^{t} (R_\lambda \psi_l, \varphi_l) = \sum_{l=1}^{t} \sum_{k=1}^{\infty} \frac{\lambda_k(\psi_l, e_k)(e_k, \varphi_l)}{\lambda_k - \lambda^n}.$$

In the right hand side we apply the identity

$$\frac{1}{\lambda_k - \lambda^n} = -\frac{1}{\lambda^n} + \frac{\lambda_k}{\lambda^n(\lambda_k - \lambda^n)} \tag{14}$$

N times. It leads us to the formula

$$\operatorname{Tr}(R_\lambda P) = -\sum_{l=1}^{t} \sum_{k=1}^{N} \frac{(T^{-k}\psi_l, \varphi_l)}{\lambda^{nk}} + \sum_{l=1}^{t} \sum_{k=1}^{\infty} \frac{\lambda_k^{N+1}(\psi_l, e_k)(e_k, \varphi_l)}{\lambda^{nN}(\lambda_k - \lambda^n)}. \tag{15}$$

We denote the second term by $\Phi_N(\lambda)$. Employing then the trick used in the proof of Lemma 3, it is easy to show that as $\psi_l \in \mathfrak{D}(T^{-(N+1)})$, $l = 1, 2, \dots, t$, and $\lambda \in \Gamma_\nu$ the identity $\Phi_N(\lambda) = o(\lambda^{-nN})$, $\nu \to \infty$, holds true and this identity is uniform in $\arg\lambda$.

Suppose that we have set (2) of finite-dimensional operators. Let $P_l^j = (\bullet, \varphi_l^j)\psi_l^j$, then $P_j = \sum\limits_{l=1}^{n_j} P_l^j$. We consider the operator function $P_\lambda^k \overset{def}{=} R_\lambda P_1 R_\lambda P_2 \cdots R_\lambda P_k$. It is easy to obtain that

$$P_\lambda^k = \sum_{\substack{l_1, l_2, \dots, l_k \\ 0 \leqslant l_j \leqslant n_j}} R_\lambda P_{l_1}^1 R_\lambda P_{l_2}^2 \cdots R_\lambda P_{l_k}^k,$$

$$\operatorname{Tr}(P_\lambda^k) = \sum_{\substack{l_1, l_2, \dots, l_k \\ 0 \leqslant l_j \leqslant n_j}} \left\{ \prod_{j=1}^{k-1} (R_\lambda \psi_{l_{j+1}}^{j+1}, \varphi_{l_j}^j) \right\} (R_\lambda \psi_{l_1}^1, \varphi_{l_k}^k). \tag{16}$$

Suppose that

$$\psi_l^j \in \mathfrak{D}(T^{-(N+2)}), \quad N \in \mathbb{N}_0, \quad j = 1, 2, \dots, k; \quad l = 1, \dots, n_j. \tag{17}$$

Hence, by relations (15) and (16), as $\lambda \in \Gamma_\nu$, $\nu \to \infty$ we get

$$\begin{aligned}\operatorname{Tr}(P_\lambda^k) = \sum_{\substack{l_1, l_2, \dots, l_k \\ 0 \leqslant l_j \leqslant n_j}} (-1)^k \sum_{\substack{p_1 + p_2 + \dots + p_k = M \\ p_1, \dots, p_k \geqslant 1}} \lambda^{-Mn} \left\{ \prod_{j=1}^{k-1} (T^{-p_{j+1}} \psi_{l_{j+1}}^{j+1}, \varphi_{l_j}^j) \right\} (T^{-p_1} \psi_{l_1}^1, \varphi_{l_k}^k) \\ + B_\lambda^M + o(\lambda^{-n(N+k)}),\end{aligned} \tag{18}$$

where natural M satisfies the condition $k \leqslant M \leqslant N+k$, and the symbol B_λ^M stands for the sum of terms obeying $p_1+p_2+\ldots+p_k \neq M$.

It follows from (18) that for each natural M satisfying the inequalities $k \leqslant M \leqslant N+k$, the identity

$$\lim_{\nu\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_\nu}\lambda^{Mn-1}\mathrm{Tr}(P_\lambda^k)d\lambda = \sum_{\substack{l_1,l_2,\ldots,l_k\\0\leqslant l_j\leqslant n_j}}(-1)^k\sum_{\substack{p_1+p_2+\ldots+p_k=M\\p_1,\ldots,p_k\geqslant 1}}\left\{\prod_{j=1}^{k-1}(T^{-p_{j+1}}\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})\right\}(T^{-p_1}\psi^1_{l_1},\varphi^k_{l_k})$$
$$=(-1)^k\sum_{\substack{p_1+p_2+\ldots+p_k=M\\p_1,\ldots,p_k\geqslant 1}}\mathrm{Tr}(T^{-p_1}P_1\cdots T^{-p_k}P_k) \tag{19}$$

holds true. Thus, we arrive at

Lemma 5. *Suppose that we are given finite-dimensional operators $P_1,\ldots,P_k$ in Hilbert space $\mathfrak{H}$ and condition (17) is satisfied. If natural number M is such that $k \leqslant M \leqslant N+k$, then operator function $P_\lambda^k \stackrel{def}{=} R_\lambda P_1 R_\lambda P_2\cdots R_\lambda P_k$ satisfies (19). If in the left hand side of (19) we replace Mn by some natural $s : kn < s < (N+k)n$ not being a multiple of n, it vanishes.*

4.2. Calculations with Q_λ^k. In what follows we shall make similar relations for operator function $Q_\lambda^k \stackrel{def}{=} R_\lambda^2 P_1 R_\lambda P_2\cdots R_\lambda P_k$.

Applying identity (14), as above, we obtain the identity

$$\mathrm{Tr}(R_\lambda^2 P) = \sum_{l=1}^{t}\sum_{j=2}^{N+1}(j-1)\frac{(T^{-j}\psi_l,\varphi_l)}{\lambda^{nj}} + \widetilde{F}_\lambda, \tag{20}$$

where P is a finite-dimensional operator satisfying $\psi_l \in \mathfrak{D}(T^{-(N+2)})$, $l=1,2,\ldots,t$,

$$\widetilde{F}_\lambda = F^1_\lambda + F^2_\lambda + F^3_\lambda + \sum_{l=1}^{t}\sum_{k=1}^{\infty}(N-1)\frac{\lambda_k^{N+1}(\psi_l,e_k)(e_k,\varphi_l)}{\lambda^{(N+1)n}}, \tag{20$_1$}$$

$$F^2_\lambda = \sum_{l=1}^{t}\sum_{k=1}^{\infty}\frac{\lambda_k^{N+2}(\psi_l,e_k)(e_k,\varphi_l)}{\lambda^{Nn}(\lambda_k-\lambda^n)^2}, \tag{21}$$

$$F^1_\lambda = -\sum_{l=1}^{t}\sum_{k=1}^{\infty}\frac{\lambda_k^{N+2}(\psi_l,e_k)(e_k,\varphi_l)}{\lambda^{(N+1)n}(\lambda_k-\lambda^n)},\qquad F^3_\lambda = \sum_{l=1}^{t}\sum_{k=1}^{\infty}(N-1)\frac{\lambda_k^{N+2}(\psi_l,e_k)(e_k,\varphi_l)}{\lambda^{(N+1)n}(\lambda_k-\lambda^n)}.$$

Following the same lines as in proof of Lemma 3, it is easy to show that $\widetilde{F}_\lambda = o(\lambda^{-Nn})$ as $\nu\to\infty$ uniformly in $\arg\lambda$ for $\psi_l \in \mathfrak{D}(T^{-(N+2)})$, $l=1,2,\ldots,t$, and $\lambda\in\Gamma_\nu$.

Formulae (20) have been obtained for $N \geqslant 2$. As $N=0,1$, by straightforward calculations we get

$$\mathrm{Tr}(R_\lambda^2 P) = o(\lambda^{-n})\quad \nu\to\infty,\quad \lambda\in\Gamma_\nu. \tag{22}$$

Let $k \geqslant 2$. As above, we have

$$\mathrm{Tr}(Q_\lambda^k) = \sum_{\substack{l_1,l_2,\ldots,l_k\\0\leqslant l_j\leqslant n_j}}\left\{\prod_{j=1}^{k-1}(R_\lambda\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})\right\}(R_\lambda^2\psi^1_{l_1},\varphi^k_{l_k}). \tag{23}$$

Suppose that smoothness condition (17) holds true for $N \geqslant 2$. In this case we can apply formulae (15) and (20) to operator-functions $(\bullet,\varphi^j_{l_j})R_\lambda\psi^{j+1}_{l_{j+1}}$ and $(\bullet,\varphi^k_{l_k})R_\lambda^2\psi^1_{l_1}$, respectively, Substituting the corresponding expressions for $(R_\lambda\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})$ and $(R^2_\lambda\psi^1_{l_1},\varphi^k_{l_k})$ into the right hand side of identity (23), we obtain

$$\begin{aligned}\mathrm{Tr}(Q^k_\lambda) &= \sum_{\substack{l_1,l_2,\dots,l_k\\ 0\leqslant l_j\leqslant n_j}}\prod_{j=1}^{k-1}\left\{-\sum_{p=1}^{N+1}\frac{(T^{-p}\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})}{\lambda^{np}}\right\}\left\{\sum_{p=2}^{N}(p-1)\frac{(T^{-p}\psi^1_{l_1},\varphi^k_{l_k})}{\lambda^{np}}\right\}+o(\lambda^{-n(N+k-1)})\\ &= \sum_{\substack{l_1,l_2,\dots,l_k\\ 0\leqslant l_j\leqslant n_j}}(-1)^{k-1}\sum_{\substack{p_1+p_2+\dots+p_k=M\\ p_1\geqslant 2;p_2,\dots,p_k\geqslant 1}}(p_1-1)\lambda^{-Mn}\left\{\prod_{j=1}^{k-1}(T^{-p_{j+1}}\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})\right\}(T^{-p_1}\psi^1_{l_1},\varphi^k_{l_k})\\ &\quad+G^M_\lambda+o(\lambda^{-n(N+k-1)}),\end{aligned}\tag{24}$$

where natural M satisfies the restriction $k+1\leqslant M\leqslant N+k-1$. Symbol G^M_λ stands for the sum of all the terms with $p_1+p_2+\ldots+p_k\neq M$. These relations yield

$$\begin{aligned}&\lim_{\nu\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_\nu}\lambda^{Mn-1}\mathrm{Tr}(Q^k_\lambda)d\lambda\\ &= \sum_{\substack{l_1,l_2,\dots,l_k\\ 0\leqslant l_j\leqslant n_j}}(-1)^{k-1}\sum_{\substack{p_1+p_2+\dots+p_k=M\\ p_1\geqslant 2;p_2,\dots,p_k\geqslant 1}}(p_1-1)\left\{\prod_{j=1}^{k-1}(T^{-p_{j+1}}\psi^{j+1}_{l_{j+1}},\varphi^j_{l_j})\right\}(T^{-p_1}\psi^1_{l_1},\varphi^k_{l_k})\\ &= (-1)^{k-1}\sum_{\substack{p_1+p_2+\dots+p_k=M\\ p_1\geqslant 2;p_2,\dots,p_k\geqslant 1}}(p_1-1)\mathrm{Tr}(T^{-p_1}P_1\cdots T^{-p_k}P_k).\end{aligned}\tag{25}$$

Let $k=1$. The smoothness condition is $\psi_l\in\mathfrak{D}(T^{-(N+2)})$, $l=1,2,\ldots,t$, $N\geqslant 1$. By (20)) we obtain

$$\mathrm{Tr}(R^2_\lambda P)=\sum_{l=1}^{t}\sum_{j=2}^{N+1}(j-1)\frac{(T^{-j}\psi_l,\varphi_l)}{\lambda^{nj}}+\sum_{l=1}^{t}\sum_{k=1}^{\infty}\frac{\lambda_k^{N+2}(\psi_l,e_k)(e_k,\varphi_l)}{\lambda^{Nn}(\lambda_k-\lambda^n)^2}+o(\lambda^{-(N+1)n}).$$

We consider the function

$$f(\lambda)=\frac{\lambda^{s-Nn}}{(\lambda_k-\lambda^n)^2},$$

where s is a natural number, $s\geqslant Nn$. Function $f(\lambda)$ has poles at the points $\lambda=\eta_{k_l}$: $\eta^n_{k_l}=\lambda_k$, $l=1,\ldots,n$. The residues at these poles can be easily calculated:

$$\operatorname*{Res}_{\eta_{k_l}}f(\lambda)=\frac{s-(N+1)n+1}{n^2}\eta_{k_l}^{s-(N+2)n+1}\qquad l=1,\ldots,n.$$

Hence, it is easy to see that

$$\sum_{l=1}^{n}\operatorname*{Res}_{\eta_{k_l}}f(\lambda)=0$$

if either $s+1=(N+1)n$ or $s+1$ is not a multiple of n. In the former case $\operatorname*{Res}_{\eta_{k_l}}f(\lambda)$ vanishes for each $l=1,\ldots,n$, while in the latter

$$\sum_{l=1}^{n}\operatorname*{Res}_{\eta_{k_l}}f(\lambda)=\frac{s-(N+1)n+1}{n^2}\sum_{l=1}^{n}\eta_{k_l}^{s-(N+2)n+1}=0,$$

i.e., $\eta^s_{k_1}+\eta^s_{k_2}+\ldots+\eta^s_{k_n}$ for each natural s not being a multiple of n.

The above arguments imply that

$$\lim_{\nu\to\infty}\frac{1}{2\pi i}\oint_{\Gamma_\nu}\lambda^{Mn-1}\mathrm{Tr}(R^2_\lambda P)d\lambda=N\cdot\mathrm{Tr}(T^{-(N+1)}P).\tag{26}$$

as $N \geqslant 1$ and $M = N + 1$.

Lemma 6. *Suppose that we are given finite-dimensional operators $P_1, P_2, \ldots, P_k$. Assume that smoothness condition (17) is satisfied. Then*

1. *If $k \geqslant 2$ and $N \geqslant 2$, N is the integer in the smoothness condition (17), then for each integer $M : k+1 \leqslant M \leqslant N+k-1$ identity (25) holds true. As $M \leqslant k$, the left hand side in (25) vanishes. If in the left hand side in (25) we replace Mn by an integer $s : s < (N+k-1)n$ not being a multiple of n, then the left hand side in (25) vanishes, too.*
2. *If $k = 1$, for $N \geqslant 1$, where N is the integer in condition (17), the identity (26) holds true. If in (26) instead of $(N+1)n$ we substitute $s: \; s < (N+1)n$ not being a multiple of n, then the left hand side in (26) vanishes.*
3. *If $N = 0$ in (17), then*
$$\lim_{\nu\to\infty} \oint_{\Gamma_\nu} \lambda^{n-1} \mathrm{Tr}(Q_\lambda^k) d\lambda = 0$$
for $k \geqslant 1$.

The latter identity follows from relation (22).

5. Regularized trace formulae

In what follows we assume that natural parameter s takes values from $Nn+1$ to $(N+1)n$, where $N \geqslant 0$ is integer. Moreover, finite-dimensional operators $P_0, P_1, \ldots, P_{n-1}$ satisfy smoothness condition (17).

5.1. Calculation of $\lim\limits_{\nu\to\infty} J_1^\nu(s)$.

We have
$$J_1^\nu(s) = \sum_{j=1}^{n-1} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s-1} \mathrm{Tr}(R_\lambda P_j) d\lambda.$$

It follows from 5 for $k = 1$ and $Mn = j + s = (N+1)n$ that

Lemma 7. *Let $Nn + 1 \leqslant s \leqslant (N+1)n$ with integer $N \geqslant 0$.*

1. *The identity*
$$G_1(s) \overset{def}{=} \lim_{\nu\to\infty} J_1^\nu(s) = -j_s \sum_{l=1}^{n_{j_s}} (T^{-(N+1)}\psi_l^{j_s}, \varphi_l^{j_s}) = -j_s \mathrm{Tr}(T^{-(N+1)} P_{j_s})$$
holds true, where $j_s = (N+1)n - s$.
2. *If $s = tn$ for each integer $t \geqslant 0$, then $G_1(s) = \lim\limits_{\nu\to\infty} J_1^\nu(s) = 0$.*

5.2. Calculation of $\lim\limits_{\nu\to\infty} J_2^\nu(s)$.

We have
$$J_1^\nu(s) = \sum_{j=1}^{n-1} \sum_{k=1}^{\infty} \frac{j}{2\pi i} \oint_{\Gamma_\nu} \lambda^{j+s-1} \mathrm{Tr}\left(R_\lambda P_j (R_\lambda P_0)^k\right) d\lambda.$$

Lemma 5 implies

Lemma 8. *Let $Nn + 1 \leqslant s \leqslant (N+1)n$ with integer $N \geqslant 0$.*

1. *The identity*
$$G_2(s) \overset{def}{=} \lim_{\nu\to\infty} J_2^\nu(s) = j_s \sum_{k=1}^{N} (-1)^{k+1} \sum_{\substack{p_0+p_1+\cdots+p_k=N+1\\ p_j\geqslant 0}} \mathrm{Tr}\left(T^{-p_0} P_{j_s} T^{-p_1} P_0 \cdots T^{-p_k} P_0\right)$$

holds true, where $j_s = (N+1)n - s$.

2. *As* $s = tn$, $t \geqslant 0$ *is integer, we have* $G_2(s) = \lim\limits_{\nu\to\infty} J_2^\nu(s) = 0$.

5.3. Вычисление $\lim\limits_{\nu\to\infty} J_3^\nu(s)$. We have

$$J_3^\nu(s) = \sum_{j=1}^{n-1}\sum_{k=1}^{\infty}\sum_{m=1}^{k(n-1)} \frac{j}{2\pi i}\oint\limits_{\Gamma_\nu} \lambda^{j+s+m-1}\mathrm{Tr}\left(R_\lambda P_j \sum_m P_{\alpha_1\ldots\alpha_k}\right) d\lambda.$$

It follows from (18) that

$$1 \leqslant k \leqslant j+s-n. \tag{27}$$

$$j_s \leqslant j \leqslant n-1, \quad \text{where} \quad j_s \overset{def}{=} \max\{1, n-s+1\}. \tag{28}$$

The limit of the integral in the expression $J_3^\nu(s)$ is non-zero if for each fixed index k we have $(k+t+1)n = j+s+m$ for some integer $t \geqslant 0$. It is easy to obtain

$$t_0 \leqslant t \leqslant t_1, \quad t_0 = \overline{[(j+s+1)/n-k-1]}, \quad t_1 = [(j+s-k-n)/n], \tag{29}$$

where $\overline{[a]}$ stands for the least integer greater or equal a.

By Lemma 5 we obtain the following result.

Lemma 9. 1. *The identities*

$$\begin{aligned} J_3(j,k,t,s) &\overset{def}{=} \lim_{\nu\to\infty}\frac{1}{2\pi i}\oint\limits_{\Gamma_\nu}\lambda^{j+s+m-1}\mathrm{Tr}\left(R_\lambda P_j\sum_m P_{\alpha_1\ldots\alpha_k}\right)d\lambda \\ &= (-1)^{k+1}\sum_{\substack{p_0+p_1+\cdots+p_k=k+t+1\\ p_l\geqslant 0; l=0,1,\ldots,k}}\ \sum_{\substack{\alpha_1+\cdots+\alpha_k=(k+t+1)-(j+s)\\ \alpha_l\geqslant 0; l=1,\ldots,k}} \mathrm{Tr}\left(T^{-p_0}P_jT^{-p_1}P_{\alpha_1}\cdots T^{-p_k}P_{\alpha_k}\right), \end{aligned} \tag{30}$$

$$G_3(s) \overset{def}{=} \lim_{\nu\to\infty} J_3^\nu(s) = \sum_{j=j_s}^{n-1} j \sum_{k=1}^{j+s-n}\sum_{t=t_0}^{t_1} J_3(j,k,t,s) \tag{31}$$

hold true.

2. *As* $s = 0, 1$, *the identity* $\lim\limits_{\nu\to\infty} J_3^\nu(s) = 0$ *holds true. It follows from the fact that the set of indices* j *defined by inequlaities (28) is empty for* $s = 0, 1$.

5.4. Calculation of $\lim\limits_{\nu\to\infty} J_4^\nu(s)$. We have

$$J_4^\nu(s) = \sum_{k=1}^{\infty}\frac{n}{2\pi i}\oint\limits_{\Gamma_\nu}\lambda^{n+s-1}\mathrm{Tr}\left(TR_\lambda\left(R_\lambda P_0\right)^k\right)d\lambda.$$

We employ the obvious identity $TR_\lambda = \lambda^{-n}R_\lambda - \lambda^{-n}E$ and substitute it into the formula for $J_4^\nu(s)$ to obtain

$$J_4^\nu(s) = n\sum_{k=1}^{\infty}\frac{1}{2\pi i}\oint\limits_{\Gamma_\nu}\lambda^{s-1}\mathrm{Tr}\left(R_\lambda\left(R_\lambda P_0\right)^k\right)d\lambda - n\sum_{k=1}^{\infty}\frac{1}{2\pi i}\oint\limits_{\Gamma_\nu}\lambda^{s-1}\mathrm{Tr}\left(\left(R_\lambda P_0\right)^k\right)d\lambda.$$

Applying Lemma 6 to the first term and Lemma 5 to the second term, we arrive at

Lemma 10. 1. *The identities*

$$G_4^1(s) \overset{def}{=} \lim_{\nu\to\infty} n \sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_{\Gamma_\nu} \lambda^{s-1} \mathrm{Tr}\left(R_\lambda (R_\lambda P_0)^k\right) d\lambda$$

$$= n \sum_{k=1}^{N} (-1)^{k+1} \sum_{\substack{p_1+p_2+\cdots+p_k=N+1 \\ p_l\geqslant 2; p_2,\ldots,p_k\geqslant 1}} (p_1-1)\mathrm{Tr}\left(T^{-p_1}P_0T^{-p_2}P_0\cdots T^{-p_k}P_0\right),$$

$$G_4^2(s) \overset{def}{=} -\lim_{\nu\to\infty} n \sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_{\Gamma_\nu} \lambda^{s-1} \mathrm{Tr}\left((R_\lambda P_0)^k\right) d\lambda$$

$$= n \sum_{k=1}^{N+1} (-1)^{k+1} \sum_{\substack{p_1+p_2+\cdots+p_k=N+1 \\ p_l, p_2,\ldots,p_k\geqslant 1}} \mathrm{Tr}\left(T^{-p_1}P_0T^{-p_2}P_0\cdots T^{-p_k}P_0\right)$$

hold true, i.e., $\lim_{\nu\to\infty} J_4^\nu(s) = G_4^1(s) + G_4^2(s)$.

2. *If parameter* $s \in \mathbb{N}$ *is not a multiple of* n, *in particular, as* $s = 0, 1, \ldots, n-1$, *then* $\lim_{\nu\to\infty} J_4^\nu(s) = 0$.

5.5. Calculation of $\lim_{\nu\to\infty} J_5^\nu(s)$. We have

$$J_5^\nu(s) = \sum_{k=1}^{\infty} \sum_{m=1}^{k(n-1)} \frac{n}{2\pi i} \oint_{\Gamma_\nu} \lambda^{m+n+s-1} \mathrm{Tr}\left(R_\lambda T \sum_m P_{\alpha_1\ldots\alpha_k}\right) d\lambda.$$

We apply the identity $TR_\lambda = \lambda^{-n}R_\lambda - \lambda^{-n}E$ to obtain

$$J_5^\nu(s) = n \sum_{k=1}^{\infty} \sum_{m=1}^{k(n-1)} \frac{1}{2\pi i} \oint_{\Gamma_\nu} \lambda^{m+s-1} \mathrm{Tr}\left(R_\lambda \sum_m P_{\alpha_1\ldots\alpha_k}\right) d\lambda$$
$$- n \sum_{k=1}^{\infty} \sum_{m=1}^{k(n-1)} \frac{1}{2\pi i} \oint_{\Gamma_\nu} \lambda^{m+s-1} \mathrm{Tr}\left(\sum_m P_{\alpha_1\ldots\alpha_k}\right) d\lambda.$$

We denote by $J_5^{1\nu}(s)$ and $J_5^{2\nu}(s)$ the terms in the right hand side of the latter formula, respectively.

By (6) we get the range of index k:

$$1 \leqslant k \leqslant s-n. \tag{32}$$

As in the proof of Lemma 10, we assume $(t+k+1)n = m+s$ for integer $t \geqslant 0$, where

$$\widetilde{t}_0 \overset{def}{=} \overline{[(s-(k+1)n+1)/n]} \leqslant t \leqslant [(s-k-n)/n] \overset{def}{=} \widetilde{t}_1, \tag{33}$$

Applying Lemma 6, we obtain

$$G_5^1(s) \overset{def}{=} \lim_{\nu\to\infty} J_5^{1\nu}(s) = n \sum_{k=1}^{s-n} \sum_{t=\widetilde{t}_0}^{\widetilde{t}_1} (-1)^{k+1}$$
$$\cdot \sum_{\substack{p_1+\ldots+p_k=k+t+1 \\ p_l\geqslant 2, p_2,\ldots,p_k\geqslant 1}} (p_1-1) \sum_{\substack{\alpha_1+\ldots+\alpha_k=(k+t+1)n-s \\ 0\leqslant\alpha_l\leqslant n-1; l=1,\ldots,k}} \mathrm{Tr}\left(T^{-p_1}P_{\alpha_1}\cdots T^{-p_k}P_{\alpha_k}\right). \tag{34}$$

To calculate $\lim_{\nu\to\infty} J_5^{2\nu}(s)$, we reproduce the same calculations but at the last step we apply Lemma 5.

We have $1 \leqslant k \leqslant s$. For each k, the identity holds $m+s=(k+\mu)n$ for some integer $\mu \geqslant 0$, where

$$1+s \leqslant (k+\mu)n \leqslant k(n-1)+s, \quad \widetilde{\mu}_0 \stackrel{def}{=} \overline{[(s-nk+1)/n]} \leqslant \mu \leqslant [(s-k)/n] \stackrel{def}{=} \widetilde{\mu}_1. \tag{35}$$

Applying Lemma 5, in view of (35) we get

$$\begin{aligned} G_5^2(s) \stackrel{def}{=} \lim_{\nu\to\infty} J_5^{2\nu}(s) = n \sum_{k=1}^{s} \sum_{\mu=\widetilde{\mu}_0}^{\widetilde{\mu}_1} (-1)^{k-1} \\ \cdot \sum_{\substack{p_1+\ldots+p_k=k+\mu \\ p_l,p_2,\ldots,p_k\geqslant 1}} \sum_{\substack{\alpha_1+\ldots+\alpha_k=(k+\mu)n-s \\ 0\leqslant\alpha_l\leqslant n-1; l=1,\ldots,k}} \operatorname{Tr}\left(T^{-p_1}P_{\alpha_1}\cdots T^{-p_k}P_{\alpha_k}\right). \end{aligned} \tag{36}$$

Lemma 11. 1. *The identity* $\lim_{\nu\to\infty} J_5^\nu(s) = G_5^1(s)+G_5^2(s)$ *holds true, where* $G_5^1(s)$ *and* $G_5^2(s)$ *are determined by identities (34) and (36).*

2. $\lim_{\nu\to\infty} J_5^\nu(0) = 0$. *It can be proven easily in the same way as second items in previous lemmata 7–10.*

Corollary 3. *Now we can prove Lemma 4 in accordance with which function* $F(s)$ *defined by formula (11) satisfies* $F(0)=0$. *But in view of (13) it is implied by the identity* $\lim_{\nu\to\infty} J_k^\nu(0)=0$ $\forall k=1,2,3,4,5$. *These identities were justified in Lemmata 7–11.*

We summarize all the above arguments in the following theorem.

Theorem 1. *Suppose that we are given operator pencil*

$$L_\lambda = E - P_0 - \lambda P_1 \cdots \lambda^{n-1}P_{n-1} - \lambda^n T$$

in separable Hilbert space $\mathfrak{H}$, *where operators* $P_0, P_1, \ldots, P_{n-1}$ *are finite-dimensional and read as* $P_j = \sum_{l=1}^{n_j}(\bullet, \varphi_l^j)\psi_l^j$, *while* T *is an injective self-adjoint compact operator in* $\mathfrak{H}$. *Suppose that the counting function of the eigenvalues of pencil* $T_\lambda = E - \lambda T$ *satisfies "sparseness" condition (6).*

Suppose that $s \in \mathbb{N}\cap[Nn+1,(N+1)n]$ *with integer* $N \geqslant 0$. *If (17) holds, there exists a monotonous sequence of the natural series* $\{N_\nu\}_{\nu=m_0}^\infty$ *satisfying the regularized trace formula*

$$\lim_{\nu\to\infty} \sum_{m=1}^{N_\nu} (\mu_m^s - \eta_m^s - c_m(s)) = F(s),$$

where μ_m *and* η_m *are the eigenvalues of pencils* L_λ *and* T_{λ^n}, *respectively taken counting multiplicities,* $c_m(s)=0$, *and*

$$F(s) = -G_1(s) - G_2(s) - G_3(s) - G_4^1(s) - G_4^2(s) - G_5^1(s) - G_5^2(s),$$

where values $G_j^i(s)$ *are determined in Lemmata 7–11.*

6. Example

We consider the second order pencil $L_\lambda = E - P_0 - \lambda P_1 - \lambda^2 T$. Assuming the hypothesis of Theorem 1, let us write down regularized trace formulae of first, second, and third orders. Quantities $G_j^l(s)$ defined in Theorem 1 read as

1. $G_1(1) = -\operatorname{Tr}(T^{-1}P_1)$, $G_2(1)=G_3(1)=G_4^1(1)=G_4^2(1)=G_5^1(1)=0$, $G_5^2(1)=2\operatorname{Tr}(T^{-1}P_1)$;
2. $G_1(2)=G_2(2)=G_4^1(2)=G_5^1(2)=0$, $G_3(2)=\operatorname{Tr}((T^{-1}P_1)^2)$, $G_4^2(2)=2\operatorname{Tr}(T^{-1}P_0)$, $G_5^2(2)=-2\operatorname{Tr}((T^{-1}P_1)^2)$;
3. $G_1(3) = -\operatorname{Tr}((T^{-2}P_1)^2)$, $G_2(3)=\operatorname{Tr}(T^{-1}P_1T^{-1}P_0)$, $G_3(3)=-\operatorname{Tr}((T^{-1}P_1)^3)$, $G_4^1(3)=G_4^2(3)=0$, $G_5^1(3)=2\operatorname{Tr}(T^{-2}P_1)$, $G_5^2(1)=2\operatorname{Tr}(T^{-2}P_1)-4\operatorname{Tr}(T^{-1}P_1T^{-1}P_0)+2\operatorname{Tr}((T^{-1}P_1)^3)$.

The first group of formulae by Theorem 1 implies

$$\lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m-\eta_m)=-\mathrm{Tr}(T^{-1}P_1). \tag{37}$$

The second group of formulae yields

$$\lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m^2-\eta_m^2)=-\mathrm{Tr}((T^{-1}P_1)^2)-2\mathrm{Tr}(T^{-1}P_0). \tag{38}$$

It follows from the third group of formulae:

$$\lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m^3-\eta_m^3)=-\mathrm{Tr}((T^{-1}P_1)^3)-3\mathrm{Tr}(T^{-2}P_1)+3\mathrm{Tr}(T^{-1}P_1T^{-1}P_0). \tag{39}$$

We apply the obtained result to the Sturm-Liouville problem for a loaded equation. We consider the boundary value problem

$$\begin{aligned}&-y''(x)+q(x)y(x)-a(x)y(x_0)-\lambda b(x)y(x_1)-\lambda^2y(x)=0, \quad 0<x<\pi,\\ &y(0)=y(\pi)=0, \quad x_0,\ x_1\in(0,\pi).\end{aligned}$$

By A we denote the self-adjoint operator in $L_2(0,\pi)$: $Ay(x)=-y''(x)+q(x)y(x)$, $D(A)=\{y\in W_2^2(0,\pi):\ y(0)=y(\pi)\}$.

Let $G(x,\xi)$ be the Green function of operator A. Then we have the identities

$$y(x_0)a(x)=a(x)\int_0^\pi G(x_0,\xi)Ay(\xi)d\xi, \quad y(x_1)b(x)=b(x)\int_0^\pi G(x_1,\xi)Ay(\xi)d\xi.$$

Thus, the boundary value problem generates operator pencil $N_\lambda=A-Q_0-\lambda Q_1-\lambda^2E$, where

$$Q_0y(x)=\int_0^\pi a(x)G(x_0,\xi)Ay(\xi)d\xi, \quad Q_1y(x)=\int_0^\pi b(x)G(x_1,\xi)Ay(\xi)d\xi.$$

Pencil $L_\lambda=N_\lambda A^{-1}$ satisfies the hypothesis of Theorem 1. Thus, if $a(x),b(x)\in D(A^2)$, in accordance with formulae (37), (38)

$$\lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m-\eta_m)=-b(x_1), \quad \lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m^2-\eta_m^2)=b^2(x_1)-2a(x_0).$$

If $a(x),b(x)\in D(A^3)$, formula (39) implies

$$\lim_{\nu\to\infty}\sum_{m=1}^{N_\nu}(\mu_m^3-\eta_m^3)=-b^3(x_1)-3[-b''(x_1)q(x_1)b(x_1)]+3a(x_1)b(x_0).$$

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