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GENERALIZED SOLUTIONS AND EULER-DARBOUX TRANSFORMATIONS

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Abstract. We introduce Euler-Darboux transformation for non-homogeneous differential equations with the right-hand side being a generalized function. As an example, we construct the fundamental solutions for Klein-Gordon-Fock and Schrödinger equations with variable coefficients describing a particle in external scalar field.

Keywords: Euler-Darboux transformation, Klein-Gordon-Fock equation, Schrödinger equation, fundamental solution

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1. Euler-Darboux transform of inhomogeneous equations and generalized solutions

We consider the linear inhomogeneous differential equation

$$Lu = Au + Bu = f, \tag{1}$$

where A is a differential operator in one variable x:

$$A = \sum_{i=0}^{K} a_i(x) D_x^i, \tag{2}$$

B is the differential operator in variables $y_1, \dots, y_n$ reading as

$$B = \sum_{|\alpha|\geqslant 0}^{M} b_\alpha(y) D_y^\alpha, \tag{3}$$

and $f(x, y_1, \dots, y_n)$ is a generalized function. In what follows we make use of the standard theory of generalized functions [1] and we introduce the notations: $\alpha = (\alpha_1, \dots, \alpha_n)$ is an integer multi-index, $D_x^i = \dfrac{\partial^i}{\partial x^i}$, $D_y^\alpha = \dfrac{\partial^{|\alpha|}}{\partial {y_1}^{\alpha_1} \dots \partial {y_n}^{\alpha_n}}$ are generalized derivatives. For the classical functions we shall also employ the notations of the derivatives (also generalized in the general situation) obvious by the context: h', γ_y. Functions $a_i(x)$ and $b_\alpha(y)$ are assumed to be smooth in corresponding domains. Moreover, we assume that all functions multiplying generalized functions are infinitely differentiable. Following work [2], we denote by $E_{K,M}$ the class of equations (1).

If $h(x)$, $g(y)$ are classical solutions to the equations

$$\begin{aligned} &Ah = ch, \\ &Bg + cg = 0, \qquad \text{where} \quad c \in \mathbb{R}^1, \end{aligned} \tag{4}$$

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then function $u_1 = gh$ solves homogeneous equation (1). Function u_1 generates a transformation of equation (1).

Theorem 1. *Class of equations $E_{K,M}$ possesses the following properties:*
1. If γ is a smooth function reading as $\gamma = p(x)q(y) \neq 0$, then the transformation

$$u \to v = u/\gamma$$

maps generalized solutions of equation (1) into generalized solutions of the equation

$$\hat{L}v = vL(\gamma)/\gamma + A_1 v + B_1 v = f/\gamma,$$

where

$$A_1 = \sum_{i=1}^{K} a_i^1(x) D_x^i, \qquad B_1 = \sum_{|\alpha|\geqslant 1}^{M} b_\alpha^1(y) D_y^\alpha.$$

As $\gamma = u_1 \neq 0$, equation $\hat{L}v = f/\gamma$ reads as

$$L_1 v = A_1 v + B_1 v = f/\gamma. \tag{5}$$

2. The transformation $v \to w = v_x$ maps generalized solutions to equation (5) into generalized solution of the equation

$$L_2 w = \sum_{i=1}^{K} (D_x(a_i^1) D_x^{i-1} w + a_i^1 D_x^i w) + \sum_{|\alpha|\geqslant 1}^{M} b_\alpha^1 D_y^\alpha w = D_x(f/\gamma). \tag{6}$$

Proof. We observe that product γv, where v is a generalized function, satisfies Leibnitz formula for the derivative of a product. Taking this fact the into consideration as well as the identity $(Lu, \varphi) = (L(\gamma v), \varphi)$ implied by the identity $(u, \varphi) = (\gamma v, \varphi)$, we obtain

$$Lu = L(\gamma v) = vL(\gamma) + \widetilde{A}v + \widetilde{B}v = f, \tag{7}$$

where

$$\widetilde{A}v = \sum_{i=0}^{K} \widetilde{a}_i(x, \gamma, \gamma_x, \ldots) D_x^i v, \qquad \widetilde{B}v = \sum_{|\alpha|\geqslant 1}^{M} \widetilde{b}_\alpha(y, \gamma, \gamma_y, \ldots) D_y^\alpha v,$$

and φ is a function in the space of test functions. Coefficients $\widetilde{a}_i$ can depend only on x, γ, and the derivatives of γ w.r.t. x, while coefficients $\widetilde{b}_\alpha$ can depend only on y, γ and its derivatives w.r.t. $y_1, \ldots, y_n$. Function γ and its derivatives can be involved in coefficients $\widetilde{a}_i$, $\widetilde{b}_\alpha$ only linearly.

We multiply (7) by $1/\gamma$ to obtain the equation

$$\widetilde{L}v = \frac{1}{\gamma} L(\gamma) v + A_1 v + B_1 v = f/\gamma,$$

where operators A_1, B_1 read as

$$A_1 = \sum_{i=0}^{K} \widetilde{a}_i(x, p, p_x, \ldots) D_x^i, \qquad B_1 = \sum_{|\alpha|\geqslant 1}^{M} \widetilde{b}_\alpha(y, q, q_y, \ldots) D_y^\alpha.$$

As $\gamma = u_1$, we obtain equation (5). In order to prove second property, it is sufficient to differentiate (5) w.r.t. x and to introduce a new generalized function $w = D_x v$. As a result, we arrive at equation (6).

We observe that all the equations $Lu = f$, $L_1 v = f/\gamma$, $L_2 = D_x(f/\gamma)$ belong to the same class $E_{K,M}$. $\square$

Corollary. *Let h be a non-trivial solution to equation (4), r be a smooth function of x. Then the transformation*

$$v = \frac{1}{r}\left(D_x u - \frac{h'}{h}u\right) \tag{8}$$

maps generalized solutions of equation (1) into generalized solutions of the same class $E_{K,M}$.

Indeed, the transformation

$$v = p(x)q(y)D_x\left(\frac{u}{u_1}\right) \tag{9}$$

is a combination of the transformations considered in Theorem 1 and hence, it preserves the class of the equation. Here p, q are smooth arbitrary functions, u_1 is a solution to equation (1) obtained by the separation of variables $u_1 = h(x)g(y)$. If we let $q = g$, $p = h/r$, by (9) we obtain (8).

Following work [2], let us prove

Lemma 1. *The transformation*

$$u_k = \mathcal{M}_k u = \frac{W(h_1, \ldots, h_k, u)}{W(h_1, \ldots, h_k)} \tag{10}$$

maps a generalized solution to equation (1) into a generalized solution to equation of the same class $E_{K,M}$.

Despite the proof of the lemma given in [2] works for also for the case of generalized solution, we provide it here since it is employed in the proof of Theorem 3.

In order to make sure that Lemma 1 is valid, we observe that if we know solutions $h_1, \ldots$, h_k to equation (4) for different c_1, …, c_k, as it was shown in [2], we can construct an operator of kth order being a superposition of first Euler-Darboux operators $\mathcal{L}_h = hD_x(1/h)$ as well as the associated transformation acting on $E_{K,M}$. Indeed, let h_1, …, h_k be smooth linearly independent functions of x. We construct a sequence of functions and operators

$$\begin{aligned} &p_1 = h_1, && p_2 = \mathcal{L}_{p_1} h_2, && \ldots && p_N = \mathcal{L}_{p_{N-1}} \ldots \mathcal{L}_{p_1} h_N, \\ &\mathcal{M}_1 = \mathcal{L}_{p_1}, && \mathcal{M}_2 = \mathcal{L}_{p_2}\mathcal{M}_1, && \ldots && \mathcal{M}_N = \mathcal{L}_{p_N}\mathcal{M}_{N-1}. \end{aligned} \tag{11}$$

It follows from the construction of operators $\mathcal{M}_k$ that functions h_1, …, h_k satisfy kth order differential equation

$$\mathcal{M}_k h = 0. \tag{12}$$

Thus, they make a basis of solutions to equation (12). Therefore, the action of operator M_k on an arbitrary function is give by [3]

$$\mathcal{M}_k u = D_x^k u + a_{k-1} D_x^{k-1} u + \ldots + a_0 u = \frac{W(h_1, \ldots, h_k, u)}{W(h_1, \ldots, h_k)}. \tag{13}$$

It remains to take solutions to equation (4) for different h_1, …, h_k.

2. Transformation of equations in class $E_{2,M}$

In the present section we consider Euler-Darboux transformations of special type in the class of equations $E_{2,M}$. We consider the equation

$$F D_x^2 u + G D_x u + H u = B u + f, \tag{14}$$

where F, G, H are smooth functions of x, f is a generalized function, and B is a linear operator given by (3).

Theorem 2. *The Euler-Darboux transformation given by identity (8) maps generalized solutions to equation (14) into generalized solution to the equation*

$$FD_x^2v + GD_xv + Hv = Bv + f_1, \tag{15}$$

where

$$G_1 = G + F' + 2F\frac{r'}{r}, \tag{16}$$

$$H_1 = H + \frac{(Fr' + Gr)'}{r} + F'(\ln h)' + 2F(\ln h)'', \tag{17}$$

$$f_1 = \frac{1}{r}(D_xf - \frac{h'}{h}f), \tag{18}$$

while function $h(x)$ solves the ordinary differential equation

$$Fh_{xx} + Gh_x + (H + c)h = 0, \quad \textit{where} \quad c \in \mathbb{R}^1. \tag{19}$$

Proof. We introduce the notations

$$v = Ru = \frac{1}{r}(D_xu + su), \quad \text{where} \quad s = -h'/h,$$
$$Au = FD_x^2u + GD_xu + Hu, \quad A_1u = F_1D_x^2u + G_1D_xu + H_1u.$$

Then original equations (14) and (15) cast into the form $Au = Bu + f$ and $A_1v = Bv + f_1$. In order to prove the theorem, we need to show that

$$(A^* - B^*)R^*\varphi = R^*(A_1^* - B^*)\varphi. \tag{20}$$

Here the star indicates the formal adjoint operator defined for operators A and B as follows:

$$A^*\varphi = \sum_{i=0}^{K}(-1)^iD_x^i(a_i(x)\varphi), \qquad B^*\varphi = \sum_{|\alpha|\geqslant 0}^{M}(-1)^{|\alpha|}D_y^\alpha(b_\alpha(y)\varphi).$$

Indeed,

$$\begin{aligned}(R(A - B)u, \varphi) =&(u, (A^* - B^*)R^*\varphi) = (u, R^*(A_1^* - B^*)\varphi)\\ =&(Ru, (A_1^* - B^*)\varphi) = (v, (A_1^* - B^*)\varphi) = ((A_1 - B)v, \varphi) = (Rf, \varphi).\end{aligned} \tag{21}$$

Here we have employed the commutation of operators B and R and as one can see easily, it implies $B^*R^*\varphi = R^*B^*\varphi$. It remains to show that $A^*R^*\varphi = R^*A_1^*\varphi$. We have

$$A^*R^*\varphi = D_x^2\left[F(-D_x(\varphi/r) + \frac{s}{r}\varphi)\right] - D_x\left[G(-D_x(\varphi/r) + \frac{s}{r}\varphi)\right] + H\left[-D_x(\varphi/r) + \frac{s}{r}\varphi\right],$$
$$R^*A_1^*\varphi = -D_x\left[\frac{1}{r}(D_x^2(F_1\varphi) - D_x(G_1\varphi) + H_1\varphi)\right] + \frac{s}{r}[D_x^2(F_1\varphi) - D_x(G_1\varphi) + H_1\varphi].$$

The left hand side of the equation $A^*R^*\varphi - R^*A_1^*\varphi = 0$ is a polynomial w.r.t. $\varphi_{xxx}, \varphi_{xx}, \varphi_x, \varphi$. The coefficients at these quantities must vanish. Equating the coefficients at φ_{xxx}, φ_{xx}, we obtain respectively $F_1 = F$ and $G_1 = G + F' + 2F(r'/r)$. Substituting F_1 and G_1 into the coefficient at φ_x, we arrive at (17).

Equating the coefficient at φ to zero, in view of found F_1, G_1, and H_1 we obtain

$$Fs'' + (F' - 2Fs + G)s' - F's^2 + G's - H' = (Fs' + Gs - Fs^2 - H)' = 0. \tag{22}$$

As $s = -h'/h$, this identity becomes $(-Fh''/h - Gh'/h - H)' = 0$ that implies equation (19). □

Let us consider higher Euler-Darboux transformations. If we know k solutions $h_1, \ldots, h_k$ to equation (19) for different $c_1, \ldots, c_k$, we can construct Euler-Darboux transformation of order k.

Theorem 3. *Let $h_1, \ldots, h_k$ be solutions to equation (19) associated with different constants $c_1, \ldots, c_k$. Then transformation (13) maps generalized solutions to equation (14) into the generalized solutions to equation*

$$FD_x^2 u_k + G_k D_x u_k + H_k u_k = Bu_k + f_k, \tag{23}$$

At that, the coefficients and function f_k are given by the formulae

$$G_k = G + kF', H_k = H + kG' + \frac{k(k-1)}{2}F'' + F'(\ln W)' + 2F(\ln W)'' \tag{24}$$

and

$$f_k = \mathcal{M}_k f = \frac{W(h_1, \ldots, h_k, f)}{W(h_1, \ldots, h_k)}. \tag{25}$$

Here W is the Wronskian for functions $h_1, \ldots, h_k$.

Proof. We employ the results of Theorem 2. The expression for G_k is obtain by induction by applying formula (16) for $r = 1$. Employing (17) and construction (11) of functions $p_1, \ldots, p_k$, it is easy to see that the inductive construction of coefficients H_k leads us to the formulae

$$H_k = H + kG' + \frac{k(k-1)}{2}F'' + F'(\ln p_1 \ldots p_k)' + 2F(\ln p_1 \ldots p_k)''. \tag{26}$$

Let us find the product $p_1 \ldots p_k$. Since in accordance (11) and (13) the identities

$$p_{i+1} = \mathcal{M}_i h_{i+1} = \frac{W(h_1, \ldots, h_i, h_{i+1})}{W(h_1, \ldots, h_i)}$$

hold true, we have the identities

$$p_1 \ldots p_k = h_1 \frac{W(h_1, h_2)}{h_1} \cdots \frac{W(h_1, \ldots, h_k)}{W(h_1, \ldots, h_{k-1})} = W(h_1, \ldots, h_k)$$

that implies formula (24) for coefficient H_k. The validity of the formula for f_k is obvious thanks to (13) and (18). □

3. Construction of fundamental solutions

Let us construct fundamental solutions to Klein-Gordon-Fock equations (KGF) and to Schrödinger equation with variable coefficients. For the sake of simplicity we restrict ourselves by one-dimensional spatial problem. The generalized formulation of the Cauchy problem employed below was discussed in details in [1]. KGF equation reads as [4]

$$D_t^2 u + m^2 u = a^2 D_x^2 u, \quad \text{where} \quad a, m \in \mathbb{R}^1. \tag{27}$$

In order to construct the fundamental solution, we consider the generalized Cauchy problem for equation (27) with source [1]

$$D_t^2 u + m^2 u = a^2 D_x^2 u + f(x,t), \tag{28}$$

where function $f(x,t)$ reads as

$$f = u_0(x) \cdot \delta'(t) + u_1(x) \cdot \delta(t) \tag{29}$$

Here $\cdot$ stands for the Cartesian product of functions.

Under Euler-Darboux transformation, by Theorem 2 equation (28) is mapped into the equation

$$D_t^2 v + m^2 v = a^2 D_x^2 v + H_1(x) v + f_1 \tag{30}$$

with

$$f_1 = D_x f - \frac{h'}{h} f. \tag{31}$$

Function $H_1(x)$ is determined by formula (17). In order to the solution to Cauchy problem for equation (28) to be mapped into the fundamental solution of equation (30), we suppose the following condition

$$D_x f - \frac{h'}{h} f = \delta(x-y)\cdot\delta(t).$$

These conditions can be rewritten as ordinary differential equations for functions u_0 and u_1

$$u_0' - \frac{h'}{h} u_0 = 0, \tag{32}$$

$$u_1' - \frac{h'}{h} u_1 = \delta(x-y). \tag{33}$$

Solutions to equations (32) and (33) are chosen as follows (for the sake of simplicity of fundamental solution)

$$u_0 = 0, \tag{34}$$

$$u_1(x,y) = \frac{\theta(x-y)h(x)}{h(y)}, \tag{35}$$

where $\theta(x-y)$ is the Heaviside theta-function. The solution to the generalized Cauchy problem for equation (28) under the choice $u_0 = 0$ is the convolution of the fundamental solution to equation (27) and function u_1. Fundamental solution to KGF equation can be chosen as [1]

$$E(x,y,t,\tau) = \frac{1}{2a}\theta(at-|x-y|)J_0\left(\frac{m}{a}\sqrt{a^2(t-\tau)^2-(x-y)^2}\right), \tag{36}$$

where J_0 is the Bessel function. The solution to the generalized Cauchy problem is

$$u(x,t) = \int\limits_{-\infty}^{\infty} u_1(\xi)E(x,\xi,t,0)d\xi. \tag{37}$$

Omitting intermediate calculation, we write down the solution to the generalized Cauchy problem for KGF equation:

$$u(x,y,t) = \frac{1}{2ah(y)}\int\limits_{-at}^{at}\theta(x-y-z)h(x-z)J_0\left(\frac{m}{a}\sqrt{a^2t^2-z^2}\right)dz. \tag{38}$$

We find fundamental solution to equation (30) by the formula

$$E_1(x,y,t) = D_x u(x,y,t) - \frac{h'(x)}{h(x)}u(x,y,t). \tag{39}$$

By simple calculation we get

$$E_1(x,y,t) = \begin{cases} 0, & \text{if } x-y<-at, \\ \frac{1}{2a}J_0(\frac{m}{a}\sqrt{a^2t^2-(x-y)^2}) + \\ \frac{1}{2ah(y)}\int\limits_{-at}^{x-y}(h'(x-z)-\frac{h'(x)}{h(x)}h(x-z))J_0(\frac{m}{a}\sqrt{a^2t^2-z^2})dz, & \text{if } -at\leqslant x-y\leqslant at, \\ \frac{1}{2ah(y)}\int\limits_{-at}^{at}(h'(x-z)-\frac{h'(x)}{h(x)}h(x-z))J_0(\frac{m}{a}\sqrt{a^2t^2-z^2})dz, & \text{if } x-y>at. \end{cases}$$

In these formulae the prime denotes the differentiation w.r.t. the complex argument written in the brackets. The provided formulae can be easily generalized for higher Euler-Darboux transformations. In order to do it, we need to take function u_1 satisfying equation

$$\frac{W(h_1,\dots,h_k,u_1)}{W(h_1,\dots,h_k)} = \delta(x-y). \tag{40}$$

The solution to this equation is given by the formula

$$u_1(x,y) = \frac{\theta(x-y)}{W_y(h_1,\dots,h_k)} \sum_{i=1}^{k} W_y(h_1,\dots,h_{i-1},h_{i+1},\dots,h_k)h_i(x). \tag{41}$$

Here we have introduced the notation $W_y(h_1,\dots,h_k) = W(h_1(y),\dots,h_k(y))$. The coefficients of the transformed equation are determined by Theorem 3, see formula (24).

The construction of the fundamental solution to Schrödinger equation with variable coefficients can be done in the same way as for KGF equation. Beginning with the equation

$$iD_t u = -D_x^2 u, \tag{42}$$

we consider the generalized Cauchy problem with the following source

$$iD_t u = -D_x^2 u + u_0(x)\cdot\delta(t). \tag{43}$$

We assume that in accordance with formula (18) of Theorem 2 function u_0 is transformed into Dirac delta function. It holds true, if the mentioned function satisfies the following equation

$$u_0' - \frac{h'}{h}u_0 = \delta(x-y), \tag{44}$$

whose solution is determined by formula (35). Fundamental solution to equation (42) is [1]

$$E(x,\xi,t) = \frac{\theta(t)}{2\sqrt{\pi t}} \exp\left(\frac{i(x-\xi)^2}{4t} - \frac{i\pi}{4}\right). \tag{45}$$

Then the solution to the generalized Cauchy problem can be written as the convolution

$$u(x,t) = \frac{\theta(t)}{2\sqrt{\pi t}} \frac{e^{-i\pi/4}}{h(y)} \int\limits_{-\infty}^{\infty} \theta(\xi-y)h(\xi)\exp\left(\frac{i(x-\xi)^2}{4t}\right)d\xi. \tag{46}$$

The solution to generalized Cauchy problem for equation (43) is transformed into the fundamental solution of the equation

$$iD_t v = -D_x^2 v + H_1(x)v \tag{47}$$

by formula (39). As in the case of KGF equation, coefficient $H_1(x)$ is given by formula (17). Let us write down the solution to transformed equation (45)

$$E_1(x,y,t) = \frac{\theta(t)}{2\sqrt{\pi t}} \frac{e^{-i\pi/4}}{h(y)} \int\limits_{y}^{\infty} h(\xi)\left[i\frac{x-\xi}{2t} - \frac{h'(x)}{h(x)}\right]\exp\left(\frac{i(x-\xi)^2}{4t}\right)d\xi. \tag{48}$$

It is obvious that the latter formula determines the fundamental solution only in the case of existence of appropriate integrals.

Similar to KGF equation, the construction of the fundamental solution for Schrödinger equation is also generalized for higher Euler-Darboux transformations.

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