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# SPECTRAL PROPERTIES OF TWO PARTICLE HAMILTONIAN ON ONE-DIMENSIONAL LATTICE

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**Abstract.** We consider a system of two arbitrary quantum particles on a one-dimensional lattice with special dispersion functions (describing site-to-site particle transport), where the particles interact by a chosen attraction potential. We study how the number of eigenvalues of operator family $h(k)$ depends on the particle interaction energy and the total quasimomentum $k \in \mathbb{T}$ (where $\mathbb{T}$ is a one-dimensional torus). Subject to the particle interaction energy, we obtain conditions for existence of multiple eigenvalues below the essential spectrum of operator $h(k)$.

**Keywords:** two-particle Hamiltonian on one dimensional lattice, eigenvalue, multiple eigenvalue.

**Mathematics Subject Classification:** 44A55, 81Q10

## 1. Introduction

In the continuous case the study of spectral properties of the complete Hamiltonian for a two-particle system is reduced to the study of a two-particle Schrödinger operator by means of choosing the energy of the mass center so that one-particle bound states are eigenfunctions for the energy operator with a separated complete momentum (in fact, such operator is independent of total momentum) [1]. Dealing with a lattice, "choosing of the mass center" corresponds to the realization of the Hamiltonian as "fibered operator", i.e., the "direct integral" of the family of operators $h(k)$ describing the energy of two particles depending on total quasi-momentum $k \in \mathbb{T}^d$ ($\mathbb{T}^d$ is a $d$-dimensional torus) [2, 3]. Discrete Hamiltonians were studied in works [4, 5]. In work [4] the emergence of bound states levels was shown for some values of quasi-momentum and these levels were separated from the continuous spectrum by a certain distance. Spectral properties of a two-particle operator depending on the total quasi-momentum were studied in [5].

It was shown in work [3] that in the case when operator $h(\mathbf{0})$ has a virtual level at the left threshold of the essential spectrum, the discrete spectrum of operator $h(k)$ located to the left of the essential spectrum is always non-empty for each $k \in \mathbb{T}^d \setminus \{\mathbf{0}\}$. Assuming that dispersion laws of particles $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ were linearly dependent functions, in work [6] there was shown that the positivity of $h(\mathbf{0})$ implies the positivity of $h(k)$ for each $k \in \mathbb{T}^3 \setminus \{\mathbf{0}\}$.

In [7] there was studied a system of two particles on a three-dimensional lattice with some dispersion law describing the transport of the particle from a note to a neighboring site interacting by an attractive potential only on the closest neighboring sites. The spectral properties of operator family $h(k)$ we studied subject to the particles interaction energy and total quasi-momentum $k \in \mathbb{T}^3$ ($\mathbb{T}^3$ is a 3-dimensional torus).

In the work we consider a two-particle Schrödinger operator $h(k)$, $k \in \mathbb{T}$, associated with a system of two particles on the one-dimensional lattice, where the potential is described by some

M.E. Muminov, A.M. Khurramov, Spectral properties of two particle Hamiltonian on one-dimensional lattice.



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$(2N+1)$-dimensional integral operator and the dispersion law is studied depending on $N$. We study the existence of eigenvalues for operator family $h(k)$ depending on particles interaction energy and total quasi-momentum $k$.

## 2. Formulation of main results

Let $\mathbb{Z}$ be the one-dimensional integer lattice, $(\mathbb{Z})^2 = \mathbb{Z} \times \mathbb{Z}$ be the Cartesian power of $\mathbb{Z}$, and $\ell_2((\mathbb{Z})^2)$ be the Hilbert space of square integrable functions defined on $(\mathbb{Z})^2$.

We consider the coordinate representation for the Hamiltonian of the system of two arbitrary particles interacting with a pair short range potential $\hat{v}(\cdot)$ on the one-dimensional lattice acting in space $\ell_2((\mathbb{Z})^2)$ by the formula

$$\hat{h} = \hat{h}_0 - \hat{v},$$

where the action of $\hat{h}_0$ and $\hat{v}$ is described by the rules:

$$(\hat{h}_0\hat{\psi})(n_1, n_2) = \sum_{s\in\mathbb{Z}} [\hat{\varepsilon}_1(s)\hat{\psi}(n_1 + s, n_2) + \hat{\varepsilon}_2(s)\hat{\psi}(n_1, n_2 + s)],$$

$$(\hat{v}\hat{\psi})(n_1, n_2) = \hat{v}(n_1 - n_2)\hat{\psi}(n_1, n_2).$$

Here $\hat{\varepsilon}_1(s)$ and $\hat{\varepsilon}_2(s)$ are real-valued even functions describing the transport of a particle from the site to the neighboring site; these functions are defined as

$$\hat{\varepsilon}_i(s) = \begin{cases} \dfrac{1}{m_i} & \text{as} \quad s = 0, \\ -\dfrac{1}{2m_i} & \text{as} s = \pm 2n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{v}(s) = \begin{cases} 2\pi\mu_0 & \text{as} \quad s = 0, \\ \pi\mu_l & \text{as} \quad s = \pm l, \quad l = \overline{1, N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_i > 0$ is the mass of $i$th particle, $i = 1, 2$, $\mu_l > 0$, $n$ is a natural number.

We introduce the polynomials

$$P_0(x) \equiv 0, \quad P_1(x) = x,$$
$$P_k(x) = P_1(P_{k-1}(x)) + 2P_{k-1}(x) - P_{k-2}(x) + 2P_1(x), \ \ k = 2, 3, 4, \ldots$$

Let $\Delta$ be the discrete Laplacian on the one-dimensional lattice. It acts in $\ell_2(\mathbb{Z})$ by the formula

$$(\Delta f)(s) = f(s + 1) + f(s - 1) - 2f(s).$$

**Proposition 1.** *For each $f \in \ell_2(\mathbb{Z})$ the identity*

$$f(s + k) + f(s - k) - 2f(s) = (P_k(\Delta)f)(s), \quad k = 2, 3, 4, \ldots$$

*holds true.*

*Proof.* Assume that as $k = l$, $l \geqslant 2$, the identity

$$f(s + l) + f(s - l) - 2f(s) = (P_l(\Delta)f)(s)$$

is valid. Then we have

$$\begin{aligned}(P_1(P_l(\Delta))f)(s) =& [f(s + l + 1) + f(s - (l - 1)) - 2f(s + 1)] \\ &+ [f(s - (l + 1)) + f(s + (l - 1)) - 2f(s - 1)] - 2(P_l(\Delta)f)(s) \\ =& (P_{l+1}(\Delta)f)(s) + (P_{l-1}(\Delta)f)(s) - 2(P_1(\Delta)f)(s) - (P_l(\Delta)f)(s).\end{aligned}$$

It yields that

$$f(s+l+1)+f(s-(l+1))-2f(s) = (P_1(\Delta)f)(s)+2(P_l(\Delta)f)(s) - (P_{l-1}(\Delta)f)(s)+2(P_1(\Delta)f)(s) = (P_{l+1}(\Delta)f)(s).$$

□

We observe that free hamiltonian $\hat{h}_0$ of two arbitrary particles system on the one-dimensional lattice acts in the space $\ell_2(\mathbb{Z}^2)$ by the formula

$$\hat{h}_0 = \frac{1}{2m_1}P_{2n}(\Delta)\times E + \frac{1}{2m_2}E\times P_{2n}(\Delta),$$

where $E$ is the identity mapping in $\ell_2(\mathbb{Z})$.

We note that considered operator $\hat{h}$ is bounded and self-adjoint in $\ell_2((Z)^2)$.

Let $\mathbb{T}=(-\pi,\pi]$, $L_2(\mathbb{T})$ be the Hilbert space of square integrable functions defined on $\mathbb{T}$. By means of Fourier transform [3], [6]

$$\Im:\ \ell_2((\mathbb{Z})^2)\to L_2((\mathbb{T})^2), \quad (\Im\hat{f})(p)=\frac{1}{2\pi}\sum_{s\in(\mathbb{Z})^2}\hat{f}(s)e^{-i(p,s)},$$

we obtain the momentum representation $h$ of operator $\hat{h}$, i.e., $h=\Im\hat{h}\Im^{-1}$. Then we expand operator $h$ into the direct operator integral

$$h=\int_{\mathbb{T}}\oplus h(k)dk,$$

where $h(k)$, $k\in\mathbb{T}$, is the self-adjoint operator acting in $L_2(\mathbb{T})$ by the formula

$$h(k)=h_0(k)-\mathbf{v}.$$

Here $h_0(k)$ the operator of multiplication by the function

$$\mathcal{E}_k(p)=\frac{1}{m_1}\varepsilon(p)+\frac{1}{m_2}\varepsilon(p-k), \quad \varepsilon(p)=\sum_{s\in Z}\hat{\varepsilon}(s)e^{ips}=1-\cos 2np$$

and $\mathbf{v}$ is an integral operator with the kernel

$$v(p-q)=\frac{1}{2\pi}\sum_{s\in Z}\hat{v}(s)e^{-i(p-q)s}=\sum_{l=0}^{N}\mu_l\cos l(p-q).$$

We note that Weyl theorem on essential spectrum [8] implies that the essential spectrum $\sigma_{ess}(h(k))$ of operator $h(k)$ does not change under compact perturbation $\mathbf{v}$ and coincides with the spectrum of unperturbed operator $h_0(k)$. At that, $\sigma_{ess}(h(k))$ consists of the range of function $\mathcal{E}_k(\cdot)$, i.e.,

$$\sigma_{ess}(h(k))=\sigma(h_0(k))=[m(k),M(k)],$$

where $m(k)=\min\limits_{p\in\mathbb{T}}\mathcal{E}_k(p)$, $M(k)=\max\limits_{p\in\mathbb{T}}\mathcal{E}_k(p)$.

Since $\mathbf{v}\geqslant 0$, then

$$\sup(h(k)f,f)\leqslant\sup(h_0(k)f,f)=M(k)(f,f), \qquad f\in L_2(\mathbb{T}).$$

This is why operator $h(k)$ has no eigenvalues to right of the essential spectrum, i.e.,

$$\sigma(h(k))\cap(M(k),\infty)=\emptyset.$$

In what follows we assume that

$$n=\begin{cases}\text{LCF}\{2,4,...,2(N-1)\} & \text{as} \quad N>1,\\ 1, & \text{as} \quad N=1,\end{cases}$$

where LCF stands for the lowest common factor. It should be mentioned that if $N$ is a power of a prime number, then number $\frac{n}{2N}$ is fractional. Otherwise number $\frac{n}{2N}$ is natural.

We introduce the notations

$$
\begin{aligned}
d(k;z) &= \int\limits_{\mathbb{T}} \frac{ds}{\tilde{\mathcal{E}}_k(s) - z}, \quad c_N(k;z) = \int\limits_{\mathbb{T}} \frac{\cos^2 Nsds}{\tilde{\mathcal{E}}_k(s) - z}, \\
s_N(k;z) &= \int\limits_{\mathbb{T}} \frac{\sin^2 Nsds}{\tilde{\mathcal{E}}_k(s) - z}, \qquad z < m(k),
\end{aligned} \tag{1}
$$

$$\tilde{\mathcal{E}}_k(p) = \frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}\cos 2np.$$

The representation for $\tilde{\mathcal{E}}_k(p)$ implies that $\min\limits_{p\in[-\pi,\pi]} \tilde{\mathcal{E}}_k(p)$ is attained at zero only. This is why the integral $\int\limits_{\mathbb{T}} \frac{\sin^2 Nsds}{\tilde{\mathcal{E}}_k(s) - m(k)}$ converges and is positive.

We denote

$$\mu^0(k) = \frac{1}{s_N(k; m(k))}. \tag{2}$$

**Assumption 1.** *Assume that $m = m_1 = m_2$ and $k = \pm\frac{\pi}{2n}$.*

We note that if $\frac{n}{2N}$ is a natural (fractional) number, then $c_N(k;z) = s_N(k;z)$ $\Big(c_N(k;z) > s_N(k;z)\Big)$.

**Theorem 1.** *Suppose that Assumption 1 fails. Then the following statements hold true:*

*1. If $\frac{n}{2N}$ is a natural number, then for each $\mu = (\mu_0, \dots, \mu_N) \in \mathbb{R}_+^{N+1}$ operator $h(k)$ has exactly $2N+1$ eigenvalues (counting multiplicty) to the left of the essential spectrum.*

*2. If $\frac{n}{2N}$ is a fractional number, then for each $\mu = (\mu_0, \dots, \mu_{N-1}) \in \mathbb{R}_+^N$ and $\mu_N \in M_\alpha$, operator $h(k)$ has exactly $2N+\alpha$ eigenvalues (counting multiplicity) to the left of the essential spectrum, where $M_0 = (0; \mu^0(k)]$, $M_1 = (\mu^0(k); \infty)$, $\alpha \in \{0,1\}$.*

**Theorem 2.** *Suppose Assumption 1 holds. Then $\tilde{\mathcal{E}}_k(p) \equiv \frac{2}{m}$ and for each $\mu = (\mu_0, \dots, \mu_N) \in \mathbb{R}_+^{N+1}$ operator $h(k)$ has exactly $2N+1$ eigenvalues reading as: $z_0 = \frac{2}{m} - 2\mu_0\pi$, $z_l = \frac{2}{m} - \mu_l\pi$, $l = \overline{1,N}$. At that, $z_0$ is a simple eigenvalue and $z_l$, $l \geqslant 1$, is a double eigenvalue.*

**Remark.** *It should be mentioned that if $1 - \frac{\mu^*}{2} d(k, z^*) = 0$, $z^* < m(k)$, $\mu^* > 0$ (see Lemma 2) and $\mu_l = \mu^*$ for each $l \in \{1, 2, 3, ..., N-1\}$, then number $z = z^*$ an eigenvalue of operator $h(k)$ having the multiplicity at least $2N-2$.*

## 3. Eigenvalues of $h(k)$

We introduce an operator $\tilde{h}(k)$ acting in $L_2(\mathbb{T})$ by the rule

$$\tilde{h}(k) = \tilde{h}_0(k) - \mathbf{v},$$

where $\tilde{h}_0(k)$ is the operator of multiplication by the function

$$\tilde{\mathcal{E}}_k(p) = \frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}\cos 2np.$$

Suppose that the unitary operator $U : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is defined by the formula

$$(Uf)(p) = f(p - \frac{1}{2n}\theta(k)),$$

where

$$\theta(k) = \arccos \frac{\frac{1}{m_1} + \frac{1}{m_2}\cos 2nk}{\sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}}.$$

Then

$$(U^{-1}f)(p) = f(p + \frac{1}{2n}\theta(k)), \quad f \in L_2(\mathbb{T}).$$

**Lemma 1.** *Operator $h(k)$ is unitarily equivalent to operator $\tilde{h}(k)$, i.e.,*

$$\tilde{h}(k) = U^{-1}h(k)U.$$

*Proof.* Since the representation

$$\mathcal{E}_k(p) = \frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}\cos(2np - \theta(k))$$

holds true, then

$$(h_0(k)Uf)(p) = \left(\frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}\cos(2np - \theta(k))\right) f\left(p - \frac{1}{2n}\theta(k)\right).$$

It is easy to check that

$$(U^{-1}h_0(k)Uf)(p) = \Big(\frac{1}{m_1} + \frac{1}{m_2} - \sqrt{\frac{1}{m_1^2} + \frac{2}{m_1 m_2}\cos 2nk + \frac{1}{m_2^2}}\cos 2np\Big) f(p),$$

i.e.,

$$U^{-1}h_0(k)U = \tilde{h}_0(k).$$

It is clear that

$$\begin{aligned}(U^{-1}\mathbf{v}Uf)(p) =& U^{-1}\left(\int\limits_{\mathbb{T}} v(s-p)f(s - \frac{1}{2n}\theta(k))ds\right) \\ =& \int\limits_{\mathbb{T}} v\left(s - \left(p + \frac{1}{2n}\theta(k)\right)\right) f\left(s - \frac{1}{2n}\theta(k)\right) ds.\end{aligned}$$

In this latter integral we make the change $s - \frac{1}{2n}\theta(k) = t$ and get

$$(U^{-1}\mathbf{v}Uf)(p) = \int\limits_{\mathbb{T}} v(t-p)f(t)dt,$$

i.e.,

$$U^{-1}\mathbf{v}U = \mathbf{v}.$$

The proof is complete. □

**Lemma 2.** *A number $z$, $z < m(k)$ is an eigenvalue of operator $\tilde{h}(k)$ if and only if $\Delta(k; z) = 0$, where*

$$\Delta(k; z) = (1 - \mu_0 d(k; z)) \prod_{l=1}^{N} \left(1 - \frac{\mu_l}{2} d(k; z)\right)^2$$

*for natural $\frac{n}{2N}$, and*

$$\Delta(k; z) = (1 - \mu_0 d(k; z)) \prod_{l=1}^{N-1} \left(1 - \frac{\mu_l}{2} d(k; z)\right)^2 (1 - \mu_N c_N(k; z))\,(1 - \mu_N s_N(k; z))$$

*for fractional* $\frac{n}{2N}$*. At that, the order of zero of function* $\Delta(k;\cdot)$ *coincides with the multiplicity of the eigenvalue of operator* $h(k)$*.*

*Proof.* Let $z < m(k)$ be an eigenvalue of operator $\tilde{h}(k)$ and $f$ the associated eigenvector, i.e., the equation

$$\tilde{h}(k)f = zf$$

has a non-trivial solution $f$. Then

$$f = r_0(z)\mathbf{v}f, \tag{3}$$

where $r_0(z)$ is the operator of multiplication by function $\frac{1}{\tilde{\mathcal{E}}_k(p)-z}$. Introducing the notations

$$\varphi_l = \int\limits_{\mathbb{T}} \cos ls f(s)ds, \tag{4}$$

$$\psi_l = \int\limits_{\mathbb{T}} \sin ls f(s)ds, \tag{5}$$

we rewrite identity (3) as

$$f(p) = \frac{1}{\tilde{\mathcal{E}}_k(p)-z}\sum_{l=0}^{N}\mu_l(\varphi_l \cos lp + \psi_l \sin lp). \tag{6}$$

Substituting (6) into (4) and (5) and employing the evenness of function $\tilde{\mathcal{E}}_k(\cdot)$, we obtain the system of linear equations

$$\varphi_l = \int\limits_{\mathbb{T}} \sum_{r=1}^{N}\mu_r\varphi_r\frac{\cos ls\cos rs}{\tilde{\mathcal{E}}_k(s)-z}ds, \quad l = 0,\dots,N, \tag{7}$$

$$\psi_l = \int\limits_{\mathbb{T}} \sum_{r=1}^{N}\mu_r\psi_r\frac{\sin ls\sin rs}{\tilde{\mathcal{E}}_k(s)-z}ds, \quad l = 1,\dots,N. \tag{8}$$

It follows from the definition of number $n$ that number $\frac{n}{2l}$ is natural for each $l = 1,\dots,N-1$. It implies that function $\tilde{\mathcal{E}}_k(\cdot)$ is periodic with the period $\frac{\pi}{2l}$ for each $l = 1,\dots,N-1$. Let us show that for each $l = 1,\dots,2N-1$ the identity

$$\int\limits_{\mathbb{T}} \frac{\cos ls}{\tilde{\mathcal{E}}_k(s)-z}ds = 0 \tag{9}$$

holds true. Indeed, if $l$ an odd (even) number, we make the change of variable $s = t+\pi (s = t+\frac{\pi}{l})$ in the integral in the left hand side of identity (9) and we have

$$I_l(z) = -\int\limits_{\mathbb{T}} \frac{\cos lt}{\tilde{\mathcal{E}}_k(t)-z}dt = -I_l(z).$$

It yields identity (9). By the primitive identities

$$\cos ls\cos rs = \frac{1}{2}(\cos(l+r)s + \cos(l-r)s), \quad \sin ls\sin rs = \frac{1}{2}(\cos(l-r)s - \cos(l+r)s),$$

and (9) we obtain that

$$\int\limits_{\mathbb{T}} \frac{\cos ls\cos rs}{\tilde{\mathcal{E}}_k(s)-z}ds = 0, \quad \int\limits_{\mathbb{T}} \frac{\sin ls\sin rs}{\tilde{\mathcal{E}}_k(s)-z}ds = 0, \quad l \neq r, \quad l,r = 1,\dots,N. \tag{10}$$

In view of (10), identities (7) and (8) cast into the form

$$\varphi_l = \mu_l \varphi_l \int\limits_{\mathbb{T}} \frac{\cos^2 ls}{\tilde{\mathcal{E}}_k(s) - z} ds, \quad l = 0, \dots, N,$$

$$\psi_l = \mu_l \psi_l \int\limits_{\mathbb{T}} \frac{\sin^2 ls}{\tilde{\mathcal{E}}_k(s) - z} ds, \quad l = 1, \dots, N.$$

The determinant of the system of linear equations w.r.t. unknowns $\varphi_0$, $\varphi_1$,..., $\varphi_N$, $\psi_1$, ..., $\psi_N$ reads as

$$\Delta(k; z) = \prod_{l=0}^{N} \left( 1 - \mu_l \int\limits_{\mathbb{T}} \frac{\cos^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} \right) \prod_{l=1}^{N} \left( 1 - \mu_l \int\limits_{\mathbb{T}} \frac{\sin^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} \right).$$

At that, if $z < m(k)$ is an eigenvalue of operator $\tilde{h}(k)$, then

$$\Delta(k; z) = (1 - \mu_0 d(k; z)) \prod_{l=1}^{N} \left( 1 - \mu_l \int\limits_{\mathbb{T}} \frac{\cos^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} \right) \left( 1 - \mu_l \int\limits_{\mathbb{T}} \frac{\sin^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} \right) = 0.$$

It is easy to show that in accordance with (10), for each $l \leqslant N$ under natural $n/2N$ the identity

$$\int\limits_{\mathbb{T}} \frac{\cos^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} = \int\limits_{\mathbb{T}} \frac{\sin^2 lsds}{\tilde{\mathcal{E}}_k(s) - z} = \frac{1}{2} \int\limits_{\mathbb{T}} \frac{ds}{\tilde{\mathcal{E}}_k(s) - z} = \frac{1}{2} d(k; z)$$

holds true. Hence, the identity

$$\Delta(k; z) = (1 - \mu_0 d(k; z)) \prod_{l=1}^{N} \left( 1 - \frac{\mu_l}{2} d(k; z) \right)^2$$

is valid.

Vice versa, suppose that $\Delta(k; z) = 0$. Then for some $l \in \{0, ..., N\}$ and $z < m(k)$ at least one of the factors in $\Delta(k; z)$ vanishes, i.e., either $1 - \mu_0 \int\limits_{\mathbb{T}} \frac{ds}{\tilde{\mathcal{E}}_k(s) - z} = 0$ or $\left(1 - \frac{\mu_l}{2} d(k; z)\right)^2 = 0$. And it is easy to make sure that number $z < m(k)$ is an eigenvalue of operator $\tilde{h}(k)$ and either

$$\frac{1}{\tilde{\mathcal{E}}_k(p) - z} \quad \text{or} \quad \frac{\cos lp}{\tilde{\mathcal{E}}_k(p) - z} \quad \text{and} \quad \frac{\sin lp}{\tilde{\mathcal{E}}_k(p) - z}$$

are the associated eigenfunctions.

For a fractional $\frac{n}{2N}$ we argue in the same way.

We observe that the order of zero of function $\Delta(k; \cdot)$ coincides with the multiplicity of the eigenvalue of operator $\tilde{h}(k)$. The proof is complete. □

*Proof of Theorem 1.* Suppose that Assumption 1 fails. Then for each $k \in \mathbb{T}$ number $\mu^0(k)$ defined by formula (2) is finite.

The relations

$$1-\mu_0 d(k;z)=\begin{cases}1 & \text{as}\quad z\to-\infty,\\ -\infty & \text{as}\quad z\to m(k),\\ \text{decays monotonously} & z\in(-\infty,m(k)),\end{cases}$$
$$1-\frac{\mu_l}{2}d(k;z)=\begin{cases}1 & \text{as}\quad z\to-\infty,\\ -\infty & \text{as}\quad z\to m(k),\\ \text{decays monotonously} & z\in(-\infty,m(k)),\end{cases}$$
$$1-\mu_N c_N(k;z)=\begin{cases}1 & \text{as}\quad z\to-\infty,\\ -\infty & \text{as}\quad z\to m(k),\\ \text{decays monotonously} & z\in(-\infty,m(k)),\end{cases}$$
$$1-\mu_N s_N(k;z)=\begin{cases}1 & \text{as}\quad z\to-\infty,\\ \geqslant 0 & \text{as}\quad \mu_N\in(0,\mu^0(k)]\quad\text{for each}\quad z\in(-\infty,m(k)),\\ <0 & \text{as}\quad \mu_N>\mu^0(k),\quad z=m(k).\end{cases}$$

hold true.

We observe that functions $d(k;\cdot)$, $c_N(k;\cdot)$, $s_N(k;\cdot)$ defined by formula (1) are positive and monotonically increasing on $(-\infty,m(k))$. This is the above relations imply

$$\begin{aligned}&1-\mu_0 d(k;\cdot)\quad\text{has the unique zero for each}\quad \mu_0>0,\\ &1-\frac{\mu_l}{2}d(k;\cdot)\quad\text{has the unique zero for each}\quad \mu_l>0,\\ &1-\mu_N c_N(k;\cdot)\quad\text{has the unique zero for each}\quad \mu_N>0,\\ &1-\mu_N s_N(k;\cdot)=\begin{cases}\text{has no zeroes as} & \mu_N\in(0;\mu^0(k)],\\ \text{has the unique zero as} & \mu_N\in(\mu^0(k);\infty).\end{cases}\end{aligned}$$

By applying Lemmata 2 and 1 we complete the proof. □

*Proof of Theorem 2.* Suppose that Assumption 1 holds true. Then $\tilde{\mathcal{E}}_k(p)\equiv\frac{2}{m}$ and

$$\begin{aligned}\Delta(k;z)=&(1-\mu_0 d(k;z))\prod_{l=1}^N\left(1-\mu_l\int_{\mathbb{T}}\frac{\cos^2 ls}{\tilde{\mathcal{E}}_k(s)-z}ds\right)\left(1-\mu_l\int_{\mathbb{T}}\frac{\sin^2 ls}{\tilde{\mathcal{E}}_k(s)-z}ds\right)\\ =&\left(1-\frac{2\mu_0\pi}{\frac{2}{m}-z}\right)\prod_{l=1}^N\left(1-\frac{\mu_l\pi}{\frac{2}{m}-z}\right)^2=0.\end{aligned}$$

It allows us to find easily the zeroes of function $\Delta(k;\cdot)$: $z_0=\frac{2}{m}-2\mu_0\pi$ is a simple zero, $z_l=\frac{2}{m}-\mu_l\pi$ is a double zero, $l\in\{1,...,N\}$. In accordance with Lemmata 1 and 1 these numbers are eigenvalues of $h(k)$. It is easy to check that the associated eigenfunctions read as

$$\varphi_0=\frac{1}{2\mu_0\pi},\quad \varphi_l^+=\frac{\cos lp}{\mu_l\pi},\quad \varphi_l^-=\frac{\sin lp}{\mu_l\pi},\quad l=\overline{1,N}.$$

The proof is complete. □

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