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ON SOME PROPERTIES OF CAUCHY PROBLEM FOR NON-STATIONARY THIRD ORDER COMPOSITE TYPE EQUATION

A.R. KHASHIMOV, S. YAKUBOV

Abstract. In the paper we construct a solution to the Cauchy problem for a non-stationary third order composite type equation and we study some of its properties.

Keywords: Cauchy problem, third order equations, non-stationary PDE, Airy functions, increasing at infinity solutions.

Mathematics Subject Classification: 35A02, 35A09, 35B40

1. Introduction

The aim of the present work is to study some properties of solution to equation

$$\sum_{i=1}^{n}\frac{\partial^3}{\partial x_i^3}-\frac{\partial u}{\partial t}=0 \tag{1}$$

in the domain $D=\{(x_i;t):-\infty<x_i<\infty, 0<t\leqslant T\}$ subject to the initial condition

$$u(x_1,x_2,\dots,x_n,0)=\varphi(x_1,x_2,\dots,x_n), \quad -\infty<x_i<\infty. \tag{2}$$

If $n=1$ in (1), we obtain the equation

$$u_{xxx}-u_t=0, \tag{3}$$

which was studied in work [2]. In this work, there were constructed the fundamental solution to equation (3) and potential theory, as well as there were developed the method of studying boundary value problems and Cauchy problem for equation (3). Later the solution to the Cauchy problem for equation (3) was constructed in work [16] in a wider class and some of its properties were studied. By the same method there was constructed a solution to the Cauchy problem for the equation of high odd order [15]

$$\frac{\partial^{2k+1}u}{\partial x^{2k+1}}+(-1)^k\frac{\partial u}{\partial t}=0.$$

If we let $n=2$ in (1), we obtain the equation

$$u_{xxx}+u_{yyy}-u_t=0. \tag{4}$$

We note that solutions to equation (4) and linear Zakharov-Kuznetsov equation (see [4], [5])

$$u_t+u_{xxx}+u_{xyy}=0 \tag{5}$$

have similar aysmptotic properties at infinity. Zakharov-Kuznetsov equation (5) is one of possible generalizations to Korteweg-de-Vries equation in a multi-dimensional space and it

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describes ion-acoustic wave processes in plasme [20]. The equation to the Cauchy problem for equation (4) was constructed in work [8].

The solvability class for the Cauchy problem in the classes of functions growing at infinity was determined first in work by A.N. Tikhonov [18] for the heat equation. Further study for differential equation of odd order in the classes of functions growing at infinity was made by means of the theory of generalized functions [9, 10, 11, 12, 13, 19]. At present, the most complete theory for linear equations of even order (for instance, for linear equations of parabolic type) was developed in [3, 14, 17].

In work [1] there was constructed the fundamental solution to equation (1) in space $\mathbb{R}^{n+1}$

$$U(x_1-\xi_1,x_2-\xi_2,\ldots,x_n-\xi_n;t-\tau) = \frac{1}{(t-\tau)^{\frac{n}{3}}} f\left(\frac{x_1-\xi_1}{(t-\tau)^{\frac{1}{3}}}\right)\ldots f\left(\frac{x_n-\xi_n}{(t-\tau)^{\frac{1}{3}}}\right),\ \ x_i\neq\xi_i,\ \ t>\tau,\ \ i=\overline{1,n}, \tag{6}$$

where $f(z)=\int\limits_0^\infty cos(\lambda^3-\lambda z)d\lambda$, $-\infty<z<\infty$, is the Airy function satisfying the equation

$$f(z)+\frac{1}{3}zf(z)=0. \tag{7}$$

Function $f(z)$ satisfies the following identities

$$\int\limits_{-\infty}^{\infty} f(z)dz=\pi,\quad \int\limits_{-\infty}^{0} f(z)dz=\frac{\pi}{3},\quad \int\limits_{0}^{\infty} f(z)dz=\frac{2\pi}{3}. \tag{8}$$

2. Main results

Theorem 1. *Let $\varphi(x_1,\ldots,x_n)$ be a piece-wise continuous function with a compact support $D_{(a_i,b_i)}=\{x_i: a_i\leqslant x_i\leqslant b_i\}$, $i=\overline{1,n}$, $D_{(a_i,b_i)}\subset\mathbb{R}^n$ and having an bounded variation. Then the function*

$$u(x_1,\ldots,x_n,t)=\frac{1}{\pi^n}\int\limits_{\mathbb{R}^n} U(x_1-\xi_1,\ldots,x_n-\xi_n;t)\varphi(\xi_1,\ldots,\xi_n)d\xi_1\ldots d\xi_n$$

satisfies equation (1) as $t>0$ and for each $x_i^0\in(a_i,b_i)$

$$\lim_{t\to+0} u(x_1^0,\ldots,x_n^0,t)=\frac{2}{3}\varphi(x_1^0-0,\ldots,x_n^0-0)+\frac{1}{3}\varphi(x_1^0+0,\ldots,x_n^0+0).$$

The validity of the first part of the theorem follows immediately from the properties of the fundamental solution to equation (1) and the hypothesis of the theorem. The second part is proven separately w.r.t. each spatial variable. Since the proof of this part of the theorem is similar to work [16] and makes no essential troubles, we do not dwell on it.

Theorem 2. *Let function $\varphi(x_1,\ldots,x_n)$ be continuous and have a bounded variation on each bounded domain $D_{(a_i,b_i)}=\{x_i: a_i\leqslant x_i\leqslant b_i\}$, $i=\overline{1,n}$, $D_{(a_i,b_i)}\subset\mathbb{R}^n$, and the variation of the function*

$$P(y)=y^{\frac{3}{4}+\delta_1}\psi(y) \tag{9}$$

be bounded as $y<a_0$ for each $a_0=const$. Moreover, let

$$\varphi(x_1,\ldots,x_n)\sim\prod_j\psi(x_j)\exp\left\{const\sum_{i\neq j}|x_i|^{\frac{3}{2}-\delta_2}\right\}$$

as $x_i \to \infty$, $x_j < a_j$, $j = \overline{1,n}$, $i + j = n$; *and*

$$\varphi(x_1, \dots, x_n) \sim \prod_j \psi(x_j)$$

as $x_j < a_j$, *where* δ_1, δ_2 *are positive numbers. Then the function*

$$u(x_1, \dots, x_n, t) = \frac{1}{\pi^n} \int\limits_{\mathbb{R}^n} U(x_1 - \xi_1, \dots, x_n - \xi_n; t)\varphi(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n \tag{10}$$

satisfies equation (1) as $t > 0$ *and the condition*

$$\lim_{(x_1,\dots,x_n,t)\to(x_1^0,\dots,x_n^0,+0)} u(x_1, \dots, x_n, t) = \varphi(x_1^0, \dots, x_n^0). \tag{11}$$

Proof. We begin by proving the first part of the theorem. We differentiate expression (10) w.r.t. x_j to obtain

$$\pi^n \frac{\partial^3 u}{\partial x_j^3} = \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} \frac{\partial^3 U(x_1 - \xi_1, \dots, x_n - \xi_n; t)}{\partial x_j^3} \varphi(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n$$
$$= \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} \frac{\partial^3 U(\xi_1, \dots, \xi_n; t)}{\partial x_j^3} \varphi(x_1 - \xi_1, \dots, x_n - \xi_n) d\xi_1 \dots d\xi_n.$$

On the other hand,

$$\frac{\partial^3 U(x_1, \dots, x_n; t)}{\partial x_j^3} = -\frac{t^{-1}}{3} \left\{ U + x_j U_{x_j} \right\}.$$

While calculating the derivates we have made use of the identity (7). We get

$$\pi^n t \frac{\partial^3 u}{\partial x_j^3} = -\frac{1}{3} \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} f(z_1) \dots f(z_n) \varphi\left(\xi_1 - z_1 t^{\frac{1}{3}}, \dots, \xi_n - z_n t^{\frac{1}{3}}\right) dz_1 \dots dz_n$$
$$- \frac{1}{3} \int\limits_{-\infty}^{\infty} \dots \int\limits_{-\infty}^{\infty} f(z_1) \dots f(z_{j-1}) f(z_{j+1}) \dots f(z_n) dz_1 \dots dz_{j-1} dz_{j+1} \dots dz_n \tag{12}$$
$$\cdot \int\limits_{-\infty}^{\infty} z f'(z_j) \varphi\left(\xi_1 - z_1 t^{\frac{1}{3}}, \dots, \xi_n - z_n t^{\frac{1}{3}}\right) dz_j = -\frac{1}{3}\{u_{j1} + u_{j2}\}, \;\; z_j = \frac{x_j - \xi_j}{t^{\frac{1}{3}}}.$$

Let us prove that under the hypothesis of the theorem the integral in the right hand side of (12) obtained by the formal differentiation converges. Let us study the convergence of integral (12) as $j = 1$; other cases can be studied in the same way. Airy function satisfies the identities [16]:

$$f(z) \sim \begin{cases} |z|^{-\frac{1}{4}} \exp\left(-\frac{2}{3}|z|^{\frac{3}{2}}\right) \left(\sqrt{\pi} + O\left(|z|^{-\frac{3}{2}}\right)\right), \\ |z|^{\frac{1}{4}} \exp\left(-\frac{2}{3}|z|^{\frac{3}{2}}\right) \left(\sqrt{\pi} + O\left(|z|^{-\frac{3}{2}}\right)\right), \end{cases} \tag{13}$$

for sufficiently large negative z;

$$f(z) \sim \begin{cases} z^{-\frac{1}{4}} \cos\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right) \left(\sqrt{\pi} + O\left(|z|^{-\frac{3}{2}}\right)\right), \\ z^{\frac{1}{4}} \sin\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right) \left(\sqrt{\pi} + O\left(|z|^{-\frac{3}{2}}\right)\right), \end{cases} \tag{14}$$

for sufficiently large positive z.

Let $j = 1$, $t \geqslant t_0 > 0$, $a_i \leqslant x_i \leqslant b_i$, $i = 2, 3, \ldots, n$.

We first consider the second term in the right hand side of (12). Then by (12) we have

$$u_{12}(x_1, x_2, \ldots, x_n, t) = \left\{ \int\limits_{-\infty}^{-r_2} + \int\limits_{-r_2}^{r_2} + \int\limits_{r_2}^{\infty} \right\} f(z_2) dz_2 \ldots \left\{ \int\limits_{-\infty}^{-r_n} + \int\limits_{-r_n}^{r_n} + \int\limits_{r_n}^{\infty} \right\} f(z_n) dz_n$$
$$\cdot \left\{ \int\limits_{-\infty}^{-r_1} + \int\limits_{-r_1}^{r_1} + \int\limits_{r_1}^{\infty} \right\} z_1 f'(z_1) \varphi \left(x_1 - z_1 t^{\frac{1}{3}}, \ldots, x_n - z_n t^{\frac{1}{3}} \right) dz_1 \tag{15}$$
$$= \left\{ \int\limits_{-\infty}^{-r_2} + \int\limits_{-r_2}^{r_2} + \int\limits_{r_2}^{\infty} \right\} f(z_2) dz_2 \ldots \left\{ \int\limits_{-\infty}^{-r_n} + \int\limits_{-r_n}^{r_n} + \int\limits_{r_n}^{\infty} \right\} [J_1 + J_2 + J_3] f(z_n) dz_n,$$

where r_j are sufficiently large positive numbers. First we consider the integrals involving the expression $J_1(x_1, \ldots, x_n; z_2, \ldots, z_n; t)$ for sufficiently large positive r_1.

Let $z_\in[-\infty; -r_i, \ \ i = \overline{2, n}]$. Then by condition (13) we have

$$\int\limits_{-\infty}^{-r_2} f(z_2) dz_2 \ldots \int\limits_{-\infty}^{-r_n} J_1(x_1, \ldots, x_n; z_2, \ldots, z_n; t) f(z_n) dz_n$$
$$= \int\limits_{-\infty}^{-r_2} f(z_2) dz_2 \ldots \int\limits_{-\infty}^{-r_n} f(z_n) dz_n \int\limits_{-\infty}^{-r_1} z_1 f'(z_1) \varphi \left(x_1 - z_1 t^{\frac{1}{3}}, \ldots, x_n - z_n t^{\frac{1}{3}} \right) dz_1$$
$$\sim O \left(\int\limits_{r_2}^{\infty} z_2^{-\frac{1}{4}} \exp \left[-z_2^{\frac{3}{2}} \left(\frac{2}{3} - C z_2^{-\delta_2} \left(\frac{x_2}{z_2} + t^{\frac{1}{3}} \right)^{\frac{3}{2} - \delta_2} \right) \right] dz_2 \right) \cdot \ldots$$
$$\cdot O \left(\int\limits_{r_n}^{\infty} z_n^{-\frac{1}{4}} \exp \left[-z_n^{\frac{3}{2}} \left(\frac{2}{3} - C z_n^{-\delta_2} \left(\frac{x_n}{z_n} + t^{\frac{1}{3}} \right)^{\frac{3}{2} - \delta_2} \right) \right] dz_n \right)$$
$$\cdot O \left(\int\limits_{r_1}^{\infty} z_1^{-\frac{1}{4}} \exp \left[-z_1^{\frac{3}{2}} \left(\frac{2}{3} - C z_1^{-\delta_2} \left(\frac{x_1}{z_1} + t^{\frac{1}{3}} \right)^{\frac{3}{2} - \delta_2} \right) \right] dz_1 \right).$$

We see that this integral converges uniformly to zero as $r_j \to \infty$.

Suppose now that $z_2 \in [r_2, \infty]$, $z_k \in [-\infty, -r_k]$, $k = \overline{3, n}$. Then by (13) and the hypothesis of the theorem we have

$$\int\limits_{r_2}^{\infty} f(z_2) dz_2 \int\limits_{-\infty}^{-r_3} f(z_3) dz_3 \ldots \int\limits_{-\infty}^{-r_n} J_1(x_1, \ldots, x_n; z_2, \ldots, z_n; t) f(z_n) dz_n =$$
$$= \int\limits_{r_2}^{\infty} f(z_2) dz_2 \int\limits_{-\infty}^{-r_3} f(z_3) dz_3 \ldots \int\limits_{-\infty}^{-r_n} f(z_n) dz_n \int\limits_{-\infty}^{-r_1} z_1 f'(z_1) \varphi \left(x_1 - z_1 t^{\frac{1}{3}}, \ldots, x_n - z_n t^{\frac{1}{3}} \right) dz_1$$
$$\sim O \left(\int\limits_{r_1}^{\infty} z_1^{\frac{5}{4}} \exp \left[-z_1^{\frac{3}{2}} \left(\frac{2}{3} - C z_1^{-\delta_2} \left(\frac{x_1}{z_1} + t^{\frac{1}{3}} \right)^{\frac{3}{2} - \delta_2} \right) \right] dz_1 \right) \cdot \ldots$$

$$\cdot O\left(\int\limits_{r_n}^{\infty} z_n^{-\frac{1}{4}} \exp\left[-z_n^{\frac{3}{2}}\left(\frac{2}{3} - C z_n^{-\delta_2}\left(\frac{x_n}{z_n} + t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)\right] dz_n\right) \int\limits_{r_2}^{\infty} \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) f(z_2) dz_2$$
$$= J_{11}(x_1,t)\dots J_{1n}(x_n,t) J_{12}(x_2,t).$$

The convergence of integrals $J_{11}(x_1,t)$, $\dots$, $J_{1n}(x_n,t)$ to zero as $r_j \to \infty$, $j = 1,3,\dots,n$, is obvious. We consider integral $J_{12}(x_2,t)$:

$$\int\limits_{r_2}^{\infty} \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) f(z_2) dz_2 \sim \int\limits_{r_2}^{\infty} z_2^{-\frac{1}{4}} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2$$
$$+ O\left(\int\limits_{r_2}^{\infty} z_2^{-\frac{7}{4}} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2\right).$$

Here

$$\left|\int\limits_{r_2}^{\infty} z_2^{-\frac{1}{4}} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2\right|$$
$$= \left|\int\limits_{r_2}^{\infty} z_2^{-1-\delta_1} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \left|\frac{x_2}{z_2} - t^{\frac{1}{3}}\right|^{-\frac{3}{4}-\delta_1} \left|x_2 - z_2 t^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta_1} \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2\right|$$
$$\leqslant M \int\limits_{r_2}^{\infty} z_2^{-1-\delta_1} dz_2 = \left(\frac{M}{\delta_1}\right) r_2^{-\delta_1},$$
$$\left|\int\limits_{r_2}^{\infty} z_2^{-\frac{7}{4}} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2\right|$$
$$= \left|\int\limits_{r_2}^{\infty} z_2^{-\frac{5}{2}-\delta_1} \cos\left(\frac{2}{3} z_2^{\frac{3}{2}} - \frac{\pi}{4}\right) \left|\frac{x_2}{z_2} - t^{\frac{1}{3}}\right|^{-\frac{3}{4}-\delta_1} \left|x_2 - z_2 t^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta_1} \psi\left(x_2 - z_2 t^{\frac{1}{3}}\right) dz_2\right|$$
$$\leqslant M \int\limits_{r_2}^{\infty} z_2^{-\frac{5}{2}-\delta_1} dz_2 = \left(\frac{M}{\delta_1}\right) r_2^{-\frac{3}{2}-\delta_1}.$$

This is why integral $J_{12}(x_2,t)$ converges to zero as $r_2 \to \infty$. In the same we prove the convergence of the other integrals involving the expression $J_1(x_1,\dots,x_n,z_2,\dots,z_n,t)$.

In what follows we shall make use of the following theorem.

Theorem 3 ([7]). *Suppose that the variation of a function $P(x)$ is bounded on an interval (a,b) and*

$$\left|\int\limits_a^b Q(x) dx\right| < M.$$

Then

$$\left|\int\limits_a^b Q(x) P(x) dx\right| < M\left\{|P(x)| + V_a^b(P(x))\right\},$$

where V_a^b is the variation of a function on interval (a,b).

We proceed to the integrals involving the expression $J_3(x_1,\dots,x_n,z_2,\dots,z_n,t)$ for sufficiently large positive r_1.

Let $z_i' \in [-\infty,-r_i']$. Then by (13) the hypothesis of the theorem we have

$$\int\limits_{-\infty}^{-r_2} f(z_2)dz_2\dots\int\limits_{-\infty}^{-r_n} J_3(x_1,\dots,x_n,z_2,\dots,z_n,t)dz_n$$

$$=\int\limits_{-\infty}^{-r_2} f(z_2)dz_2\dots\int\limits_{-\infty}^{-r_n} f(z_n)dz_n\int\limits_{r_1}^{\infty} z_1f'(z_1)\varphi\left(x_1-z_1t^{\frac{1}{3}},\dots,x_n-z_nt^{\frac{1}{3}}\right)dz_1$$

$$\sim O\left(\int\limits_{r_2}^{\infty} z_2^{-\frac{1}{4}}\exp\left[-z_2^{\frac{3}{2}}\left(\frac{2}{3}-Cz_2^{-\delta_2}\left(\frac{x_2}{z_2}+t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)\right]dz_2\right)\cdot\dots$$

$$\cdot O\left(\int\limits_{r_n}^{\infty} z_n^{-\frac{1}{4}}\exp\left[-z_n^{\frac{3}{2}}\left(\frac{2}{3}-Cz_n^{-\delta_2}\left(\frac{x_n}{z_n}+t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)\right]dz_n\right)$$

$$\cdot\int\limits_{r_1}^{\infty}\psi\left(x_1-z_1t^{\frac{1}{3}}\right)z_1f'(z_1)dz_1=J_{32}(x_2,t)\dots J_{3n}(x_n,t)J_{31}(x_1,t).$$

It is obvious that as $r_i'\to\infty$, integral $J_{32}(x_2,t),\dots,J_{3n}(x_n,t)$ converges to zero. We consider integral $J_{31}(x_1,t)$:

$$\int\limits_{r_1}^{\infty}\psi(x_1-z_1t^{\frac{1}{3}})z_1f'(z_1)dz_1$$

$$\sim\int\limits_{r_1}^{\infty} z_1^{\frac{5}{4}}\sin\left(\frac{2}{3}z_1^{\frac{3}{2}}-\frac{\pi}{4}\right)\psi\left(x_1-z_1t^{\frac{1}{3}}\right)dz_1+O\left(\int\limits_{r_1}^{\infty} z_1^{-\frac{1}{4}}\sin\left(\frac{2}{3}z_2^{\frac{3}{2}}-\frac{\pi}{4}\right)\psi\left(x_1-z_1t^{\frac{1}{3}}\right)dz_1\right).$$

It is clear that as $r_1\to\infty$, the second integral in the right hand side of this relation converges to zero. This is why it is sufficient to study just the first integral in the right hand side:

$$\int\limits_{r_1}^{\infty} z_1^{\frac{5}{4}}\sin\left(\frac{2}{3}z_1^{\frac{3}{2}}-\frac{\pi}{4}\right)\psi\left(x_1-z_1t^{\frac{1}{3}}\right)dz_1$$

$$=\int\limits_{r_1}^{\infty} z_1^{\frac{1}{2}-\delta_1}\sin\left(\frac{2}{3}z_1^{\frac{3}{2}}-\frac{\pi}{4}\right)\left(\frac{x_1}{z_1}-t^{\frac{1}{3}}\right)^{-\frac{3}{4}-\delta_1}\left(x_1-z_1t^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta_1}\psi\left(x_1-z_1t^{\frac{1}{3}}\right)dz_1$$

$$=\frac{2}{3}\int\limits_{\rho}^{\infty}\upsilon^{-\frac{2}{3}\delta_1}\sin\left(\frac{2}{3}\upsilon-\frac{\pi}{4}\right)\mu(\upsilon)\left\{\left(x_1-\upsilon^{\frac{2}{3}}t^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta_1}\psi\left(x_1-\upsilon^{\frac{2}{3}}t^{\frac{1}{3}}\right)\right\}d\upsilon,$$

where $\rho=r_1^{\frac{3}{2}}$, $\mu(\upsilon)=\left(\frac{x_1}{\upsilon^{\frac{2}{3}}}-t^{\frac{1}{3}}\right)^{-\frac{3}{4}-\delta_1}$. For sufficiently large positive r_1, the absolute value of this integral is bounded by

$$\frac{2}{3}\left\{\left|x_1-r_1t^{\frac{1}{3}}\right|^{\frac{3}{4}+\delta_1}\left|\psi\left(x_1-r_1t^{\frac{1}{3}}\right)\right|+V\left[\left(x_1-\upsilon^{\frac{2}{3}}t^{\frac{1}{3}}\right)^{\frac{3}{4}+\delta_1}\psi\left(x_1-\upsilon^{\frac{2}{3}}t^{\frac{1}{3}}\right);\upsilon\geqslant r_1^{\frac{3}{2}}\right]\right\}$$

$$\cdot \sup\left\{\left|\int\limits_m^n \upsilon^{-\frac{2}{3}\delta_1}\sin\left(\frac{2}{3}\upsilon-\frac{\pi}{4}\right)d\upsilon\right|\right\},$$

where $r_1^{\frac{3}{2}} \leqslant m < n$.

The existence of the integrals (see [6])

$$\int\limits_0^\infty x^p\sin(ax+b)dx = a^{\frac{1}{p+1}}\Gamma(1+p)\cos\left(b+\frac{p\pi}{2}\right), \qquad a>0, \quad -1<p<0,$$

$$\int\limits_0^\infty x^p\cos(ax+b)dx = -a^{\frac{1}{p+1}}\Gamma(1+p)\sin\left(b+\frac{p\pi}{2}\right), \qquad a<0, \quad -1<p<0$$

means that the expression under the sup converges to zero as $r_1 \to \infty$. Therefore, integral $J_{31}(x_1,t)$ converges to zero uniformly as $r_1 \to \infty$.

Suppose that $z_2 \in [r_2,\infty)$, $z_k \in (-\infty,-r_k]$, $k=\overline{3,n}$. Then by (13) and the hypothesis of the theorem we have

$$\int\limits_{r_2}^\infty f(z_2)dz_2\int\limits_{-\infty}^{-r_3} f(z_3)dz_3\ldots\int\limits_{-\infty}^{-r_n} J_3(x_1,\ldots,x_n,z_2,\ldots,z_n,t)dz_n$$

$$=\int\limits_{r_2}^\infty f(z_2)dz_2\ldots\int\limits_{-\infty}^{-r_n} f(z_n)dz_n\int\limits_{r_1}^\infty z_1f'(z_1)\varphi\left(x_1-z_1t^{\frac{1}{3}},\ldots,x_n-z_nt^{\frac{1}{3}}\right)dz_1$$

$$\sim O\left(\int\limits_{r_3}^\infty z_3^{-\frac{1}{4}}\exp\left[-z_3^{\frac{3}{2}}\left(\frac{2}{3}-Cz_3^{-\delta_2}\left(\frac{x_3}{z_3}+t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)\right]dz_3\right)\cdot\ldots$$

$$\cdot O\left(\int\limits_{r_n}^\infty z_n^{-\frac{1}{4}}\exp\left[-z_n^{\frac{3}{2}}\left(\frac{2}{3}-Cz_n^{-\delta_2}\left(\frac{x_n}{z_n}+t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)\right]dz_n\right)$$

$$\cdot\int\limits_{r_2}^\infty \psi\left(x_2-z_2t^{\frac{1}{3}}\right)f(z_2)dz_2\int\limits_{r_1}^\infty\psi\left(x_1-z_1t^{\frac{1}{3}}\right)z_1f'(z_1)dz_1$$

$$=J_{33}(x_2,t)\ldots J_{3n}(x_n,t)J_{32}(x_2,t)J_{31}(x_1,t).$$

The convergence of integrals $J_{33}(x_2,t)$,..., $J_{3n}(x_n,t)$, $J_{32}(x_2,t)$, $J_{31}(x_1,t)$ follows from (16) and Theorem 3.

In the same we prove the convergence of the integrals involving the expression $J_3(x_1,\ldots,x_n,z_1,\ldots,z_n,t)$.

Thus, we have proven that integral (12) converges uniformly in $D_{(a_i,b_i)}$. Therefore, by the arbitrariness of a_i, b_i, and t_0 integral (12) converges uniformly in D.

Let us prove identity (11). We consider functions $\bar\varphi(x_1,\ldots,x_n)$ with a compact support. We assume that $a_i+1\leqslant x_i^0\leqslant b_i-1$. We let $\bar\varphi(x_1,\ldots,x_n)=\Phi(a_i,b_i)\varphi(x_1,\ldots,x_n)$, where

$$\Phi(a_i,b_i)=\begin{cases}1, & \text{if} \quad x_i\in D_{(a_i,b_i)},\\ 0, & \text{if} \quad x_i\notin D_{(a_i,b_i)}.\end{cases}$$

Let

$$\bar u(x_1,\ldots,x_n,t)=\frac{1}{\pi^n}\int\limits_{\mathbb{R}^n}U(x_1-\xi_1,\ldots,x_n-\xi_n;t)\bar\varphi(\xi_1,\ldots,\xi_n)d\xi_1\ldots d\xi_n.$$

We consider the difference

$$\upsilon(x_1,\dots,x_n,t)=u(x_1,\dots,x_n,t)-\bar{u}(x_1,\dots,x_n,t),\ \ a_i+1\leqslant x_i\leqslant b_i-1.$$

We have

$$\begin{aligned}\upsilon(x_1,\dots,x_n,t)=&\int\limits_{-\infty}^{-k_1} f(z_1)dz_1\dots\int\limits_{-\infty}^{-k_n} f(z_n)\varphi(x_1-z_1t^{\frac{1}{3}},\dots,x_n-z_nt^{\frac{1}{3}})dz_n\\ &+\int\limits_{h_1}^{\infty} f(z_1)dz_1\dots\int\limits_{h_n}^{\infty} f(z_n)\varphi(x_1-z_1t^{\frac{1}{3}},\dots,x_n-z_nt^{\frac{1}{3}})dz_n\\ =&\upsilon_1(x_1,\dots,x_n,t)+\upsilon_2(x_1,\dots,x_n,t),\end{aligned}\tag{16}$$

where $k_i=(b_i-x_i)t^{-\frac{1}{3}}$, $h_i=(x_i-a_i)t^{-\frac{1}{3}}$.

By identities (13) and the hypothesis of the theorem for sufficiently large k_i we obtain

$$\begin{aligned}\upsilon_1(x_1,\dots,x_n,t)=&O\left(\int\limits_{k_1}^{\infty} z_1^{-\frac{1}{4}}\exp\left(-\frac{2}{3}z_1^{\frac{3}{2}}+C\left(x_1-z_1t^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)dz_1\right)\cdot\dots\\ &\cdot O\left(\int\limits_{k_1}^{\infty} z_n^{-\frac{1}{4}}\exp\left(-\frac{2}{3}z_n^{\frac{3}{2}}+C\left(x_n-z_nt^{\frac{1}{3}}\right)^{\frac{3}{2}-\delta_2}\right)dz_n\right).\end{aligned}$$

Hence, $\upsilon_1(x_1,\dots,x_n,t)$ tends to zero as $t\to+0$, $k_i\to\infty$.

By Theorem 3, the second integral in (16) can be estimated as

$$\begin{aligned}|\upsilon_2(x_1,\dots,x_n,t)|\leqslant\Big\{&\left|\varphi(x_1-h_1t^{\frac{1}{3}}\dots x_n-h_nt^{\frac{1}{3}})\right|\\ &+V\left(\varphi(x_1-z_1t^{\frac{1}{3}}\dots x_n-z_nt^{\frac{1}{3}});z_i\geqslant h_i\right)\Big\}A(\alpha_i,\beta_i),\end{aligned}$$

where

$$A(\alpha_i,\beta_i)=\sup\left\{\left|\int\limits_{\alpha_1}^{\beta_1} f(z_1)dz_1\dots\int\limits_{\alpha_n}^{\beta_n} f(z_n)dz_n\right|:\ \ h_i\leqslant\alpha_i\leqslant\beta_i\right\}.$$

Under the hypothesis of the theorem the first factor is bounded. Let us study $A(\alpha_i,\beta_i)$ for sufficiently large $h_i\leqslant\alpha_i\leqslant\beta_i$. We have

$$\int\limits_{\alpha_1}^{\beta_1} f(z_1)dz_1\dots\int\limits_{\alpha_n}^{\beta_n} f(z_n)dz_n.$$

Let us estimate the first integral; other integrals can be estimates in the same way:

$$\begin{aligned}\int\limits_{\alpha_1}^{\beta_1} f(z_1)dz_1\sim&\int\limits_{\alpha_1}^{\beta_1} z_1^{-\frac{1}{4}}\cos\left(\frac{2}{3}z_1^{\frac{3}{2}}-\frac{\pi}{4}\right)\left(\sqrt{\pi}+O\left(z_1^{-\frac{3}{2}}\right)\right)dz_1\sim\\ &\sim\int\limits_{\gamma}^{\tau}\nu^{-\frac{1}{2}}\cos\left(\frac{2}{3}\nu-\frac{\pi}{4}\right)\left(\sqrt{\pi}+O\left(\nu^{-1}\right)\right)d\nu,\end{aligned}\tag{17}$$

where $\gamma=\alpha_1^{\frac{3}{2}}$, $\tau=\beta_1^{\frac{3}{2}}$.

The first term in (17) can be estimated as follows:

$$\left|\int_{\gamma}^{\tau} \nu^{-\frac{1}{2}} \cos\left(\frac{2}{3}\nu - \frac{\pi}{4}\right) d\nu\right| \leqslant C \left|\nu^{-\frac{1}{2}} \sin\left(\frac{2}{3}\nu - \frac{\pi}{4}\right)\right|_{\nu=\gamma}^{\nu=\tau}$$

$$+ C\left|\int_{\gamma}^{\tau} \nu^{-\frac{3}{2}} \sin\left(\frac{2}{3}\nu - \frac{\pi}{4}\right)\right| \leqslant \gamma^{-\frac{1}{2}} + \tau^{-\frac{1}{2}} + C\left(\gamma^{-\frac{1}{2}} - \tau^{-\frac{1}{2}}\right).$$

The second term is estimated by the expression $2\left(\gamma^{-\frac{1}{2}} - \tau^{-\frac{1}{2}}\right)$. We finally have

$$\sup\left|\int_{\alpha_1}^{\beta_1} f(z_1)dz_1\right| \leqslant C\left(\beta_1^{-\frac{3}{4}} + \alpha_1^{-\frac{3}{4}}\right).$$

Thus, as $t \to +0$, $h_i \to \infty$, integral (17) uniformly converges to zero. □

By means of the above theorem we can study the character of the growth for solutions $u(x_1, \ldots, x_n, t)$ to the problem. For the sake of simplicity we study the such growth w.r.t. variable x_1.

By (10), for sufficiently large positive numbers r_1 we get

$$\begin{aligned} u(x_1, \ldots, x_n, t) &= \frac{1}{\pi^n}\int_{\mathbb{R}^n} U(x_1 - \xi_1, \ldots, x_n - \xi_n; t)\varphi(\xi_1, \ldots, \xi_n) d\xi_1 \ldots d\xi_n \\ &= \frac{1}{\pi^n}\int_{\mathbb{R}^n} U(\xi_1, \ldots, \xi_n; t)\varphi(\xi_1 - \xi_1, \ldots, \xi_n - \xi_n) d\xi_1 \ldots d\xi_n \\ &= \int_{-\infty}^{\infty} f(z_2)dz_2 \ldots \int_{-\infty}^{\infty} f(z_n)dz_n \left\{\int_{-\infty}^{-r_1} + \int_{-r_1}^{r_1} + \int_{r_1}^{\infty}\right\} f(z_1)\varphi(x_1 - z_1 t^{\frac{1}{3}}, \ldots, x_n - z_n t^{\frac{1}{3}}) dz_1 \\ &= u_1(x_1, \ldots, x_n, t) + u_2(x_1, \ldots, x_n, t) + u_3(x_1, \ldots, x_n, t). \end{aligned}$$

In view of (13), (14) and the hypothesis of Theorem 2 we have

$$|u_1(x_1, \ldots, x_n, t)| \leqslant K \exp\left\{|x_1|^{\frac{3}{2} - \delta_2}\right\}, \tag{18}$$

$$|u_3(x_1, \ldots, x_n, t)| \leqslant M |x_1|^{-\frac{3}{4} - \delta_1}. \tag{19}$$

Since $u_2(x_1, \ldots, x_n, t)$ is a bounded function, estimates (18), (19) follow that the solution to the Cauchy problem can grows exponentially at the infinity the growth rate does not exceed $\exp\left\{|x_1|^{\frac{3}{2} - \delta_2}\right\}$, where $\delta_2 > 0$.

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Abdukomil Risbekovich Khashimov,
Tashkent Finance Institute,
A. Temur str., 60A,
100000, Tashkent, Uzbekistan
E-mail: `abdukomil@yandex.ru`

Sobitkhon Yakubov,
Tashkent Finance Institute,
A. Temur str., 60A,
100000, Tashkent, Uzbekistan
E-mail: `abdukomil@yandex.ru`