BOUNDARY VALUE PROBLEMS FOR EQUATIONS OF VISCOUS HEAT-CONDUCTING GAS IN TIME-INCREASING NON-CYLINDRICAL DOMAINS

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Abstract. In this paper we prove global solvability of the initial-boundary value problems for the complete system of equations describing one-dimensional non-stationary flow of the viscous heat-conducting gas in time-increasing non-cylindrical domains. Local existence and uniqueness of these problems are proved in earlier articles by Kazhikhov A.V. and Kaliev I.A. This is why, the proof of the global in time existence and uniqueness theorem is connected with obtaining a priori estimates, in which the constant depend only on the data of the problem and the value of the time interval T, but do not depend on the period of existence of a local solution. The study is made in terms of Eulerian variables.

Keywords: Navier-Stokes equations system, heat-conducting gas, global solvability, time-increasing non-cylindrical domains.

Mathematics Subject Classification: 35Q30, 76D05,76N10

1. INTRODUCTION

The complete system of equations of viscous heat-conduction gas motion, or Navier-Stokes system of equations is an interesting and important class of partial differential equations. In the theory of such systems, one of the central problems is the unique solvability "in general" both in time and the data.

The study of issues on well-definiteness of initial-boundary value problems for Navier-Stokes system of equations began with the work by J. Serrin in 1959 [1]. There were formulation the main boundary value problems and proven uniqueness theorems in the class of smooth functions. We also mention an earlier paper by D. Graffi in 1953 [2] on the uniqueness of classical solutions for a barotropic gas.

The first result on solvability of Navier-Stokes equations was obtained by J. Nash in 1962 [3]. He showed the existence of classical solution to Cauchy problem for small times. By other methods, this result was reproduced and generalized in works by N. Itaya [4], A.I. Volpert and S.I. Khudyaev [5].

For initial boundary value problems theorem on local in time existence and uniqueness were proven by V.A. Solonnikov [6] and A. Tani [7].

The first result on the unique global solvability in time and in data was established by Ya.I. Kanel in 1968 [8] for the Cauchy problem for the equations of one-dimensional viscous barotropic gas ($p = R\rho^\gamma$). For the Bürgers model ($p = const$) the solvability of Cauchy problem and initial boundary value problems was proven in works by N. Itaya [9], [10] and A. Tani [11].

In 1976, A.V. Kazhikhov [12] first obtained the result on global solvability for the equations of one-dimensional viscous heat-conducting gas. In what follows in a series of works by A.V. Kazhikhov [13]–[16], V.V. Shelukhin [17]–[19], S.Ya. Belov [20], V.A. Vaigant [21], [22] there was constructed a rather complete theory on global solvability of main initial boundary value problems and Cauchy problem for equations of viscous gas motion.

In works by I.A. Kaliev and A.V. Kazhikhov [23], [24] there were studied issues on the unique solvability of a problem with a free boundary modeling the process of phase transition between a viscous gas and a solid state. At that, there appears an auxiliary problem describing the motion of a viscous heat-conducting gas in a curvilinear domain and the existence and the unique solvability of local solution was proven.

As a rule, the domain in which one proves the global existence in time, is either the strip $\{(x,t)\,|-\infty < x < \infty, 0 < t < T\}$ or a cylinder $\{(x,t)\,|a < x < b, 0 < t < T\}$; a, b, T are given constants. In our work, for the complete system of equations of one-dimensional non-stationary motion of a viscous heat-conducting gas, we prove the global solvability of initial boundary value problems in non-cylindrical domains $\{(x,t)\,|0 < x < s(t), 0 < t < T\}$ increasing with time; here $x = s(t)$ is a given smooth increasing function.

For a viscous gas, the results on global solvability are known for the problem with a free boundary on gas outflow into vacuum [12], [25] and for the problem on a piston moving by a prescribed law [25]. In both the problems the velocity of boundary $s(t)$ of domain occupied by the gas coincided with the velocity of the with the coordinates $s(t)$, i.e., $u(s(t), t) = ds(t)/dt$, $0 < t < T$. In other words, the gas did not flow through the boundary $s(t)$ and this fact played an essential role in the proof of the existence theorems since the domain of the solution becomes a fixed cylinder in Lagrangian coordinates.

In the present work $u(s(t), t) = 0, ds(t)/dt > 0$, i.e., $u(s(t), t) - ds(t)/dt < 0$, and the gas inflows via the moving boundary of the domain $x = s(t)$. We make the study in terms of Eulerian variables.

The case $ds(t)/dt \leqslant 0$ was considered in works by I.A. Kaliev and M.S. Pokuiko [26], [27].

2. FORMULATION OF THE PROBLEM AND THE MAIN RESULTS

Given a smooth function $x = s(t)$, we suppose that a non-cylindrical domain $\Omega_T = \{(x,t)\,|\,0 < x < s(t), 0 < t < T\}$, is occupied by a viscous heat-conducting gas. In the work we study the case when the domain widens in time, i.e., $ds(t)/dt > 0$. The one-dimensional non-stationary motion of a viscous heat-conducting gas in domain Ω_T is described by the system of equations [25]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \quad (x,t) \in \Omega_T, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \mu\frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x}, \quad p = R\rho\theta, \quad (x,t) \in \Omega_T, \tag{2}$$

$$\rho\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x}\right) = \kappa\frac{\partial^2 \theta}{\partial x^2} + \mu\left(\frac{\partial u}{\partial x}\right)^2 - p\frac{\partial u}{\partial x}, \quad (x,t) \in \Omega_T. \tag{3}$$

Here $\rho(x,t)$, $u(x,t)$, $p(x,t)$ and $\theta(x,t)$ are the density, velocity, pressure and absolute temperature of the gas; μ, R, κ are positive constants denoting the viscosity, gas constant, and heat conductivity factor, respectively.

At the initial time we specify u, θ, ρ:

$$u(x,t)\big|_{t=0} = u_0(x), \quad \theta(x,t)\big|_{t=0} = \theta_0(x), \quad \rho(x,t)\big|_{t=0} = \rho_0(x), \quad x \in [0, s_0], \tag{4}$$

where $s_0 = s(0)$. On the known boundaries $x = 0$ and $x = s(t)$ we impose the conditions

$$u(x,t)\big|_{x=0} = 0, \quad u(x,t)\big|_{x=s(t)} = 0, \quad t \in [0, T], \tag{5}$$

$$\theta(x,t)\big|_{x=0} = \theta_1(t), \quad \theta(x,t)\big|_{x=s(t)} = \theta_2(t), \quad t \in [0, T], \tag{6}$$

$$\rho(x,t)\big|_{x=s(t)} = \rho_2(t), \quad t \in [0, T]. \tag{7}$$

We assume that the inequalities

$$0 < m \leqslant \rho_0(x), \rho_2(t), \theta_0(x), \theta_1(t), \theta_2(t) \leqslant M < +\infty, \tag{8}$$

$$0 < s_0, \quad 0 < m \leqslant \frac{ds}{dt}(t) \leqslant M, \tag{9}$$

hold true for each $t \in [0, T]$ and $x \in [0, s_0]$, where m, M are some positive constants.

Problem Gas. Find functions $\rho(x,t)$, $u(x,t)$, $\theta(x,t)$ solving system of equations (1)–(3), if the initial conditions and boundary conditions (4)–(7) are satisfied.

Theorem 1. *Suppose that initial and boundary data in problem Gas belong to Hölder spaces*

$$\rho_0(x) \in C^{1+\alpha}([0, s_0]), \qquad u_0(x) \in C^{2+\alpha}([0, s_0]), \qquad \theta_0(x) \in C^{2+\alpha}([0, s_0]),$$

$$s(t), \rho_2(t) \in C^{1+\alpha}([0, T]), \qquad \theta_1(t), \theta_2(t) \in C^{(2+\alpha)/2}([0, T]),$$

$0 < \alpha = const < 1$; *conditions* (8), (9) *and conditions of zero order and first order matching at the points* $(0, 0)$, $(s_0, 0)$ *are satisfied. Then problem Gas has the unique classical solution possessing the properties*

$$\rho(x, t) \in C^{1+\alpha}(\overline{\Omega}_T), \quad u(x, t) \in C^{2+\alpha, (2+\alpha)/2}(\overline{\Omega}_T), \quad \theta(x, t) \in C^{2+\alpha, (2+\alpha)/2}(\overline{\Omega}_T),$$

at that,

$$0 < m_1 \leqslant \rho(x, t) \leqslant M_1 < +\infty, \quad 0 < m_2 \leqslant \theta(x, t) \leqslant M_2 < +\infty, \quad (x, t) \in \Omega_T; \qquad (10)$$

where m_1, M_1, m_2, M_2 *are some positive constants.*

Local existence and uniqueness theorem for problem Gas was proven in [23], [24]. This is why the proof of the above theorem is related with obtaining apriori estimates, where constants depend only on data of the problem and time interval T but are independent of the existence interval for the local solution.

3.   Auxiliary propositions and apriori estimates

We assume that $\rho(x, t) > 0$, $\theta(x, t) > 0$ (there is a local in time existence theorem with appropriate estimates) [23], [24].

Lemma 1. *For each* $t \in [0, T]$, *the estimates*

$$\int_0^{s(t)} \rho(x, t) dx = \int_0^{s_0} \rho_0(x) dx + \int_0^t \rho_2(\tau) \frac{ds(\tau)}{d\tau} \leqslant M_0,$$

where

$$M_0 = \int_0^{s_0} \rho_0(x) dx + \int_0^T \rho_2(\tau) \frac{ds(\tau)}{d\tau} d\tau.$$

Proof. Employing conditions (5), (7), integrate equation (1) w.r.t. x from 0 to $s(t)$:

$$\frac{d}{dt} \int_0^{s(t)} \rho(x, t) dx - \rho_2(t) \frac{ds}{dt} = 0.$$

Integrating w.r.t. t, we complete the proof:

$$\int_0^{s(t)} \rho(x, t) dx = \int_0^{s_0} \rho_0(x) dx + \int_0^t \rho_2(\tau) \frac{ds(\tau)}{d\tau} d\tau \leqslant M_0,$$

$$M_0 = \int_0^{s_0} \rho_0(x) dx + \int_0^T \rho_2(\tau) \frac{ds(\tau)}{d\tau} d\tau.$$

$\square$

In what follows, while obtaining estimates for functions ρ, u, θ in a domain occupied by a viscous gas, we employ the methods developed by B.A. Vaigant [21]. We observe that in [21] the domain occupied by gas is the rectangle $(0, 1) \times (0, T)$, while in our case the domain occupied by gas is the curvilinear trapezoid $\Omega_T = \{(x, t) | 0 < x < s(t), 0 < t < T\}$, where $x = s(t)$ is a given increasing function. Nevertheless, we succeed to prove all desired apriori estimates.

In Ω_T we introduce an auxiliary function $B(x, t)$ as follows:

$$\frac{\partial B}{\partial x} = \frac{1}{\mu} \rho u, \quad \frac{\partial B}{\partial t} = \frac{\partial u}{\partial x} - \frac{1}{\mu} R \rho \theta - \frac{1}{\mu} \rho u^2,$$

$$B\big|_{t=0} = B_0(x) = \frac{1}{\mu} \int_0^x \rho_0(\xi) u_0(\xi) d\xi, \quad 0 \leqslant x \leqslant s_0.$$

In [21] for function $B(x,t)$ there were obtained the identities

$$\frac{\partial}{\partial t}(B + \ln \rho) + u\frac{\partial}{\partial x}(B + \ln \rho) + \frac{1}{\mu}R\rho\theta = 0, \tag{11}$$

$$\frac{\partial}{\partial t}(\rho e^B) + u\frac{\partial}{\partial x}(\rho e^B) + \frac{1}{\mu}R\rho^2\theta e^B = 0, \tag{12}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}e^{-B}\right) + u\frac{\partial}{\partial x}\left(\frac{1}{\rho}e^{-B}\right) - \frac{1}{\mu}R\theta e^{-B} = 0. \tag{13}$$

Lemma 2. *There exists a constant C depending on boundary data and T such that for each $(x,t) \in \Omega_T$ the inequality*

$$|B(x,t)| \leqslant C\left(1 + \left(\int_0^{s(t)} \rho u^2 dx\right)^{1/2} + \int_0^t \int_0^{s(\tau)} \rho\theta dx d\tau + \int_0^t \int_0^{s(\tau)} \rho u^2 dx d\tau\right)$$

holds true.

Proof. We integrate function $\partial B/\partial t$ over the domain $\Omega_t = \{(x,\tau)|0 < x < s(\tau),\ 0 < \tau < t\}$. Sometimes it will be more convenient for us to write domain Ω_t as $\Omega_t = \{(x,\tau)|0 < x < s(t),\ h(x) < \tau < t\}$, where $h(x) = 0$ as $x \in [0,s_0]$ and $s(h(x)) = x$ as $s_0 < x \leqslant s(t)$. We have

$$\iint_{\Omega_t} B_t d\tau dx = \int_0^{s_0} B(x,\tau)\Big|_{\tau=0}^{\tau=t} dx + \int_{s_0}^{s(t)} B(x,\tau)\Big|_{\tau=h(x)}^{\tau=t} dx$$

$$= \int_0^{s_0} B(x,t)dx - \int_0^{s_0} B(x,0)dx + \int_{s_0}^{s(t)} B(x,t)dx - \int_{s_0}^{s(t)} B(x,h(x))dx$$

$$= \int_0^{s(t)} B(x,t)dx - \frac{1}{\mu}\int_0^{s_0}\int_0^x \rho_0(\xi)u_0(\xi)d\xi dx - \int_{s_0}^{s(t)} B(x,h(x))dx.$$

On the other hand,

$$\iint_{\Omega_t} B_t dx d\tau = \iint_{\Omega_t}\left(u_x - \frac{1}{\mu}R\rho\theta - \frac{1}{\mu}\rho u^2\right) dx d\tau$$

$$= -\frac{R}{\mu}\int_0^t\int_0^{s(\tau)} \rho\theta dx d\tau - \frac{1}{\mu}\int_0^t\int_0^{s(\tau)} \rho u^2 dx d\tau.$$

As a result we get

$$\int_0^{s(t)} B(x,t)dx = \frac{1}{\mu}\int_0^{s_0}\int_0^x \rho_0(\xi)u_0(\xi)d\xi dx + \int_{s_0}^{s(t)} B(x,h(x))dx$$

$$- \frac{R}{\mu}\int_0^t\int_0^{s(\tau)} \rho\theta dx d\tau - \frac{1}{\mu}\int_0^t\int_0^{s(\tau)} \rho u^2 dx d\tau.$$

By (8), (9) it implies the inequality

$$\left|\int_0^{s(t)} B(x,t)dx\right| \leqslant \frac{s_0^2 M}{2\mu}\max_{x\in[0,s_0]}|u_0(\xi)| + \frac{R}{\mu}\int_0^t\int_0^{s(\tau)} \rho\theta dx d\tau$$

$$+ \frac{1}{\mu}\int_0^t\int_0^{s(\tau)} \rho u^2 dx d\tau + \left|\int_{s_0}^{s(t)} B(x,h(x))dx\right|.$$

In the latter integral we change the variable $x = s(\tau)$, $dx = \frac{ds}{d\tau}d\tau$, $B(x,h(x)) = B(s(\tau),\tau)$ and we employ (9):

$$\left|\int_0^{s(t)} B(x,t)dx\right| \leqslant C + \frac{R}{\mu}\int_0^t\int_0^{s(\tau)} \rho\theta dx d\tau + \frac{1}{\mu}\int_0^t\int_0^{s(\tau)} \rho u^2 dx d\tau + M\int_0^t |B(s(\tau),\tau)|\, d\tau. \tag{14}$$

Hereinafter by C we denote constants depending on boundary data and T.

Since for each $t \in [0, T]$ there exists a point $x_0 = x_0(t) \in [0, s(t)]$ such that

$$B(x_0(t), t) = \frac{1}{s(t)} \int_0^{s(t)} B(x, t) dx,$$

we obtain

$$
\begin{aligned}
|B(x,t)| &\leqslant |B(x_0(t), t)| + \int_0^{s(t)} |B_x| dx \\
&\leqslant \frac{1}{s(t)} \left| \int_0^{s(t)} B(x,t) dx \right| + \frac{1}{\mu} \left(\int_0^{s(t)} \rho dx \right)^{1/2} \left(\int_0^{s(t)} \rho u^2 dx \right)^{1/2} \\
&\leqslant \frac{1}{s_0} \left| \int_0^{s(t)} B(x,t) dx \right| + \frac{1}{\mu} \sqrt{M_0} \left(\int_0^{s(t)} \rho u^2 dx \right)^{1/2},
\end{aligned}
\tag{15}
$$

M_0 is a constant in Lemma 1.

Employing Cauchy inequality, for $x = s(\tau)$ and $t = \tau$ by (15) we have

$$|B(s(\tau), \tau)| \leqslant \frac{1}{s_0} \left| \int_0^{s(\tau)} B(x, \tau) dx \right| + \frac{M_0}{4\mu^2} + \int_0^{s(\tau)} \rho u^2 dx. \tag{16}$$

We substitute (16) into (14):

$$\left| \int_0^{s(t)} B(x,x) \right| \leqslant C + \frac{R}{\mu} \int_0^t \int_0^{s(\tau)} \rho \theta dx d\tau + \left(\frac{1}{\mu} + M \right) \int_0^t \int_0^{s(\tau)} \rho u^2 dx d\tau + \frac{M}{s_0} \int_0^t \left| \int_0^{s(\tau)} B(x,\tau) dx \right| d\tau.$$

By Grönwall inequality for the function

$$y(t) = \left| \int_0^{s(t)} B(x,t) dx \right|$$

it implies the estimate

$$\left| \int_0^{s(t)} B(x,t) dx \right| \leqslant C \left(1 + \int_0^t \int_0^{s(\tau)} \rho u^2 dx d\tau + \int_0^t \int_0^{s(\tau)} \rho \theta dx d\tau \right).$$

Substituting this inequality in (15), we complete the proof:

$$|B(x,t)| \leqslant C \left[1 + \int_0^t \int_0^{s(\tau)} \rho u^2 dx d\tau + \int_0^t \int_0^{s(\tau)} \rho \theta dx d\tau + \left(\int_0^{s(t)} \rho u^2 dx \right)^{1/2} \right].$$

$\square$

Lemma 3. *There exists a constant C depending on boundary data and T such that for each $t \in [0, T]$ the inequality*

$$\int_0^{s(t)} (\rho \ln \rho - \rho + 1) dx + \frac{R}{\mu} \int_0^t \int_0^{s(\tau)} \rho^2 \theta dx d\tau \leqslant C(1 + \max_{(x,\tau) \in \Omega_t} |B(x,\tau)|). \tag{17}$$

holds true.

Proof. We multiply equation (11) by $\rho(x,t)$ and integrate w.r.t. x from 0 to $s(t)$. Then by equation (1) we get

$$\frac{d}{dt} \int_0^{s(t)} \rho(B + \ln \rho) dx - \frac{ds(t)}{dt} \rho_2(t) \left[B(s(t), t) + \ln \rho_2(t) \right]$$

$$+ \left[\rho u (B + \ln \rho) \right] \Big|_{x=0}^{x=s(t)} + \frac{R}{\mu} \int_0^{s(t)} \rho^2 \theta dx - \frac{d}{dt} \int_0^{s(t)} \rho dx + \frac{ds(t)}{dt} \rho_2(t) \pm \frac{d}{dt} \int_0^{s(t)} dx = 0.$$

By (5) and (7) it yields

$$\frac{d}{dt} \int_0^{s(t)} (\rho B + \rho \ln \rho - \rho + 1)\, dx + \frac{R}{\mu} \int_0^{s(t)} \rho^2 \theta dx$$
$$- \frac{ds(t)}{dt} \left\{ \rho_2(t)[B(s(t),t) + \ln \rho(s(t),t) - 1] + 1 \right\} = 0.$$

Integrating w.r.t. the time, we obtain

$$\int_0^{s(t)} (\rho \ln \rho - \rho + 1)dx + \frac{R}{\mu} \int_0^t \int_0^{s(\tau)} \rho^2 \theta dx d\tau$$
$$= \int_0^t \frac{ds(\tau)}{d\tau} [\rho_2(\tau) \ln \rho_2(\tau) - \rho_2(\tau) + 1] d\tau - \int_0^{s(t)} \rho B dx$$
$$+ \int_0^{s_0} (\rho_0 B_0 + \rho_0 \ln \rho_0 - \rho_0 + 1)dx + \int_0^t \frac{ds(\tau)}{d\tau} \rho_2(\tau) B(s(\tau),\tau)d\tau.$$

We denote

$$C = \int_0^T \frac{ds(\tau)}{d\tau} [\rho_2(\tau) \ln \rho_2(\tau) - \rho_2(\tau) + 1] d\tau + \int_0^{s_0} (\rho_0 B_0 + \rho_0 \ln \rho_0 - \rho_0 + 1)dx.$$

Taking into consideration that $\rho \ln \rho - \rho + 1 \geqslant 0$, $ds/dt > 0$, we get

$$\int_0^{s(t)} (\rho \ln \rho - \rho + 1)dx + \frac{R}{\mu} \int_0^t \int_0^{s(\tau)} \rho^2 \theta dx d\tau \leqslant C + \int_0^{s(t)} \rho |B| dx + \int_0^t \frac{ds(\tau)}{d\tau} \rho_2(\tau) |B(s(\tau),\tau)| d\tau$$
$$\leqslant C \left(1 + \max_{(x,\tau) \in \Omega_t} |B(x,\tau)| \right).$$

It completes the proof. $\hspace{2cm} \square$

Lemma 4. *For each $t \in [0,T]$ the estimates*

$$\max_{0 \leqslant x \leqslant s(t)} \rho(x,t) \leqslant M \exp\{2 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)|\}, \tag{18}$$

$$\max_{0 \leqslant x \leqslant s(t)} \frac{1}{\rho(x,t)} \leqslant C \left[\exp\{2 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)|\} + \exp\{4 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)|\} \int_0^t \max_{0 \leqslant x \leqslant s(\tau)} \theta(x,\tau)d\tau \right] \tag{19}$$

hold true.

Proof. We multiply identity (12) by $\rho(\rho e^B)^{n-1}$, where n is a natural number and integrate w.r.t. x from 0 to $s(t)$ employing (1):

$$\frac{1}{n} \frac{d}{dt} \int_0^{s(t)} \rho(\rho e^B)^n dx - \frac{1}{n} \frac{ds(t)}{dt} \rho_2(t)[\rho_2(t)e^{B(s(t),t)}]^n + \frac{1}{n} \rho u (\rho e^B)^n \Big|_{x=0}^{x=s(t)} + \frac{R}{\mu} \int_0^{s(t)} \rho^2 (\rho e^B)^n \theta dx = 0.$$

By (5) and (7) we get

$$\frac{1}{n} \frac{d}{dt} \int_0^{s(t)} \rho(\rho e^B)^n dx - \frac{1}{n} \frac{ds(t)}{dt} \rho_2(t)[\rho_2(t)e^{B(s(t),t)}]^n + \frac{R}{\mu} \int_0^{s(t)} \rho^2 \theta (\rho e^B)^n dx = 0.$$

The third term is non-negative and thus

$$\frac{d}{dt} \int_0^{s(t)} \rho^{n+1} e^{nB} dx - \frac{ds(t)}{dt} \rho_2(t)[\rho_2(t)e^{B(s(t),t)}]^n \leqslant 0.$$

Integrating w.r.t. the time from 0 to t, we obtain

$$\int_0^{s(t)} \rho^{n+1} e^{nB} dx \leqslant \int_0^{s_0} \rho_0^{n+1} e^{nB_0} dx + \int_0^t \frac{ds(\tau)}{d\tau} \rho_2^{n+1}(\tau) e^{nB(s(\tau),\tau)} d\tau$$
$$\leqslant M^{n+1}(s_0 + TM) \left[\max_{(x,\tau) \in \Omega_t} e^{B(x,\tau)} \right]^n.$$

Then

$$\int_0^{s(t)} \rho^{n+1} dx \leqslant CM^{n+1} \left[\max_{(x,\tau) \in \Omega_t} e^{B(x,\tau)} \right]^n \left[\min_{(x,\tau) \in \Omega_t} e^{B(x,\tau)} \right]^{-n} \leqslant CM^{n+1} \left[\exp\{2 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)|\} \right]^n.$$

Passing to the limit as $n \to \infty$ and employing the identity

$$\lim_{n \to \infty} \left(\int_a^b |f(x)|^n dx \right)^{1/n} = \max_{x \in [a,b]} |f(x)|,$$

we obtain estimate (18):

$$\max_{0 \leqslant x \leqslant s(t)} \rho(x,t) \leqslant M \exp\left\{ 2 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)| \right\}.$$

In order to prove estimate (19), we multiply identity (13) by $\rho(\rho e^B)^{-n}$, where n is a natural number, and integrate w.r.t. x from 0 to $s(t)$ employing (1), (7):

$$\frac{1}{n+1} \frac{d}{dt} \int_0^{s(t)} \rho \left(\frac{1}{\rho e^B} \right)^{n+1} dx - \frac{1}{n+1} \frac{ds(t)}{dt} \rho_2(t) \left(\frac{1}{\rho_2(t) e^{B(s(t),t)}} \right)^{n+1}$$

$$+ \frac{1}{n+1} \rho u \left(\frac{1}{\rho e^B} \right)^{n+1} \Big|_{x=0}^{x=s(t)} = \frac{R}{\mu} \int_0^{s(t)} \rho e^{-B} \theta (\rho e^B)^{-n} dx.$$

By (5) and (7) we get

$$\frac{d}{dt} \int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^n \frac{1}{e^B} dx = \frac{ds(t)}{dt} \left(\frac{1}{\rho_2(t) e^{B(s(t),t)}} \right)^n \frac{1}{e^{B(s(t),t)}} + \frac{R(n+1)}{\mu} \int_0^{s(t)} e^{-2B} \theta \left(\frac{1}{\rho e^B} \right)^{n-1} dx.$$

Integrating w.r.t. the time from 0 to t, we arrive at

$$\int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^n \frac{1}{e^B} dx \leqslant s_0 \left(\frac{1}{m} \right)^n \left[\max_{(x,\tau) \in \Omega_t} e^{-B(x,\tau)} \right]^{n+1} + MT \left(\frac{1}{m} \right)^n \left[\max_{(x,\tau) \in \Omega_t} e^{-B(x,\tau)} \right]^{n+1}$$

$$+ \frac{R(n+1)}{\mu} \left[\max_{(x,\tau) \in \Omega_t} e^{-2B(x,\tau)} \right] \int_0^t \int_0^{s(\tau)} \left(\frac{1}{\rho e^B} \right)^{n-1} \theta \, dx \, d\tau.$$

Then

$$\int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^n dx \leqslant (s_0 + MT) \left(\frac{1}{m} \right)^n \left[\max_{(x,\tau) \in \Omega_t} e^{B(x,\tau)} \right] \left[\max_{(x,\tau) \in \Omega_t} e^{-B(x,\tau)} \right]^{n+1}$$

$$+ \frac{R(n+1)}{\mu} \left[\max_{(x,\tau) \in \Omega_t} e^{B(x,\tau)} \right] \left[\max_{(x,\tau) \in \Omega_t} e^{-2B(x,\tau)} \right] \int_0^t \int_0^{s(\tau)} \left(\frac{1}{\rho e^B} \right)^{n-1} \theta \, dx \, d\tau$$

$$\leqslant C \left(\frac{1}{m} \right)^n \exp\left\{ (n+2) \max_{(x,\tau) \in \Omega_t} |B(x,\tau)| \right\} + \frac{R(n+1)}{\mu} \exp\left\{ 3 \max_{(x,\tau) \in \Omega_t} |B(x,\tau)| \right\}$$

$$\cdot \int_0^t \max_{0 \leqslant x \leqslant s(\tau)} \theta(x,\tau) \int_0^{s(\tau)} \left(\frac{1}{\rho e^B} \right)^{n-1} dx \, d\tau. \tag{20}$$

We denote

$$y(t) = \left(\int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^n dx \right)^{1/n}.$$

Applying Hölder inequality to the last integral in (20), employing the inequality $s(\tau) \leqslant s(T)$, we obtain

$$\int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^{n-1} dx \leqslant \left(\int_0^{s(t)} \left(\frac{1}{\rho e^B} \right)^{(n-1) \cdot \frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \cdot \left(\int_0^{s(t)} 1^n dx \right)^{\frac{1}{n}} \leqslant y^{n-1}(t) \cdot \sqrt[n]{s(T)}.$$

Substituting the obtained estimate into (20), we have

$$
\begin{aligned}
y^n(t) \leqslant &C\left(\frac{1}{m}\right)^n \exp\left\{(n+2)\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\} \\
&+ \frac{R(n+1)\sqrt[n]{s(T)}}{\mu}\exp\left\{3\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\}\int_0^t \max_{0\leqslant x\leqslant s(\tau)}\theta(x,\tau)y^{n-1}(\tau)d\tau.
\end{aligned}
\tag{21}
$$

In order to estimate function $y(t)$ we make use of the following lemma.

Lemma 5 ([22]). *If a continuous non-negative on $[0,T]$ function $y(t)$ satisfies the inequality*

$$
y^n(t) \leqslant a + b\int_0^t c(\tau)y^{n-1}(\tau)d\tau,
$$

where $a, b = const \geqslant 0$, $n = const \geqslant 1$, $c(t)$ is a given non-negative function in class $L_1[0,T]$, then the estimate

$$
y(t) \leqslant \sqrt[n]{a} + \frac{b}{n}\int_0^t c(\tau)d\tau
\tag{22}
$$

holds true.

If we apply estimate (22) to inequality (21), we obtain

$$
\begin{aligned}
y(t) \leqslant &\frac{\sqrt[n]{C}}{m}\exp\left\{\left(1+\frac{2}{n}\right)\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\} \\
&+ \frac{R\sqrt[n]{s(T)}}{\mu}\left(1+\frac{1}{n}\right)\exp\left\{3\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\}\int_0^t \max_{0\leqslant x\leqslant s(\tau)}\theta(x,\tau)d\tau.
\end{aligned}
$$

Passing to the limit as $n\to\infty$, we get

$$
\max_{0\leqslant x\leqslant s(t)}\frac{1}{\rho e^B} \leqslant \frac{1}{m}\exp\left\{\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\} + \frac{R}{\mu}\exp\left\{3\max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right\}\int_0^t \max_{0\leqslant x\leqslant s(\tau)}\theta(x,\tau)d\tau.
$$

It implies estimate (19). The proof is complete. $\qquad\square$

Lemma 6 (Estimate of total energy). *There exists a constant $C > 0$ depending on boundary data and T such that*

$$
\max_{0\leqslant t\leqslant T}\int_0^{s(t)}\left(\rho\theta + \frac{\rho u^2}{2}\right)dx \leqslant C.
$$

Proof. In order to estimate the total energy, we introduce the auxiliary function $A(x,t)$ as solution to the boundary value problem

$$
\rho(A_t + uA_x) = \kappa A_{xx}, (x,t)\in\Omega_T,
\tag{23}
$$

$$
A\big|_{x=0} = \theta_1(t), \quad A\big|_{x=s(t)} = \theta_2(t), \quad A\big|_{t=0} = \theta_0(x).
$$

By the maximum principle we obtain

$$
0 < m \leqslant A(x,t) \leqslant M < +\infty.
\tag{24}
$$

The function

$$
\varphi(x,t) = \theta(x,t)A^{-1}(x,t),
$$

which takes values

$$
\varphi\big|_{x=0} = \varphi\big|_{x=s(t)} = 1, \varphi\big|_{t=0} = 1,
$$

satisfies by (3) and (23) the equation

$$
A\rho(\varphi_t + u\varphi_x) = \kappa(A\varphi_x)_x + \kappa A_x\varphi_x + \mu u_x^2 - R\rho A\varphi u_x.
$$

We multiply this equation by $(1-\frac{1}{\varphi})$ and integrate w.r.t. x from 0 to $s(t)$. Employing (23), we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_0^{s(t)}&A\rho(\varphi - \ln\varphi - 1)dx + \int_0^{s(t)}\kappa A\frac{\varphi_x^2}{\varphi^2}dx + \int_0^{s(t)}\frac{1}{\varphi}\mu u_x^2 dx \\
&= \int_0^{s(t)}(\mu u_x^2 - R\rho A\varphi u_x)dx + \int_0^{s(t)}\frac{1}{\varphi}R\rho A\varphi u_x dx.
\end{aligned}
\tag{25}
$$

Multiplying equation (2) by $u(x,t)$ and integrating w.r.t. x from 0 to $s(t)$ employing (1), we find

$$\frac{1}{2}\frac{d}{dt}\int_0^{s(t)}\rho u^2 dx + \int_0^{s(t)} u_x(\mu u_x - p)dx = 0.$$

We sum up the obtained identity with (25)

$$\frac{d}{dt}\int_0^{s(t)}\left(A\rho(\varphi - \ln\varphi - 1) + \frac{\rho u^2}{2}\right)dx + \int_0^{s(t)}\kappa A\frac{\varphi_x^2}{\varphi^2}dx + \int_0^{s(t)}\mu\frac{u_x^2}{\varphi}dx = \int_0^{s(t)} R\rho Au_x dx. \quad (26)$$

Let us estimate the terms in the right hand side:

$$\int_0^{s(t)} R\rho Au_x dx \leqslant \int_0^{s(t)}\left(\frac{\mu u_x^2}{2\varphi} + \frac{R^2 A}{2\mu}\rho^2 A\varphi\right)dx \leqslant \frac{\mu}{2}\int_0^{s(t)}\frac{u_x^2}{\varphi}dx + \frac{R^2 M}{2\mu}\int_0^{s(t)}\rho^2\theta dx.$$

As a result, we arrive at the inequality

$$\frac{d}{dt}\int_0^{s(t)}\left(A\rho(\varphi - \ln\varphi - 1) + \frac{\rho u^2}{2}\right)dx + \int_0^{s(t)}\kappa A\frac{\varphi_x^2}{\varphi^2}dx + \frac{\mu}{2}\int_0^{s(t)}\frac{u_x^2}{\varphi}dx \leqslant C\int_0^{s(t)}\rho^2\theta dx.$$

Integrating this inequality w.r.t. the time, we get

$$\int_0^{s(t)}\left(A\rho(\varphi - \ln\varphi - 1) + \frac{\rho u^2}{2}\right)dx + \int_0^t\int_0^{s(\tau)}\kappa A\frac{\varphi_x^2}{\varphi^2}dxd\tau + \frac{\mu}{2}\int_0^t\int_0^{s(\tau)}\frac{u_x^2}{\varphi}dxd\tau$$
$$\leqslant C\left(1 + \int_0^t\int_0^{s(\tau)}\rho^2\theta dxd\tau\right).$$

In view of the inequality

$$\varphi - \ln\varphi - 1 \geqslant \frac{1}{2}\varphi - \ln 2$$

and the boundedness of $A(x,t)$, by (24) and Lemma 1 we obtain

$$\frac{1}{2}\int_0^{s(t)}\rho\theta dx + \frac{1}{2}\int_0^{s(t)}\rho u^2 dx + \int_0^t\int_0^{s(\tau)}\kappa A\frac{\varphi_x^2}{\varphi^2}dxd\tau + \frac{\mu}{2}\int_0^t\int_0^{s(\tau)}\frac{u_x^2}{\varphi}dxd\tau \leqslant$$
$$\leqslant C\left(1 + \int_0^t\int_0^{s(\tau)}\rho^2\theta dxd\tau\right). \quad (27)$$

By estimate (17) in Lemma 3 we have

$$\int_0^t\int_0^{s(\tau)}\rho^2\theta dxd\tau \leqslant C\left(1 + \max_{(x,\tau)\in\Omega_t}|B(x,\tau)|\right).$$

Employing the inequality in Lemma 2, we obtain

$$\int_0^t\int_0^{s(\tau)}\rho^2\theta dxd\tau \leqslant C\left(1 + \max_{0\leqslant\tau\leqslant t}\left(\int_0^{s(\tau)}\rho u^2 dx\right)^{1/2} + \int_0^t\int_0^{s(\tau)}\rho\theta dxd\tau + \int_0^t\int_0^{s(\tau)}\rho u^2 dxd\tau\right).$$

By applying this estimate and neglecting in the left hand side of (27) the integrals involving the derivatives, we get

$$\frac{1}{2}\int_0^{s(t)}\rho\theta dx + \frac{1}{2}\int_0^{s(t)}\rho u^2 dx$$
$$\leqslant C\left(1 + \max_{0\leqslant\tau\leqslant t}\left(\int_0^{s(\tau)}\rho u^2 dx\right)^{1/2} + \int_0^t\int_0^{s(\tau)}\rho\theta dxd\tau + \int_0^t\int_0^{s(\tau)}\rho u^2 dxd\tau\right). \quad (28)$$

We introduce two functions on $[0,T]$:

$$a(t) = \max_{0\leqslant\tau\leqslant t}\int_0^{s(\tau)}\rho(x,\tau)u^2(x,\tau)dx, \quad b(t) = \max_{0\leqslant\tau\leqslant t}\int_0^{s(\tau)}\rho(x,\tau)\theta(x,\tau)dx.$$

Then by (28) we get

$$\frac{1}{2}a(t) + \frac{1}{2}b(t) \leqslant C\left(1 + \sqrt{a(t)} + \int_0^t a(\tau)d\tau + \int_0^t b(\tau)d\tau\right). \qquad (29)$$

Since $C\sqrt{a(t)} \leqslant C^2 + \frac{a(t)}{4}$, it follows from (29) that

$$a(t) + b(t) \leqslant C\left(1 + \int_0^t a(\tau)d\tau + \int_0^t b(\tau)d\tau\right).$$

By Grönwall inequality it yields $a(t) \leqslant C$, $b(t) \leqslant C$ for each $t \in [0, T]$, where C is a positive constant. Returning back to inequality (27), we obtain the following series of estimates

$$\max_{0 \leqslant t \leqslant T} \int_0^{s(t)} \rho(x,t)u^2(x,t)dx \leqslant C, \qquad (30)$$

$$\max_{0 \leqslant t \leqslant T} \int_0^{s(t)} \rho(x,t)\theta(x,t)dx \leqslant C, \qquad (31)$$

$$\int_0^T \int_0^{s(t)} \left(\kappa A \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{\varphi}\right) dxd\tau \leqslant C. \qquad (32)$$

Estimates (30), (31) provide the estimate for the total energy. $\qquad \square$

4. Upper and lower bounds for density and lower bounds for temperature

Lemma 7. *There exists a constant $M_1 > 0$ depending on boundary data and T such that*

$$\max_{(x,t) \in \Omega_T} \rho(x,t) \leqslant M_1. \qquad (33)$$

Proof. Lemma 2 and estimate (30), (31) imply the boundedness of function $B(x,t)$. The inequality (18) in Lemma 4 completes the proof. $\qquad \square$

Lemma 8. *There exists a constant $m_2 > 0$ depending on boundary data and T such that*

$$\min_{(x,t) \in \Omega_T} \theta(x,t) \geqslant m_2. \qquad (34)$$

Proof. We write equation (3) for the temperature as follows

$$\rho(\theta_t + u\theta_x) = \kappa\theta_{xx} + \mu\left(u_x - \frac{R\rho\theta}{2\mu}\right)^2 - \frac{R^2\rho^2\theta^2}{4\mu}.$$

Dividing by $\rho\theta^2$, in domain Ω_T we obtain the equation for the function $q(x,t) = 1/\theta(x,t)$:

$$q_t + uq_x - \frac{\kappa}{\rho}q_{xx} = \frac{R^2\rho}{4\mu} - \left[\frac{2\kappa}{\rho}\theta q_x^2 + \frac{\mu}{\rho}q^2\left(u_x - \frac{R\rho\theta}{2\mu}\right)^2\right]. \qquad (35)$$

We pass from function $q(x,t)$ to a new function $v(x,t)$, by the identity

$$q(x,t) = v(x,t)e^t.$$

Thanks to (35), function $v(x,t)$ satisfies the equation

$$v_t + uv_x - \frac{\kappa}{\rho}v_{xx} + v = \frac{R^2\rho}{4\mu}e^{-t} - \left[\frac{2\kappa}{\rho}\theta v_x^2 e^t + \frac{\mu}{\rho}v^2 e^t\left(u_x - \frac{R\rho\theta}{2\mu}\right)^2\right].$$

By Lemma 7 and the non-negativeness of terms in the square brackets for function $v(x,t)$ we have the differential inequality

$$v_t + uv_x - \frac{\kappa}{\rho}v_{xx} + v \leqslant Ce^{-t}. \qquad (36)$$

Suppose that function $v(x,t)$ attains its positive maximum at an interior point (x_0, t_0) in domain Ω_T or as $t_0 = T$. Then at this point

$$v_t \geqslant 0, \quad v_x = 0, \quad v_{xx} \leqslant 0. \qquad (37)$$

By (36), (37) we obtain the estimate

$$\max_{(x,t)\in\Omega_T} v(x,t) \leqslant Ce^{-t_0} \leqslant C.$$

Therefore, for each $(x,t) \in \Omega_T$ the inequalities

$$\frac{q(x,t)}{e^t} \leqslant C, \ q(x,t) \leqslant Ce^t \leqslant Ce^T, \ \frac{1}{\theta(x,t)} \leqslant Ce^T$$

hold true or

$$\theta(x,t) \geqslant m_2 = \frac{1}{Ce^T}.$$

The proof is complete. $\hfill\square$

Lemma 9. *There exists a constant $m_1 > 0$ depending on boundary data and T such that*

$$\min_{(x,t)\in\Omega_T} \rho(x,t) \geqslant m_1.$$

Proof. Estimates (24), (32) yield

$$\int_0^t \int_0^{s(\tau)} \frac{\varphi_x^2}{\varphi^2} dx d\tau \leqslant C. \tag{38}$$

By Lemma 4, the estimate $|B(x,t)| \leqslant C$ and in view of (24) we have

$$
\begin{aligned}
\max_{0\leqslant x\leqslant s(t)} \frac{1}{\rho(x,t)} &\leqslant C\left(1 + \int_0^t \max_{0\leqslant x\leqslant s(\tau)} \theta(x,\tau)d\tau\right) \\
&\leqslant C\left(1 + \int_0^t \max_{0\leqslant x\leqslant s(\tau)} A(x,\tau)\varphi(x,\tau)d\tau\right) \leqslant C + CM\int_0^t \max_{0\leqslant x\leqslant s(\tau)} \varphi(x,\tau)d\tau.
\end{aligned} \tag{39}
$$

Taking into consideration the inequalities

$$
\begin{aligned}
\varphi(x,t) = (\sqrt{\varphi(x,t)})^2 &\leqslant \left(1 + \int_0^{s(t)} |(\sqrt{\varphi})_x| dx\right)^2 \leqslant \left(1 + \frac{1}{2}\int_0^{s(t)} \left|\frac{\varphi_x}{\sqrt{\varphi}}\right| dx\right)^2 \\
&\leqslant \left[1 + \frac{1}{2}\left(\int_0^{s(t)} \frac{\varphi_x^2}{\varphi^2} dx\right)^{1/2} \left(\int_0^{s(t)} \varphi dx\right)^{1/2}\right]^2 \leqslant 2\left(1 + \int_0^{s(t)} \frac{\varphi_x^2}{\varphi^2} dx \int_0^{s(t)} \varphi dx\right),
\end{aligned}
$$

$$\int_0^{s(t)} \varphi(x,t)dx = \int_0^{s(t)} \frac{\rho\theta}{\rho A} dx \leqslant \frac{1}{m} \max_{0\leqslant x\leqslant s(t)} \frac{1}{\rho(x,t)} \int_0^{s(t)} \rho\theta dx \leqslant C \max_{0\leqslant x\leqslant s(t)} \frac{1}{\rho(x,t)},$$

by (38) and (39) we arrive at the inequality

$$\max_{0\leqslant x\leqslant s(t)} \frac{1}{\rho(x,t)} \leqslant C + C\int_0^t \max_{0\leqslant x\leqslant s(\tau)} \frac{1}{\rho(x,\tau)} d\tau.$$

Applying Grönwall inequality, we complete the proof. $\hfill\square$

In the above proof there were obtained the estimates

$$\int_0^t \max_{0\leqslant x\leqslant s(\tau)} \theta(x,\tau)d\tau \leqslant C + C\int_0^t \max_{0\leqslant x\leqslant s(\tau)} \frac{1}{\rho(x,\tau)} d\tau, \quad \int_0^{s(t)} \varphi(x,t)dx \leqslant C \max_{0\leqslant x\leqslant s(t)} \frac{1}{\rho(x,t)},$$

which imply

$$\int_0^T \max_{0\leqslant x\leqslant s(t)} \theta(x,t)dt \leqslant C, \tag{40}$$

$$\int_0^{s(t)} \theta(x,t)dx = \int_0^{s(t)} A(x,t)\varphi(x,t)dx \leqslant M\int_0^{s(t)} \varphi(x,t)dx \leqslant C, \tag{41}$$

$$\int_0^T \int_0^{s(t)} \theta^2(x,t)dxdt \leqslant \int_0^T \max_{0\leqslant x\leqslant s(t)} \theta(x,t) \int_0^{s(t)} \theta(x,t)dxdt \leqslant C. \tag{42}$$

ESTIMATES FOR DERIVATIVES

Lemma 10. *There exists a constant $C > 0$ depending on boundary data and T such that*

$$\max_{t \in [0,T]} \int_0^{s(t)} \rho(x,t) u^2(x,t) dx + \int_0^T \int_0^{s(t)} u_x^2(x,t) dx dt \leqslant C. \tag{43}$$

Proof. We multiply equation (2) by $u(x,t)$ and integrate it w.r.t. x from 0 to $s(t)$ employing (1), (5), and (7):

$$\frac{1}{2} \frac{d}{dt} \int_0^{s(t)} \rho u^2 dx + \mu \int_0^{s(t)} u_x^2 dx - \int_0^{s(t)} R \rho \theta u_x dx = 0.$$

Applying Young inequality with ε and integrating w.r.t. t, we get

$$\frac{1}{2} \int_0^{s(t)} \rho u^2 dx + \mu \int_0^t \int_0^{s(\tau)} u_x^2 dx d\tau \leqslant C_1 + \frac{R^2}{\varepsilon} \int_0^t \int_0^{s(\tau)} \rho^2 \theta^2 dx d\tau + \varepsilon \int_0^t \int_0^{s(\tau)} u_x^2 dx d\tau.$$

Choosing ε sufficiently small and applying Lemma 7 and inequality (42), we obtain

$$\int_0^{s(t)} \rho u^2 dx + \int_0^t \int_0^{s(\tau)} u_x^2 dx d\tau \leqslant C.$$

The proof is complete. $\qquad \square$

As a corollary of Lemmata 9 and 10 we obtain the estimate

$$\max_{t \in [0,T]} \int_0^{s(t)} u^2(x,t) dx + \int_0^T \int_0^{s(t)} u_x^2(x,t) dx dt \leqslant C. \tag{44}$$

Lemma 11. *For each $t \in [0,T]$ the inequality*

$$\int_0^{s(t)} u_x^2(x,t) dx + \int_0^t \int_0^{s(\tau)} \left(u_\tau^2(x,\tau) + u_{xx}^2(x,\tau) \right) dx d\tau$$

$$\leqslant C \left(1 + \int_0^t \int_0^{s(\tau)} \theta_x^2(x,\tau) dx d\tau + \int_0^t \max_{x \in [0,s(\tau)]} \theta^2(x,\tau) \int_0^{s(\tau)} \rho_x^2(x,\tau) dx d\tau \right) \tag{45}$$

holds true.

Proof. We represent equation (2) as

$$\sqrt{\rho} u_t - \frac{1}{\sqrt{\rho}} \mu u_{xx} = -\sqrt{\rho} u u_x - R \sqrt{\rho} \theta_x - \frac{1}{\sqrt{\rho}} R \rho_x \theta.$$

Hence,

$$\int_0^{s(t)} \left(\rho u_t^2 + \frac{1}{\rho} \mu^2 u_{xx}^2 - 2\mu u_t u_{xx} \right) dx = \int_0^{s(t)} \left(\sqrt{\rho} u u_x + R \sqrt{\rho} \theta_x + \frac{1}{\sqrt{\rho}} R \rho_x \theta \right)^2 dx.$$

Thus,

$$\int_0^{s(t)} \rho u_t^2 dx + \mu^2 \int_0^{s(t)} \frac{1}{\rho} u_{xx}^2 dx - 2\mu u_t(s(t),t) u_x(s(t),t) + \mu \frac{d}{dt} \int_0^{s(t)} u_x^2 dx$$

$$- \mu \frac{ds(t)}{dt} u_x^2(s(t),t) \leqslant 3 \int_0^{s(t)} \left(\rho u^2 u_x^2 + R^2 \rho \theta_x^2 + \frac{1}{\rho} R^2 \rho_x^2 \theta^2 \right) dx. \tag{46}$$

Since $u(s(t),t) = 0$, then

$$u_x(s(t),t) \frac{ds(t)}{dt} = -u_t(s(t),t),$$

and by (46) it implies the inequality

$$\int_0^{s(t)} \rho u_t^2 dx + \mu^2 \int_0^{s(t)} \frac{1}{\rho} u_{xx}^2 dx + \mu \frac{d}{dt} \int_0^{s(t)} u_x^2 dx + \mu \frac{ds(t)}{dt} u_x^2(s(t),t)$$

$$\leqslant 3 \int_0^{s(t)} \left(\rho u^2 u_x^2 + R^2 \rho \theta_x^2 + \frac{1}{\rho} R^2 \rho_x^2 \theta^2 \right) dx. \tag{47}$$

We have

$$u_x^2(x,t) \leqslant u_x^2(x_0,t) + \int_{x_0}^x \left|(u_x^2)_x\right| dx \leqslant u_x^2(x_0,t) + \int_0^{s(t)} \left|(u_x^2)_x\right| dx = u_x^2(x_0,t) + 2 \int_0^{s(t)} |u_x u_{xx}| \, dx.$$

Given any $\varepsilon > 0$, we integrate this inequality w.r.t. x_0 from 0 to $s(t)$

$$
\begin{aligned}
u_x^2(x,t) &\leqslant \frac{1}{s(t)} \int_0^{s(t)} u_x^2(x,t) dx + 2 \int_0^{s(t)} |u_x u_{xx}| \, dx \\
&\leqslant \left(\frac{1}{s_0} + \frac{1}{\varepsilon}\right) \int_0^{s(t)} u_x^2 dx + \varepsilon \int_0^{s(t)} u_{xx}^2 dx.
\end{aligned}
\tag{48}
$$

Taking into consideration (48), the non-negativeness of the term $\mu \frac{ds(t)}{dt} u_x^2(s(t),t)$, the estimate

$$\int_0^{s(t)} \rho u^2 u_x^2 dx \leqslant C \max_{0 \leqslant x \leqslant s(t)} u_x^2(x,t) \leqslant \frac{C}{\varepsilon} \int_0^{s(t)} u_x^2 dx + C\varepsilon \int_0^{s(t)} u_{xx}^2 dx,$$

Lemma 10 and the boundedness of $\rho(x,t)$ from above and below, by (47) we obtain (45) under appropriate choice of $\varepsilon > 0$. □

Lemma 12. *For each $t \in [0,T]$ the inequality*

$$\int_0^{s(t)} \rho_x^2(x,t) dx \leqslant C + C \int_0^t \int_0^{s(\tau)} \frac{\theta_x^2}{\theta} dx d\tau + C\varepsilon \int_0^t \int_0^{s(\tau)} u_{xx}^2 dx d\tau$$

holds true.

Proof. By equation (1) we have

$$(\ln \rho)_t + u(\ln \rho)_x + u_x = 0.$$

Then equation (2) casts into the form

$$(\rho u)_t + (\rho u^2)_x = -\mu[(\ln \rho)_t + u(\ln \rho)_x]_x - p_x.$$

Hence,

$$\rho \left[\left(u + \mu \frac{\rho_x}{\rho^2}\right)_t + u \left(u + \mu \frac{\rho_x}{\rho^2}\right)_x\right] + p_x = 0.$$

We multiply this identity by $(u + \mu \rho_x/\rho^2)$ and integrate w.r.t. x from 0 to $s(t)$:

$$
\frac{1}{2} \frac{d}{dt} \int_0^{s(t)} \rho \left(u + \mu \frac{\rho_x}{\rho^2}\right)^2 dx - \frac{1}{2} \frac{ds(t)}{dt} \rho \left(u + \mu \frac{\rho_x}{\rho^2}\right)^2 \bigg|_{x=s(t)} +
$$

$$
+ \frac{1}{2} \rho u \left(u + \mu \frac{\rho_x}{\rho^2}\right)^2 \bigg|_{x=0}^{x=s(t)} - \int_0^{s(t)} R\rho \theta u_x dx + R\rho\theta u \bigg|_{x=0}^{x=s(t)} + \int_0^{s(t)} (R\rho\theta)_x \mu \frac{\rho_x}{\rho^2} dx = 0.
$$

Since $u = 0$ as $x = s(t)$ and $x = 0$, the obtained identity becomes

$$
\frac{1}{2} \frac{d}{dt} \int_0^{s(t)} \left(\rho u^2 + 2\mu \frac{u\rho_x}{\rho} + \mu^2 \frac{\rho_x^2}{\rho^3}\right) dx - \int_0^{s(t)} R\rho\theta u_x dx + R\mu \int_0^{s(t)} \frac{\theta \rho_x^2}{\rho^2} dx
$$

$$
+ R\mu \int_0^{s(t)} \frac{\theta_x \rho_x}{\rho} dx - \frac{\mu^2}{2} \frac{ds}{dt} \frac{\rho_x^2}{\rho^3} \bigg|_{x=s(t)} = 0.
$$

Employing Lemma 7, the replace the third term in the left hand side by a smaller one:

$$
\frac{1}{2} \frac{d}{dt} \int_0^{s(t)} (\rho u^2 + 2\mu \frac{u\rho_x}{\rho} + \mu^2 \frac{\rho_x^2}{\rho^3}) dx - \int_0^{s(t)} R\rho\theta u_x dx + \frac{R\mu}{M_1^2} \int_0^{s(t)} \theta \rho_x^2 dx
$$

$$
+ R\mu \int_0^{s(t)} \frac{\theta_x \rho_x}{\rho} dx - \frac{\mu^2}{2} \frac{ds}{dt} \frac{\rho_x^2}{\rho^3} \bigg|_{x=s(t)} \leqslant 0.
\tag{49}
$$

Let us estimate some of the terms. By equation (1) for $x = s(t)$, and (5), (7), we obtain

$$\rho_t(s(t),t) = -\rho_2(t) u_x(s(t),t).$$

On the other hand, differentiating (7) w.r.t. t, we have

$$\rho_x(s(t),t)\frac{ds(t)}{dt} + \rho_t(s(t),t) = \frac{d\rho_2(t)}{dt}, \quad \rho_x(s(t),t)\frac{ds(t)}{dt} = \frac{d\rho_2(t)}{dt} + \rho_2(t)u_x(s(t),t).$$

Then by condition (9), we get

$$\rho_x^2(s(t),t)\frac{ds(t)}{dt} = \left(\frac{d\rho_2(t)}{dt} + \rho_2(t)u_x(s(t),t)\right)^2 \frac{1}{ds(t)/dt}$$

$$\leqslant 2\left(\frac{d\rho_2(t)}{dt}\right)^2\frac{1}{m} + \frac{2M^2}{m}u_x^2(s(t),t) \leqslant C\left(1 + u_x^2(s(t),t)\right).$$

By (48) it yields

$$\rho_x^2(s(t),t)\frac{ds(t)}{dt} \leqslant C + \frac{C}{\varepsilon}\int_0^{s(t)} u_x^2 dx + \varepsilon C\int_0^{s(t)} u_{xx}^2 dx.$$

Thus,

$$\frac{\mu^2}{2}\frac{ds(t)}{dt}\frac{\rho_x^2(s(t),t)}{\rho^3(s(t),t)} \leqslant C + \frac{C}{\varepsilon}\int_0^{s(t)} u_x^2 dx + \varepsilon C\int_0^{s(t)} u_{xx}^2 dx.$$

Due to the boundedness of $\rho(x,t)$ from above and below, (40), (42), Lemma 10, and the estimates

$$\int_0^t\int_0^{s(\tau)} R\rho\theta u_x dxd\tau \leqslant C\int_0^t\int_0^{s(\tau)} \theta^2 dxd\tau + C\int_0^t\int_0^{s(\tau)} u_x^2 dxd\tau \leqslant C,$$

$$- R\mu\int_0^{s(t)}\frac{\theta_x\rho_x}{\rho}dx \leqslant \frac{C}{\varepsilon_1}\int_0^{s(t)}\frac{\theta_x^2}{\theta}dx + \varepsilon_1 C\int_0^{s(t)}\theta\rho_x^2 dx,$$

$$- \mu\int_0^{s(t)}\frac{u\rho_x}{\rho}dx \leqslant \frac{C}{\varepsilon_2}\int_0^{s(t)} u^2 dx + \varepsilon_2 C\int_0^{s(t)}\rho_x^2 dx,$$

we integrate (49) w.r.t. the time and for an appropriate choice of ε_1, ε_2 we arrive at the statement of the lemma. The proof is complete.   □

ESTIMATE FOR TEMPERATURE

Lemma 13. *There exists a constant $C > 0$ such that for each $t \in [0, T]$ the estimate*

$$\max_{0\leqslant\tau\leqslant t}\int_0^{s(\tau)} \theta^2 dx + \int_0^t\int_0^{s(\tau)} \theta_x^2 dxd\tau \leqslant C$$

holds true.

Proof. We write the equation for the energy $\theta + \frac{1}{2}u^2$. In order to do it, we multiply equation (2) by u and sum up with (3):

$$\rho\left(\theta + \frac{u^2}{2}\right)_t + \rho u\left(\theta + \frac{u^2}{2}\right)_x = \mu(u_x u)_x + \kappa\theta_{xx} - (R\rho\theta u)_x. \tag{50}$$

We multiply this identity by

$$\theta + \frac{u^2}{2} - \frac{x\theta_2(t)}{s(t)} - \left(1 - \frac{x}{s(t)}\right)\theta_1(t)$$

and integrate w.r.t. x from 0 to $s(t)$:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx - \frac{1}{2}\frac{ds(t)}{dt}\rho_2(t)\theta_2^2(t) \\
&\quad -\frac{d}{dt}\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)\left[\frac{x\theta_2(t)}{s(t)}+\left(1-\frac{x}{s(t)}\right)\theta_1(t)\right]dx + \frac{ds(t)}{dt}\rho_2(t)\theta_2^2(t) \\
&\quad +\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)\left[\frac{x\theta_2(t)}{s(t)}+\left(1-\frac{x}{s(t)}\right)\theta_1(t)\right]_t dx + \frac{\theta_2(t)-\theta_1(t)}{s(t)}\int_0^{s(t)}\rho u\left(\theta+\frac{u^2}{2}\right)dx \\
&= -\mu\int_0^{s(t)}uu_x\theta_x dx - \mu\int_0^{s(t)}u^2u_x^2 dx + \mu\frac{\theta_2(t)-\theta_1(t)}{s(t)}\int_0^{s(t)}uu_x dx - \kappa\int_0^{s(t)}\theta_x^2 dx \\
&\quad -\kappa\int_0^{s(t)}uu_x\theta_x dx + \kappa\frac{\theta_2(t)-\theta_1(t)}{s(t)}(\theta_2(t)-\theta_1(t)) + R\int_0^{s(t)}\rho\theta u\theta_x dx + R\int_0^{s(t)}\rho\theta u^2 u_x dx \\
&\quad -R\frac{\theta_2(t)-\theta_1(t)}{s(t)}\int_0^{s(t)}\rho\theta u\,dx.
\end{aligned}
\tag{51}
$$

Let us estimate some terms in (51):

$$
\frac{\theta_2(t)-\theta_1(t)}{s(t)}\int_0^{s(t)}\rho u\left(\theta+\frac{u^2}{2}\right)dx \leqslant C\int_0^{s(t)}u^2 dx + C\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx,
$$

$$
\int_0^{s(t)}uu_x\theta_x dx \leqslant \frac{1}{\varepsilon}\int_0^{s(t)}u^2u_x^2 dx + \varepsilon\int_0^{s(t)}\theta_x^2 dx,
$$

$$
R\int_0^{s(t)}\rho\theta u\theta_x dx \leqslant \frac{R}{\varepsilon}\int_0^{s(t)}\rho^2\theta^2 u^2 dx + \varepsilon R\int_0^{s(t)}\theta_x^2 dx,
$$

$$
R\int_0^{s(t)}\rho\theta u^2 u_x dx \leqslant R\int_0^{s(t)}\rho^2\theta^2 u^2 dx + R\int_0^{s(t)}u^2 u_x^2 dx.
$$

Other terms can be estimated in obvious way. Integrating (51) w.r.t. the time, we obtain the inequality

$$
\begin{aligned}
&\frac{1}{4}\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx + \frac{\kappa}{2}\int_0^t\int_0^{s(\tau)}\theta_x^2 dx d\tau \\
&\leqslant C + M_3\int_0^t\int_0^{s(\tau)}u^2u_x^2 dx d\tau + C\int_0^t\int_0^{s(\tau)}\rho^2\theta^2 u^2 dx d\tau + C\int_0^t\int_0^{s(\tau)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx d\tau,
\end{aligned}
\tag{52}
$$

where C, M_3 are positive constants depending on T, initial and boundary data.

We multiply (2) by $4u^3$, integrate w.r.t. x from 0 to $s(t)$ and employ (1)

$$
\frac{d}{dt}\int_0^{s(t)}\rho u^4 dx + 12\mu\int_0^{s(t)}u^2u_x^2 dx = 12R\int_0^{s(t)}\rho\theta u^2 u_x dx.
$$

We apply Cauchy inequality to the right hand side

$$
\frac{d}{dt}\int_0^{s(t)}\rho u^4 dx + 12\mu\int_0^{s(t)}u^2u_x^2 dx \leqslant 6\mu\int_0^{s(t)}u^2u_x^2 dx + \frac{6R^2}{\mu}\int_0^{s(t)}\rho^2\theta^2 u^2 dx.
\tag{53}
$$

We integrate (53) w.r.t. the time to obtain

$$
\int_0^{s(t)}\rho u^4 dx + 6\mu\int_0^t\int_0^{s(\tau)}u^2u_x^2 dx d\tau \leqslant C + C\int_0^t\int_0^{s(\tau)}\rho^2\theta^2 u^2 dx d\tau.
\tag{54}
$$

We multiply (54) by M_3/μ and sum up with (52)

$$
\begin{aligned}
&\frac{1}{4}\int_0^{s(t)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx + \frac{\kappa}{2}\int_0^t\int_0^{s(\tau)}\theta_x^2 dx d\tau + \frac{M_3}{\mu}\int_0^{s(t)}\rho u^4 dx + 5M_3\int_0^t\int_0^{s(\tau)}u^2u_x^2 dx d\tau \\
&\qquad\qquad \leqslant C + C\int_0^t\int_0^{s(\tau)}\rho^2\theta^2 u^2 dx d\tau + C\int_0^t\int_0^{s(\tau)}\rho\left(\theta+\frac{u^2}{2}\right)^2 dx d\tau.
\end{aligned}
\tag{55}
$$

In what follows we shall make use of the estimates

$$\theta^2(x,t) = \theta_1^2(t) + \int_0^x (\theta^2)_x dx \leqslant \theta_1^2(t) + 2\int_0^{s(t)} |\theta\theta_x| dx \leqslant \theta_1^2(t) + \frac{1}{\varepsilon}\int_0^{s(t)} \theta^2 dx + \varepsilon\int_0^{s(t)} \theta_x^2 dx.$$

Integrating w.r.t. the time and employing (42), we get

$$\int_0^t \max_{0\leqslant x\leqslant s(\tau)} \theta^2(x,\tau)d\tau \leqslant C(\varepsilon) + \varepsilon\int_0^t\int_0^{s(\tau)} \theta_x^2 dxd\tau. \tag{56}$$

Then employing (44) and (56) we estimate as follows

$$\int_0^t\int_0^{s(\tau)} \rho^2\theta^2 u^2 dxd\tau \leqslant M_1^2\int_0^t\left[\max_{0\leqslant x\leqslant s(\tau)} \theta^2(x,\tau)\int_0^{s(\tau)} u^2(x,\tau)dx\right]d\tau$$

$$\leqslant M_1^2\max_{0\leqslant \tau\leqslant t}\int_0^{s(\tau)} u^2(x,\tau)dx \cdot \int_0^t \max_{0\leqslant x\leqslant s(\tau)} \theta^2(x,\tau)d\tau \leqslant C + C\varepsilon\int_0^t\int_0^{s(\tau)} \theta_x^2 dxd\tau.$$

Employing this estimate for an appropriate choice of ε and applying Grönwall inequality, by (55) we complete the proof. □

Lemma 14. *There exist constants C_1, C_2, C_3, $M_2 > 0$ depending on T, initial and boundary data such that for each $t \in [0,T]$ the estimates*

$$\int_0^{s(t)} u_x^2(x,t)dx + \int_0^t\int_0^{s(\tau)} u_\tau^2 dxd\tau + \int_0^t\int_0^{s(\tau)} u_{xx}^2 dxd\tau \leqslant C_1, \tag{57}$$

$$\int_0^{s(t)} \rho_x^2(x,t)dx + \int_0^{s(t)} \rho_t^2(x,t)dx \leqslant C_2,$$

$$\int_0^{s(t)} \theta_x^2(x,t)dx + \int_0^t\int_0^{s(\tau)} \theta_\tau^2 dxd\tau + \int_0^t\int_0^{s(\tau)} \theta_{xx}^2 dxd\tau \leqslant C_3, \tag{58}$$

$$\max_{(x,t)\in\Omega_T} \theta(x,t) \leqslant M_2. \tag{59}$$

hold true.

Proof. Thanks to the boundedness of $\theta(x,t)$ from below, by employing Lemma 13 we have

$$\int_0^t\int_0^{s(\tau)} \frac{\theta_x^2}{\theta}dxd\tau \leqslant C.$$

Substituting this estimate into the statement of Lemma 12, we obtain

$$\int_0^{s(t)} \rho_x^2(x,t)dx \leqslant C + C\varepsilon\int_0^t\int_0^{s(\tau)} u_{xx}^2 dxd\tau. \tag{60}$$

Now estimate (57) follows from Lemmata 11, 13, (56), (60).

By (60) and (57) for each $t \in [0,T]$ we have

$$\int_0^{s(t)} \rho_x^2(x,t)dx \leqslant C.$$

Equation (1) yields that for each $t \in [0,T]$

$$\int_0^{s(t)} \rho_t^2(x,t)dx \leqslant C.$$

Estimates (58), (59) can be obtained from equation (3) in the standard way by employing obtained estimates for ρ and u. □

Once we have apriori estimates proven in Lemmata 1–14, the estimates in the theorem in Hölder norms can be obtained by the approach presented in [25].

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