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TWO-SIDED K-ORDER ESTIMATE FOR DIRICHLET SERIES IN A HALF-STRIP

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Abstract. We study Dirichlet series convergent only in a half-plane and whose sequence of exponents can be extended to some "regular" sequence. We establish the best possible k-order estimates for the sum of the Dirichlet series in the half-strip whose width depends on a special distribution density of the exponents.

Keywords: k-order of the Dirichlet series in a half-strip, entire functions of a given growth on the real axis

Mathematics Subject Classification: 30D10

Let $\Lambda = \{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ be a sequence satisfying the condition

$$\varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n} = H < \infty. \tag{1}$$

While studying entire functions

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad (s = \sigma + it) \tag{2}$$

defined everywhere by convergent Dirichlet series, the notion of R-order was introduced by J.F. Ritt. Let us recall its definition.

Ritt order (R-order) of entire function F defined by series (2) is the quantity [1]

$$\rho_R = \varlimsup_{\sigma\to+\infty} \frac{\ln\ln M(\sigma)}{\sigma},$$

where $M(\sigma) = \sup_{|t|<\infty} |F(\sigma+it)|$. We note that by condition (1), the series converges absolutely in the whole plane. It is known that $\ln M(\sigma)$ is an increasing convex function of σ, $\lim_{\sigma\to+\infty} \ln M(\sigma) = +\infty$.

We consider the strip $S(a,t_0) = \{s = \sigma+it : |t-t_0| \leqslant a\}$. We let $M_s(\sigma) = \max_{|t-t_0|\leqslant a} |F(\sigma+it)|$. The quantity

$$\rho_s = \varlimsup_{\sigma\to+\infty} \frac{\ln^+ \ln M_s(\sigma)}{\sigma} \quad (a^+ = \max(a,0))$$

is called R-order of function F in strip $S(a,t_0)$.

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Let

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < \infty, \qquad D^* = \overline{\lim_{\lambda\to+\infty}} \frac{1}{\lambda}\int_0^\lambda D(x)dx,$$

where $D(x) = \frac{n(x)}{x}$, $n(x) = \sum_{\lambda_n \leqslant x} 1$ (D is the upper density, D^* is the averaged upper density of sequence Λ). It is known that $D^* \leqslant D \leqslant eD^*$ [2]. It was shown in [2] that if

$$\underline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = h > 0,$$

then R-order ρ_s of function F in strip $S(a, t_0)$ as $a > \pi D^*$ is equal to R-order ρ_R in the whole plane. The most general result on the relation between quantities ρ_R and ρ_s was established by A.F. Leontiev [3].

Similar question in the case when $H = 0$ and the convergence domain of series (2) is the half-plane $\Pi_0 = \{s = \sigma + it : \sigma < 0\}$ were studied by A.M. Gaisin in [4].

As $H = 0$, if series (2) converges in the half-plane Π_0, it converges in Π_0 absolutely. Then the sum of series F is analytic in this half-plane. We denote by $D_0(\Lambda)$ the class of all analytic functions represented by Dirichlet series (2) convergent only in half-plane Π_0.

Let $S(a, t_0) = \{s = \sigma + it : |t - t_0| \leqslant a, \sigma < 0\}$ be a half-strip. The quantities

$$\rho_R = \overline{\lim_{\sigma\to 0-}} \frac{\ln^+ \ln M(\sigma)}{|\sigma|^{-1}}, \qquad \rho_s = \overline{\lim_{\sigma\to 0-}} \frac{\ln^+ \ln M_s(\sigma)}{|\sigma|^{-1}}$$

are called Ritt orders of function F in half-plane Π_0 and half-strip $S(a, t_0)$ [4]. In what follows, we call ρ_R and ρ_s orders in the half-plane and half-strip. If necessary, instead of ρ_R and ρ_s we shall write $\rho_R(F)$ and $\rho_s(F)$.

It was shown in [4] that if

$$\lim_{n\to\infty} \frac{\ln \lambda_n}{\lambda_n} \ln n = 0,$$

then order ρ_R of each function $F \in D_0(\Lambda)$ is equal to

$$\rho_R = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \ln^+ |a_n|. \tag{3}$$

Let sequence Λ have a finite upper density D. Then

$$L(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) \quad (z = x + iy)$$

is an entire function of exponential type. Let $h(\varphi)$ be the growth indicatrix of function $L(z)$. Then $\tau = h(\pm\frac{\pi}{2}) \leqslant \pi D^*$ [2]. It is obvious that τ is a type of function $L(z)$. Let

$$|L(x)| \leqslant e^{g(x)}\, (x \geqslant 0), \qquad \lim_{x\to+\infty} \frac{g(x)\ln x}{x} = 0, \tag{4}$$

where g is a non-negative on $\mathbb{R}_+ = [0, \infty)$ function. In this case $h(0) = h(\pi) = 0$. Therefore, the adjoint diagram of function $L(z)$ is the segment $I = [-\tau i, \tau i]$, $h(\varphi) = \tau|\sin\varphi|$.

In [4] there was proven the following

Theorem I. *Suppose that function $L(z)$ satisfies conditions (4) and has type τ, $(0 \leqslant \tau < \infty)$. We let $q = q(L)$, where*

$$q(L) = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \ln\left|\frac{1}{L'(\lambda_n)}\right|. \tag{5}$$

Then order ρ_s in the strip $S(a, t_0)$ as $a > \tau$ and order ρ_R of each function $F \in D_0(\Lambda)$ in half-plane Π_0 satisfy the estimates

$$\rho_s \leqslant \rho_R \leqslant \rho_s + q. \tag{6}$$

As $a < \tau$, for half-strip $S(a, t_0)$ the right estimate in (6) is generally speaking not valid [4].

It is clear that the left estimate in (6) is exact. Indeed, if $t_0 = 0$ and $a_n > 0$, then $M(\sigma) = M_s(\sigma)$ and $\rho_R = \rho_s$. It was shown in [4] that if Λ is a sequence of all zeroes for a function like sine, then there exists a function $F \in D_0(\Lambda)$ such that $\rho_R = \rho_s + q$ as $a > \tau$. In the general situation the right estimate in (6) is not exact, moreover, the pair of conditions (4) can fail. However, there can exist an entire function of exponential type Q with simple zeroes at points of sequence Λ satisfying conditions (4), at that $q(Q) = q^*$, where

$$q^* = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n}{\lambda_n} \int\limits_0^1 \frac{n(\lambda_n; t)}{t} dt,$$

$q(Q)$ is the quantity defined in the same way as $q(L)$ in (5), and $n(\lambda_n; t)$ is the number of points $\lambda_k \neq \lambda_n$ in the segment $\{x : |x - \lambda_n| \leqslant t\}$. Paper [5] was devoted to constructing such entire functions Q with a prescribed zero set Λ and given asymptotics on the real axis. It happened to be possible to provide general but simple and observable conditions in terms of special density $G(R)$ of distribution of sequence Λ under which the estimate

$$\rho_R \leqslant \rho_s + q^*$$

holds true (ρ_s is the order in the half-strip $S(a, t_0)$ of the width large than $2\pi G(R)$) and this estimate is not improvable in class $D_0(\Lambda)$ [6]. The aim of paper is to generalize and specify the results of works [6], [4] for the case of k-orders.

1. Preliminaries and lemmata

1^0. Special distribution densities for sequence Λ. Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence having a finite upper density, L be the class of positive continuous and unboundedly increasing on $[0, \infty)$ functions. By K we denote the subclass of functions h in L such that $h(0) = 0$, $h(t) = o(t)$ as $t \to \infty$, $\frac{h(t)}{t} \downarrow$ as $t \uparrow$ ($\frac{h(t)}{t}$ decays monotonically as $t > 0$). In particular, if $h \in K$, then $h(2t) \leqslant 2h(t)$ $(t > 0)$, $h(t) \leqslant h(1)t$ as $t \geqslant 1$.

K-density of sequence Λ is the quantity

$$G(K) = \inf_{h\in K} \overline{\lim_{t\to\infty}} \frac{\mu_\Lambda(\omega(t))}{h(t)}, \tag{7}$$

where $\omega(t) = [t, t + h(t))$ is a semi-interval, $\mu_\Lambda(\omega(t))$ is the number of points in Λ located in semi-interval $\omega(t)$.

Let $\Omega = \{\omega\}$ be the family of semi-intervals $\omega = [a, b)$. By $|\omega|$ we denote the length of ω. Each sequence $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ generates an integer-valued counting measure μ_Λ:

$$\mu_\Lambda(\omega) = \sum_{\lambda_n\in\omega} 1, \quad \omega \in \Omega.$$

Let μ_Γ be a counting measure generated by sequence $\Gamma = \{\mu_n\}$, $(0 < \mu_n \uparrow \infty.)$ Then inclusion $\Lambda \subset \Gamma$ means that $\mu_\Lambda(\omega) \leqslant \mu_\Gamma(\omega)$ for each $\omega \in \Omega$. In this case we say that μ_Γ majorizes measure μ_Λ.

By $D(K)$ we denote the infimum of numbers b, $(0 \leqslant b < \infty)$ such that there exists measure μ_Γ majorizing μ_Λ such that for some function $h \in K$

$$|M(t) - bt| \leqslant h(t) \qquad (t \geqslant 0). \tag{8}$$

Here $\Lambda = \{\lambda_n\}$, $\Gamma = \{\mu_n\}$, $M(t) = \sum\limits_{\mu_n\leqslant t} 1$.

Lemma 1 ([6]). *Quantities $D(K)$ and $G(K)$ coincide: $D(K) = G(K)$.*

2^0. Existence of entire function with regular behavior on the real axis. Let L and K be the classes of functions introduced above,

$$S = \left\{ h \in K: \ d(h) = \varlimsup_{x\to\infty} \frac{h(x)\ln h(x)}{x \ln \frac{x}{h(x)}} < \infty \right\}.$$

Theorem II [6]. *Let $\Lambda = \{\lambda_n\}$, $(0 < \lambda_n \uparrow \infty)$ be a sequence having a finite S-density $G(S)$. Then for each $b > G(S)$ there exists a sequence $\Gamma = \{\mu_n\}$ $(0 < \mu_n \uparrow \infty)$ containing Λ and having density b such that the entire function of exponential type πb*

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\mu_n^2}\right) \quad (z = x + iy) \tag{9}$$

possesses the properties:

1) $Q(\lambda_n) = 0$, $Q'(\lambda_n) \neq 0$ for each $\lambda_n \in \Lambda$;

2) there exists $H \in S$ such that

$$\ln|Q(x)| \leqslant AH(x)\ln^+ \frac{x}{H(x)} + B; \tag{10}$$

3) if $\Lambda(x) = \sum\limits_{\lambda_n \leqslant x} 1$ and

$$\Lambda(x+\rho) - \Lambda(x) \leqslant a\rho + b + \frac{\varphi(x)}{\ln^+ \rho + 1} \quad (\rho \geqslant 0) \tag{11}$$

(φ is an arbitrary nonnegative non-decaying function defined on the ray $[0,\infty)$, $1 \leqslant \varphi(x) \leqslant \alpha x \ln^+ x + \beta$), then there exists a sequence $\{r_n\}$, $0 < r_n \uparrow \infty$, $r_{n+1} - r_n = O(H(r_n))$ as $n \to \infty$ such that for $x = r_n$, $(n \geqslant 1)$

$$\ln|Q(x)| \geqslant -CH(x)\ln^+ \frac{x}{H(x)} - 2\varphi(x) - D; \tag{12}$$

4) if

$$\Delta = \varlimsup_{n\to\infty} \frac{1}{\lambda_n} \int_0^1 \frac{n(\lambda_n;t)}{t} dt < \infty,$$

then under condition (11)

$$\left| \ln\left|\frac{1}{Q'(\lambda_n)}\right| - \int_0^1 \frac{n(\lambda_n;t)}{t}dt \right| \leqslant EH(\lambda_n)\ln^+ \frac{\lambda_n}{H(\lambda_n)} + 2\varphi(\lambda_n) + F\ln\lambda_n + L \quad (n \geqslant 1), \tag{13}$$

where $n(\lambda_n;t)$ is the number of points $\lambda_k \neq \lambda_n$ in the segment $\{x : |x - \lambda_n| \leqslant t\}$.

Here all the constants are positive and finite.

We note that condition $\Delta < \infty$ is not an implication of estimate (11) even if function φ is bounded. Indeed, let $\sup\limits_{x\geqslant 0} \varphi(x) < \infty$, $0 \leqslant \rho \leqslant 1$. Then it follows from (11) that $\Lambda(x+\rho) - \Lambda(x) \leqslant C < \infty$, $(x \geqslant 0)$. Thus, if $h_n = \min\limits_{k\neq n} |\lambda_k - \lambda_n|$, then

$$\ln^+ \frac{1}{h_n} \leqslant \int_0^1 \frac{n(\lambda_n;t)}{t}dt \leqslant 2C \ln^+ \frac{1}{h_n}.$$

Since in this case $\Delta < \infty$ if and only if

$$\varlimsup_{n\to\infty} \frac{1}{\lambda_n} \ln^+ \frac{1}{h_n} < \infty.$$

If function φ is unbounded, under condition (11) there possible the situation

$$\sup_{x\geqslant 0}[\Lambda(x+1)-\Lambda(x)]=\infty.$$

Let us prove several technical lemmata. In order to do it we introduce the following notations: $\ln_0 t = t$, $\exp_0 t = t$, $\ln_k t = \underbrace{\ln\ln\ldots\ln}_{k} t$, $\exp_k t = \underbrace{\exp\exp\ldots\exp}_{k} t$ $(k\geqslant 1)$.

We consider the series

$$\sum_{n=1}^{\infty} e^{-\varepsilon\frac{\lambda_n}{\ln_m\lambda_n}},\ \lambda_n\uparrow\infty,\ \varepsilon>0,\ m\geqslant 1. \tag{14}$$

Lemma 2. *Series (14) converges for each $\varepsilon>0$ if and only if*

$$\lim_{n\to\infty}\frac{\ln n\ln_m\lambda_n}{\lambda_n}=0.$$

Proof. 2^0. *Necessity.* Suppose that series (14) converges for each $\varepsilon>0$. Since the terms of the series decays monotonically as $n\geqslant n_0$, then, as it is known,

$$\lim_{n\to\infty} ne^{-\varepsilon\frac{\lambda_n}{\ln_m\lambda_n}}=0.$$

Thus, for each $\varepsilon>0$ there exists $N(\varepsilon)$ such that for each $n\geqslant N(\varepsilon)$ the estimate

$$ne^{-\varepsilon\frac{\lambda_n}{\ln_m\lambda_n}}<1$$

is valid. Hence, as $n\geqslant N(\varepsilon)$

$$\ln n<\varepsilon\frac{\lambda_n}{\ln_m\lambda_n}$$

that completes the proof.

2^0. *Sufficiency.* Suppose that

$$\lim_{n\to\infty}\frac{\ln n\ln_m\lambda_n}{\lambda_n}=0.$$

Then for each $\varepsilon>0$ there exists $N(\varepsilon)$ such that as $n\geqslant N(\varepsilon)$

$$\frac{\ln n\ln_m\lambda_n}{\lambda_n}\leqslant\frac{\varepsilon}{2}.$$

Then for each $n\geqslant N(\varepsilon)$

$$e^{-\varepsilon\frac{\lambda_n}{\ln_m\lambda_n}}\leqslant\left(\frac{1}{n}\right)^2.$$

Hence, series (14) converges for each $\varepsilon>0$. The proof is complete. □

We consider the function

$$\varphi_m(t)=q\frac{t}{\ln_m t}-t\sigma\quad(m\geqslant 1,\ q,\sigma>0).$$

This function is defined for $t>\exp_{m-1}(0)$ except the point $p_0=\exp_m(0)$ at which the logarithm vanishes.

Let t_0 the a solution to the equation

$$\frac{q}{\ln_m t_0}=\sigma\quad(0<\sigma\leqslant 1),$$

and $\max\limits_{t\geqslant p}\varphi_m(t)=\varphi(t_\ni)$, where $p=\exp_m q$. Since $\frac{q}{\ln_m p}=1$, then $p\leqslant t_\ni\leqslant t_0$. Thus, $\varphi_m(t_\ni)\leqslant q\frac{t_\ni}{\ln_m t_\ni}\leqslant t_\ni\leqslant t_0=\exp_m\left(\frac{q}{\sigma}\right)$.

Therefore,

$$\max_{t\geqslant p}\varphi_m(t)\leqslant\exp_m\left(\frac{q}{\sigma}\right).$$

Hence, we have proven

Lemma 3. *As* $0 < \sigma \leqslant 1$*, function* $\varphi_m(t)$ *satisfies the estimate*

$$\max_{t \geqslant p} \varphi_m(t) \leqslant \exp_m \left(\frac{q}{\sigma}\right), \quad p = \exp_m q.$$

Let Q be an entire functions of exponential type (9), and γ be the function associated with it in Borel sense. The following lemma holds true.

Lemma 4. *There exists a nonnegative majorant* g *for the function* $\ln |Q(x)|$ *on* $[a, \infty)$ *satisfying the condition*

$$\lim_{x \to \infty} \frac{g(x) \ln_{k-1} x}{x} = 0 \quad (k \geqslant 2), \tag{15}$$

if and only if

$$\overline{\lim_{\delta \to 0_+}} \, \delta \ln_k |\gamma(t)| \leqslant 0 \quad (k \geqslant 2), \quad \delta = |\mathrm{Re}\, t|. \tag{16}$$

Proof. 1^0. *Necessity.* As $x \geqslant x_0 \geqslant 1$ for each $\varepsilon > 0$ we have

$$|Q(x)| \leqslant \exp\left(\varepsilon \frac{x}{\ln_{k-1} x}\right).$$

Therefore, letting $\delta = |\mathrm{Re}\, t|$, we obtain

$$|\gamma(t)| \leqslant \int_0^\infty |Q(x)| e^{-\delta x} dx \leqslant A + B \int_{x_0}^\infty \exp\left(\varepsilon \frac{x}{\ln_{k-1} x} - \delta x\right) dx.$$

It implies

$$|\gamma(t)| \leqslant A + B \exp\left[\max_{x \geqslant x_0} \left(\varepsilon \frac{x}{\ln_{k-1} x} + 2 \ln x - \delta x\right)\right]. \tag{17}$$

But

$$\exp\left[\max_{x \geqslant x_0} (\varepsilon \frac{x}{\ln_{k-1} x} + 2 \ln x - \delta x)\right] \leqslant$$
$$\leqslant B_1(\varepsilon) \exp\left[\max_{x \geqslant x_0} (2\varepsilon \frac{x}{\ln_{k-1} x} - \delta x)\right].$$

Applying Lemma 3 to the expression in the square brackets, by (17) we finally get

$$|\gamma(t)| \leqslant C(\varepsilon) \exp_k \left(\frac{2\varepsilon}{\delta}\right) \quad (0 < \delta \leqslant 1).$$

Thus, condition (16) indeed holds true.

2^0. *Sufficiency.* The sequence of all zeroes of function Q has density b. Therefore, the type of function Q is equal to πb (the adjoint diagram of Q is the segment $[-\pi b i, \pi b i]$). Then, function Q is even. For $x > 0$ we have

$$Q(x) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \gamma(t) e^{xt} dt, \tag{18}$$

where Γ_δ the boundary of rectangle with the sides on the lines $\mathrm{Re}\, t = \pm\delta$ ($0 < \delta \leqslant 1$, $\mathrm{Im}\, t = \pm(\pi b + 1)$. In view of (16) by (18) we obtain that for each $\varepsilon > 0$ as $0 < \delta \leqslant \delta_0(\varepsilon)$

$$|Q(x)| \leqslant C_\varepsilon \exp_{[k-1]} \left[e^{\varepsilon \delta^{-1}} + \delta x\right] \quad (0 < C_\varepsilon < \infty). \tag{19}$$

Estimate (19) is valid for each $\delta \in (0, \delta_0(\varepsilon)]$. In view of this fact, we let

$$\delta^{-1} = \varepsilon^{-1} \ln_{k-1} x^\alpha, \quad \alpha = \alpha(x) = 1 - \frac{\ln(\ln_{k-1} x)^2}{\ln x}.$$

We see that $\alpha(x) \to 1$ as $x \to \infty$, and as $\delta \to 0_+$

$$x^\alpha = \frac{x}{\ln_{k-1}^2 x} \to \infty.$$

Substituting the chosen value of δ^{-1} into (19), as $x \geqslant x_0(\varepsilon)$ we obtain that

$$\ln |Q(x)| \leqslant \ln C_\varepsilon + \frac{x}{\ln_{k-1}^2 x} + \frac{\varepsilon x}{\ln_{k-1}^\alpha}. \tag{20}$$

We check that as $x \to \infty$,

$$\ln_{k-1} x^\alpha = (1 + o(1)) \ln_{k-1} x \ \ (k \geqslant 2).$$

In view of this identity, by (20) we finally get

$$\ln |Q(x)| \leqslant 2\varepsilon \frac{x}{\ln_{k-1} x}, \quad x \geqslant x_1(\varepsilon).$$

It means that

$$\ln |Q(x)| \leqslant g(x) \quad (x \geqslant 0)$$

for some nonnegative (and non-decaying) function g satisfying condition (15). □

2. Formula for calculation k-order for sum of Dirichlet series

We shall call the quantity

$$\rho_k = \overline{\lim_{\sigma \to 0_-}} \frac{\ln_k M(\sigma)}{|\sigma|^{-1}} \quad (k \geqslant 2) \tag{21}$$

k-order of function $F \in D_0(\Lambda)$ in the half-plane $\Pi_0 = \{s : \ \sigma = \operatorname{Re} s < 0\}$. Here $M(\sigma) = \sup\limits_{|t|<\infty} |F(\sigma + it)|$. It makes sense to consider only the functions $F \in D_0(\Lambda)$ satisfyin $\sup\limits_{\sigma<0} M(\sigma) = \infty$. By the definition of k-order (21) we see that $\rho_2 = \rho_R$, where ρ_R is R-order in half-plane Π_0 [4].

The following theorem holds.

Theorem 1. *Condition*

$$\lim_{n \to \infty} \frac{\ln n \ln_{k-1} \lambda_n}{\lambda_n} = 0, \quad (k \geqslant 2), \tag{22}$$

is the criterion for k-order (21) of each function $F \in D_0(\Lambda)$ satisfies the formula

$$\rho_k = \overline{\lim_{n \to \infty}} \frac{\ln |a_n|}{\lambda_n} \ln_{k-1} \lambda_n \quad (k \geqslant 2;\ 0 \leqslant \rho_R \leqslant \infty). \tag{23}$$

Remark. *Suppose that the power series*

$$g(z) = \sum_{n=1}^{\infty} a_n z^{p_n} \quad (p_n \in N) \tag{24}$$

converges in the circle $\{z : \ |z| < 1\}$ and

$$M_g(r) = \max_{|z|=r} |g(z)| \to \infty, \ r \to \infty.$$

We let

$$r_k = \overline{\lim_{r \uparrow \infty}} \frac{\ln_k M_g(r)}{(1-r)^{-1}} \quad (0 \leqslant r < 1).$$

Since $1 - r = (1 + o(1))|\ln r|$ as $r \uparrow 1$, then

$$r_k = \overline{\lim_{r \uparrow \infty}} \frac{\ln_k M_g(r)}{|\ln r|^{-1}} \tag{25}$$

We make the change $z = e^s$. Then

$$f(s) = g(e^s) = \sum_{n=1}^{\infty} a_n e^{p_n s} \quad (s = \sigma + it). \tag{26}$$

It is clear the Dirichlet-Taylor series (26) converges absolutely in half-plane Π_0. Since $r = e^{\sigma}$, then $M_g(r) = M(\sigma)$ and $r_k = \rho_k$, where ρ_k is the k-order of series (26) (one can see it by (25) and the definition of ρ_k).

Let us formulate a corollary of Theorem 1.

Corollary. *k-order r_k of each function g given by (24) satisfies the formula*

$$r_k = \overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{p_n} \ln_{k-1} p_n \quad (k \geqslant 2)$$

if and only if

$$\lim_{n\to\infty} \frac{\ln n \ln_{k-1} p_n}{p_n} = 0 \quad (k \geqslant 2).$$

Proof. 1^0. *Sufficiency.* Let k-order ρ_k of function F is finite. Let us prove that then $\alpha \leqslant \rho_k$, where

$$\alpha \equiv \overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{\lambda_n} \ln_{k-1} \lambda_n \quad (k \geqslant 2).$$

Indeed, by the definition of k-order we obtain that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for $\delta < \sigma < 0$ the inequality

$$\ln M(\sigma) \leqslant \exp_{k-1}\left(\frac{\rho_k + \varepsilon}{|\sigma|}\right). \tag{27}$$

holds true. For $\sigma < 0$ we have $|a_n| \leqslant M(\sigma) e^{\lambda_n|\sigma|}$ $(n \geqslant 1)$. By (27) it follows that as $\delta < \sigma < 0$,

$$\ln|a_n| \leqslant \exp_{k-1}\left(\frac{\rho_k + \varepsilon}{|\sigma|}\right) + \lambda_n|\sigma|.$$

If we let $t = |\sigma|^{-1}$, then

$$\ln|a_n| \leqslant \exp_{k-1}(\rho_k + \varepsilon)t + \frac{\lambda_n}{t}.$$

We let $t = t_*$, where

$$t_* = \frac{1}{\rho_k + \varepsilon} \ln_{k-1} \lambda_n^{\alpha_n}, \quad \alpha_n = 1 - \frac{\ln(\ln_{k-1}\lambda_n)^2}{\lambda_n}.$$

We see that $\alpha_n \to 1$ as $n \to \infty$ and $\lambda_n^{\alpha_n} = \frac{\lambda_n}{\ln_{k-1}^2 \lambda_n}$. Since $t_* = t_*(n) \to \infty$ as $n \to \infty$ and

$$\ln|a_n| \leqslant \exp_{k-1}\left((\rho_k + \varepsilon)t_*\right) + \frac{\lambda_n}{t_*} \quad (n \geqslant N = N(\varepsilon)).$$

It yields that for each $n \geqslant N$

$$\ln|a_n| \leqslant \frac{\lambda_n}{\ln_{k-1}^2 \lambda_n} + \frac{\lambda_n(\rho_k + \varepsilon)}{\ln_{k-1}\lambda_n^{\alpha_n}} \quad (k \geqslant 2). \tag{28}$$

Since by straightforward calculations one can check that for $n \to \infty$

$$\ln_{k-1}\lambda_n^{\alpha_n} = (1 + o(1)) \ln_{k-1}\lambda_n \quad (k \geqslant 2),$$

then as $n \geqslant N_1 \geqslant N$, by (28) we obtain the estimate

$$\frac{\ln|a_n|}{\lambda_n} \ln_{k-1}\lambda_n < \rho_k + 3\varepsilon \quad (k \geqslant 2).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\alpha \leqslant \rho_k$.

Suppose now that $\alpha < \infty$. Let us prove that in this case $\rho_k \leqslant \alpha$. By the definition of α, for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for $n \geqslant N$

$$\frac{\ln |a_n|}{\lambda_n} \ln_{k-1} \lambda_n < \alpha + \varepsilon \ \ (k \geqslant 2).$$

Let $k_0 = \min\{n : \ \lambda_n > p_0 = \exp_{k-2}(0) \ (k \geqslant 2)\}$. We choose $A(\varepsilon)$ such that for each $n \geqslant k_0$ the inequality

$$|a_n| < A(\varepsilon) \exp \left[\frac{(\alpha+\varepsilon)\lambda_n}{\ln_{k-1} \lambda_n}\right] \quad (k \geqslant 2)$$

holds. Then

$$|F(s)| \leqslant \sum_{n=1}^{\infty} |a_n| e^{\lambda_n \sigma} \leqslant B + A(\varepsilon) \sum_{n=k_0}^{\infty} \exp\left(q \frac{\lambda_n}{\ln_{k-1} \lambda_n} - \lambda_n |\sigma|\right)$$

$$\leqslant B + A(\varepsilon) \max_{t \geqslant \lambda_{k_0}} \exp\left(q_1 \frac{t}{\ln_{k-1} t} - t\sigma\right) \sum_{n=k_0}^{\infty} \exp\left(-\varepsilon \frac{\lambda_n}{\ln_{k-1} \lambda_n}\right),$$

where $q = \alpha + \varepsilon$, $q_1 = \alpha + 2\varepsilon$, $\sigma = \operatorname{Re} s < 0$. In view of condition (22), we employ Lemma 2 to obtain

$$\sum_{n=k_0}^{\infty} \exp\left(-\varepsilon \frac{\lambda_n}{\ln_{k-1} \lambda_n}\right) = A_1(\varepsilon) < \infty. \tag{29}$$

Let us estimate the function

$$\varphi(t) = q_1 \frac{t}{\ln_{k-1} t} - t\sigma.$$

It follows from Lemma 3 that

$$\max_{t \geqslant \lambda_{k_0}} \varphi(t) \leqslant \exp_{k-1}\left(\frac{q_1}{\sigma}\right), \quad 0 < |\sigma| \leqslant 1. \tag{30}$$

Thus, due to (29) and (30), we have

$$|F(s)| \leqslant B + A_2(\varepsilon) \exp_k \left(\frac{q_1}{\sigma}\right), \ \ 0 < |\sigma| \leqslant 1.$$

Therefore, as $-1 \leqslant \sigma_0 < \sigma < 0$,

$$|F(s)| \leqslant \exp_k \left(\frac{q_2}{\sigma}\right), \quad q_2 = \alpha + 3\varepsilon.$$

It implies that for each $\varepsilon > 0$

$$\frac{\ln_k M(\sigma)}{|\sigma|^{-1}} \leqslant q_2, \ \ -1 \leqslant \sigma_0 < \sigma < 0.$$

It means that $\rho_k \leqslant \alpha$. Hence, $\alpha = \rho_k$. It yields that $\alpha = \infty$ if and only if $\rho_k = \infty$. The proof of sufficiency is complete.

2^0. *Necessity.* Let us show that condition (22) is necessary for the k-order of each function $F \in D_0(\Lambda)$ to satisfy formula (23). Indeed, suppose that condition (22) fails, i.e.,

$$\varlimsup_{n \to \infty} \frac{\ln n \ln_{k-1} \lambda_n}{\lambda_n} > 0 \quad (k \geqslant 2).$$

Then there exists a subsequence $\{n_m\}$, such that for each $m \geqslant 1$

$$\frac{\ln n_m \ln_{k-1} \lambda_{n_m}}{\lambda_{n_m}} \geqslant \beta > 0. \tag{31}$$

We let $a_n = e$ $(n \geqslant 1)$ and we estimate the k-order of function F defined by the series

$$F(s) = e \sum_{n=1}^{\infty} e^{\lambda_n s} \quad (s = \sigma + it). \tag{32}$$

We assume that the condition
$$\lim_{n\to\infty}\frac{\ln n}{\lambda_n}=0 \tag{33}$$
holds true. Series (32) converges (absolutely by condition (33)) in half-plane Π_0. Calculating k-order by formula (23), we get $\rho_k = 0$. Let us make sure that the k-order of function F is positive. Indeed, since $a_n > 0$, then $M(\sigma) = F(\sigma)$ $(\sigma < 0)$. Thus, for each natural N we have
$$M(\sigma) \geqslant e\sum_{k=[\frac{N}{2}]}^{N} e^{-\lambda_k|\sigma|} \geqslant e\frac{N}{2}e^{-\lambda_N|\sigma|} \geqslant Ne^{-\lambda_N|\sigma|} = \exp(\ln N - \lambda_N|\sigma|). \tag{34}$$
We write condition (31) as
$$\lambda_{n_m} \leqslant \frac{1}{\beta}\ln n_m \ln_{k-1}\lambda_{n_m} \quad (\beta > 0) \tag{35}$$
and we let $N = n_m$ in (34). Then for each $m \geqslant 1$ we have
$$M(\sigma) \geqslant \exp(\ln n_m - \lambda_{n_m}|\sigma|) \geqslant \exp(\ln n_m - \frac{|\sigma|}{\beta}\ln n_m \ln_{k-1}\lambda_{n_m}. \tag{36}$$
We see by (35) that $\ln\lambda_{n_m} \leqslant 2\ln\ln n_m$ as $m \geqslant m_0$. It follows that $\ln_{k-1}\lambda_{n_m} \leqslant 2\ln_k n_m$ as $m \geqslant m_1 \geqslant m_0$. By (36) it yields the estimate
$$M(\sigma) \geqslant \exp(\ln n_m - \frac{2|\sigma|}{\beta}\ln n_m \ln_k n_m) \quad (m \geqslant m_1). \tag{37}$$
In (34), $|\sigma| > 0$ is arbitrary. We let $\sigma = \sigma_m$, where σ_m solves the equation
$$\ln_k n_m = \frac{\beta}{4|\sigma|} \ \ (m \geqslant m_1).$$
Then by (37) we get
$$M(\sigma) \geqslant \exp\left\{\frac{1}{2}\exp_{k-1}\frac{\beta}{4|\sigma|}\right\}, \quad \sigma = \sigma_m.$$
Hence,
$$\ln M(\sigma) \geqslant \frac{1}{2}\exp_{k-1}\frac{\beta}{4|\sigma|}, \ \ \sigma = \sigma_m \ \ (m \geqslant m_1),$$
and
$$\ln\ln M(\sigma) \geqslant \ln\left(\frac{1}{2}\exp_{k-1}\frac{\beta}{4|\sigma|}\right) \geqslant \frac{1}{2}\exp_{k-2}\frac{\beta}{4|\sigma|}, \ \ \sigma = \sigma_m \ \ (m \geqslant m_2).$$
Proceeding in the same way, we finally obtain
$$\ln_k M(\sigma) \geqslant \frac{\beta}{8|\sigma}, \ \ \sigma = \sigma_m \ \ (m \geqslant m_k).$$
It means that $\rho_k \geqslant \frac{\beta}{8}$. The proof is complete. □

3. Two-sided estimate for k-order via k-order in half-strip

Before we formulate the theorem, we introduce the following classes of functions:
$$L_k = \{h \in L : \ h(x)\ln_{k-1}x = o(x), \quad x \to \infty\} \qquad (k \geqslant 2),$$
$$R_k = \{h \in S : \ h(x)\ln\frac{x}{h(x)} = o\left(\frac{x}{\ln_{k-1}x}\right), \quad x \to \infty\} \qquad (k \geqslant 2).$$

Theorem 2. *Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence satisfying the conditions:*

1). *The inequality*

$$\Lambda(x+\rho)-\Lambda(x)\leqslant c\rho+d+\frac{\varphi(x)}{\ln^{+}\rho+1}\quad (\rho\geqslant 0),\tag{38}$$

holds true, where $\Lambda(x)=\sum\limits_{\lambda_n\leqslant x}1$, φ *is a some function in* L_k $(k\geqslant 2)$*;*

2). *The relation*

$$q_k^*=\varlimsup_{n\to\infty}\frac{\ln_{k-1}\lambda_n}{\lambda_n}\int\limits_0^1\frac{n(\lambda_n;t)}{t}dt<\infty\quad (k\geqslant 2),\tag{39}$$

holds true, where $n(\lambda_n;t)$ *is the number of points* $\lambda_k\neq\lambda_n$ *in the segment* $\{x:\ |x-\lambda_n|\leqslant t\}$*.*

If R *is the density of sequence* Λ *is equal to* $G(R)$*, then* k*-order* ρ_s *of each function* $F\in D_0(\Lambda)$ *in strip* $S(a,t_0)$ *as* $a>\pi G(R_k)$ *and order* ρ_R *of this function in half-plane* Π_0 *satisfy the estimates*

$$\rho_s\leqslant\rho_k\leqslant\rho_s+q_k^*\quad (k\geqslant 2).\tag{40}$$

Proof. Since $\varphi\in L_k$, it follows from estimate (38) and the definition of R_k-density that $G(R_k)<\infty$. Indeed, if $p_0=\exp_{k-2}(0)$, $(k\geqslant 2)$, $h(x)=\frac{x}{\ln(x+1)\ln_{k-1}(x+p_0+1)}$, $(x\geqslant 0)$, then one can check easily that $h\in R_k$ and

$$\varlimsup_{t\to\infty}\frac{\mu_\Lambda(\omega(t))}{h(t)}\leqslant c,$$

where c is a constant in condition (38), $\omega(t)=[t,t+h(t))$. Therefore, $G(R_k)\leqslant c<\infty$.

We make use of Theorem II. For each b, $G(R_k)<b<\frac{a}{\pi}$, there exists a sequence $\Gamma=\{\mu_n\}$, $(0<\mu_n\uparrow\infty)$, containing Λ and having density b such that the entire function of exponential type πb

$$Q(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\mu_n^2}\right)\quad (z=x+iy)\tag{41}$$

possesses the properties:

1) $Q(\lambda_n)=0,\quad Q'(\lambda_n)\neq 0\quad (n\geqslant 1)$;

2) $\ln|Q(x)|\leqslant g(x)\ \ (x\geqslant 0),\quad g\in L_k$;

3) $q_k(Q)=q_k^*$, where q_k^* is defined by formula (39) and

$$q_k(Q)=\varlimsup_{n\to\infty}\frac{\ln_{k-1}\lambda_n}{\lambda_n}\ln\left|\frac{1}{Q'(\lambda_n)}\right|\quad (k\geqslant 2).$$

We note that estimate 2) and identity 3) follow from estimates (10) and (11) in view of fact that R_k-densities in estimates (10), (11) obey $H\in R_k$, and $\varphi\in L_k$.

We introduce A.F. Leontiev interpolating function [3]

$$\omega(\mu,\alpha,F)=e^{-\alpha\mu}\frac{1}{2\pi i}\int\limits_C\gamma(t)\left(\int\limits_0^t F(t+\alpha-\eta)e^{\mu\eta}d\eta\right)dt,$$

where $F\in D_0(\Lambda)$, γ is the function associated with entire function Q defined by (41) in the Borel sense, C is a closed contour enveloping the segment $I=[-\pi bi,\pi bi]$, which is the adjoint diagram of Q, α is an arbitrary complex parameter $\operatorname{Re}\alpha<0$. It is clear that $(t+\alpha-\eta)\in C_\alpha$, where C_α is the shift of C by vector α. As C we take the boundary of the rectangle

$$P=\{t:|\mathrm{Re}\ t|\leqslant h(0<h\leqslant 1),\quad |\mathrm{Im}\ t|\leqslant a\},\qquad \pi G(R)<\pi b<a.$$

Let us prove that $\rho_k \leqslant \rho_s + q_k^*$ (estimate $\rho_s \leqslant \rho_k$ is obvious). We have

$$|\omega(\lambda_n, \alpha, F)| \leqslant \frac{2}{\pi}(1+a)^2 |e^{-\alpha\lambda_n}| \max_{\eta \in P} |e^{\lambda_n \eta}| \max_{t \in C} |\gamma(t)| \max_{u \in P_\alpha} |F(u)|.$$

We let $\alpha = \sigma - h + it_0$ ($\sigma < 0$). Applying Lemma 4 and taking into consideration the fact that $|\gamma(t)| \leqslant M$ on the horizontal parts of the contour, as $h < h_0(\delta)$ for each $\delta > 0$ we get

$$|\omega(\lambda_n, \alpha, F)| \leqslant e^{(|\sigma|+2h)\lambda_n} \exp_k(\frac{\delta}{h}) \max_{u \in P_\alpha} |F(u)|. \tag{42}$$

Here P_α is the shift of rectangle P by vector α.

We assume that $\rho_k < \infty$. Then $\rho_s < \infty$. It follows from the definition of k-order ρ_s in the half-strip $S(a, t_0)$ that as $0 < |\sigma| < \sigma_0(\varepsilon)$ for each $\varepsilon > 0$

$$M_s(\sigma) \leqslant \exp_k[(\rho_s + \varepsilon)|\sigma|^{-1}].$$

For $0 < |\sigma| < \sigma_0(\varepsilon)$ it implies

$$\max_{u \in P_\alpha} |F(u)| \leqslant \exp_k[(\rho_s + \varepsilon)|\sigma|^{-1}]. \tag{43}$$

Letting $h = \gamma|\sigma|$, $(0 < \gamma < \infty)$ and taking into consideration (43), by (42) we obtain that

$$|\omega(\lambda_n, \alpha, F)| \leqslant e^{(1+2\gamma)\lambda_n|\sigma|} \exp\left[\exp_k(\frac{\delta}{\gamma|\sigma|}) + \exp_k(\frac{\rho}{|\sigma|})\right], \tag{44}$$

where $\rho = \rho_s + \varepsilon$, $0 < |\sigma| < \sigma_1(\delta, \varepsilon)$, $\gamma > 0$.

Let $\delta = \varepsilon^2$, $\gamma = \varepsilon$. Then employing the formulae for the coefficients [3]

$$a_n = \frac{\omega(\lambda_n, \alpha, F)}{Q'(\lambda_n)} \quad (n \geqslant 1)$$

and bearing in mind (44), we have

$$|a_n| \leqslant \left|\frac{1}{Q'(\lambda_n)}\right| \exp\left[(1+2\varepsilon)\lambda_n t^{-1} + \exp_{k-1}(\rho_1 t)\right],$$

where $t = |\sigma|^{-1}, t > t_0(\varepsilon)$, $\rho_1 = \rho + \varepsilon$. In particular, this inequality holds true as $n \geqslant n_0(\varepsilon)$ for

$$t = \frac{1}{\rho_1} \ln_{k-1}^{\alpha_n} \lambda_n, \quad \alpha_n = 1 - \frac{\ln\ln^2 \lambda_n}{\ln \lambda_n}.$$

For such t we have (cf. (28)):

$$|a_n| \leqslant \left|\frac{1}{Q'(\lambda_n)}\right| \exp\left[\frac{(1+2\varepsilon)\rho_1 \lambda_n}{\ln^{\alpha_n} \lambda_n} + \frac{\lambda_n}{\ln_{k-1}^2 \lambda_n}\right] (n \geqslant n_0(\varepsilon)).$$

Since $\ln_{k-1}^{\alpha_n} \lambda_n = (1 + o(1)) \ln_{k-1} \lambda_n$ as $n \to \infty$, by applying formula (23) for calculating order ρ_k in half-plane, we obtain that $\rho_k \leqslant q_k(Q) + (1+2\varepsilon)(\rho_s + \varepsilon)$. Since $q_k(Q) = q_k^*$, $\varepsilon > 0$, is arbitrary, then $\rho_k \leqslant \rho_s + q_k^*$ and it completes the proof. $\square$

Remark. *In the above theorem, instead of $S(a, t_0)$ one can take a curved half-strip K described by a vertical segment of length $2a$ while its center moves along a curve in half-plane Π_0 having a common point with the imaginary axis. In this case estimates (40) hold true.*

The left estimate in (40) is exact. The exactness of the right estimate as $k = 2$ was proven in [6]. In the general situation this issue will be considered in an independent paper.

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