ISSN 2304-0122 Ufa Mathematical Journal. Vol. 5. No 3 (2013). P. 30-36.

doi:10.13108/2013-5-4-30 UDC 517.5

ON GENERALIZATION OF PALEY-WIENER THEOREM FOR WEIGHTED HARDY SPACES

B.V. VINNITSKII, V.N. DILNYI

Abstract. We consider the Hardy space $H^p_\sigma(\mathbb{C}_+)$ in the half-plane with an exponential weight. In this space we study the analytic continuation from the boundary. In the previous works for the case $p \in (1, 2]$ a result on analytic continuation from the imaginary axis was obtained, and it was a generalization of Paley-Wiener theorem. But for many applications the case $p = 1$ is more interesting. For this case in the paper we obtain estimates for a function satisfying certain standard conditions.

Keywords: weighted Hardy space, Paley-Wiener theorem, angular boundary values.

Mathematics Subject Classification: 30H10, 30E20

1. Introduction

We denote by $H^p(\mathbb{C}_+)$, $1 \leqslant p < +\infty$, the space of functions analytic in $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ obeying

$$\|f\|_{H^p} := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\}^{1/p} < +\infty.$$

A.M. Sedletskii [1] showed that this space coincides with Hardy space $\widetilde{H}^p(\mathbb{C}_+)$ of analytic in $\mathbb{C}_+$ functions for which

$$\|f\|_{\widetilde{H}^p} = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\}^{1/p} < +\infty,$$

and the norms $\|\cdot\|_{H^p}$ and $\|\cdot\|_{\widetilde{H}^p}$ are equivalent. The properties of Hardy space are described rather in detail in [2], [3]. The functions in Hardy space have angular boundary values almost everywhere on $i\mathbb{R}$ which we indicate by $f(iy)$ and $f(iy) \in L^p(\mathbb{R})$. The following Paley-Wiener theorem is well-known [3, p. 94].

Theorem 1. *Function $f_0 \in L^p(i\mathbb{R})$, $1 \leqslant p \leqslant 2$, is an angular boundary function of some function $f \in H^p(\mathbb{C}_+)$ if and only if*

$$\int_{-\infty}^{+\infty} f_0(it)e^{i\tau t}dt = 0$$

for almost all negative numbers τ.

B.V. Vinnitskii, V.N. Dilnyi, On generalization of Paley-Wiener theorem for weighted Hardy spaces.

Submitted September 28, 2013.

B.V. Vinnitskii considered [4] the following generalization of Hardy space. We denote by $H^p_\sigma(\mathbb{C}_+)$, $\sigma \geqslant 0$, $1 \leqslant p < +\infty$, the space of analytic in $\mathbb{C}_+$ function for which

$$\|f\|_{H^p_\sigma} := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|} dr \right\}^{1/p} < +\infty.$$

The functions in this space have angular boundary values almost everywhere on $i\mathbb{R}$ which we denote by $f(iy)$ and $f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R})$.

The following generalization of Paley-Wiener theorem was obtained in [5].

Theorem 2. *A function $f_0 : i\mathbb{R} \to \mathbb{C}$ such that $f_0(it)e^{-\sigma|t|} \in L^p(\mathbb{R})$, $1 < p \leqslant 2$, is an angular boundary function of a function $f \in H^p_\sigma(\mathbb{C}_+)$ if and only if there exists a function f_2 such that*

1) $f_2 \in H^p_{2\sigma}(\mathbb{C}_+)$;

2) $f_3(iv) := f_1(iv) + f_2(iv) \in L^p(-\infty;0)$, $f_1(iv) := f_0(iv)e^{-\sigma v}$;

3) for almost each $\tau < 0$

$$\int_0^{+\infty} f_1(iv)e^{i\tau v}dv + \frac{1}{i}\int_0^{+\infty} f_2(u)e^{\tau u}du + \int_{-\infty}^0 f_3(iv)e^{i\tau v}dv = 0.$$

Theorem 1 is a particular case of Theorem 2 for the case $\sigma = 0$. In studies of cyclicity in some spaces and the properties of convolution equations (see [6], [7]) there appear an issue of validity of Theorem 2 for the case $p = 1$. It was shown in [5] that the necessary conditions of this theorem are true also for the case $p = 1$, but the question on sufficiency remains open. The aim of this paper is to describe the functions whose angular boundary functions satisfy conditions 1)-3) of the previous theorem.

2. Main result

We obtain the following statement.

Theorem 3. *If $f_0 : i\mathbb{R} \to \mathbb{C}$, $f_0(it)e^{-\sigma|t|} \in L^1(\mathbb{R})$ and conditions 1)–3) of Theorem 2 are satisfied, then there exists analytic in $\mathbb{C}_+$ function f, for which function f_0 is an angular boundary function and*

$$\sup\left\{ \int_0^{+\infty} \left|f(re^{i\varphi})\right| e^{-\sigma r|\sin\varphi|}dr : \varphi \in (-\pi/2;\pi/2)\backslash(-\delta;\delta) \right\} < +\infty \tag{1}$$

for each $\delta \in (0;\pi/2)$.

The proof is based mainly on the following auxiliary statements.

Lemma 1. *If the hypothesis of Theorem 3 holds and*

$$\Xi(z) := \frac{1}{2\pi}\int_0^{+\infty} \frac{f_1(iv)}{iv-z}dv + \frac{1}{2\pi}\int_{-\infty}^0 \frac{f_3(iv)}{iv-z}dv + \frac{1}{2\pi i}\int_0^{+\infty} \frac{f_2(u)}{u-z}du,$$

then $\Xi(z) = 0$ for each $z \in \mathbb{C}_- := \{z : \mathrm{Re} z < 0\}$.

Proof. We denote

$$\eta_1(\tau) = \int_0^{+\infty} f_1(iv)e^{i\tau v}dv,\ \eta_2(\tau) = \frac{1}{i}\int_0^{+\infty} f_2(u)e^{\tau u}du,\ \eta_3(\tau) = \int_{-\infty}^0 f_3(iv)e^{i\tau v}dv.$$

Then it follows from Condition 3) that

$$\int_{-\infty}^{0} e^{-\tau z}\left(\eta_1(\tau)+\eta_2(\tau)+\eta_3(\tau)\right)d\tau = 0, \quad \mathrm{Re} z<0. \tag{2}$$

But by Fubini theorem we have for $\mathrm{Re} z<0$

$$\int_{-\infty}^{0} e^{-\tau z}\eta_1(\tau)d\tau = \int_{0}^{+\infty} f_1(iv)\int_{-\infty}^{0} e^{\tau(iv-z)}d\tau dv = \int_{0}^{+\infty}\frac{f_1(iv)}{iv-z}dv.$$

In the same way

$$\int_{-\infty}^{0} e^{-\tau z}\eta_3(\tau)d\tau = \int_{-\infty}^{0}\frac{f_3(iv)}{iv-z}dv, \quad \int_{-\infty}^{0} e^{-\tau z}\eta_2(\tau)d\tau = \frac{1}{i}\int_{0}^{+\infty}\frac{f_2(u)}{u-z}du.$$

This is why by (2) we complete the proof. □

We introduce the notation $\mathbb{C}(\alpha;\beta)=\{z:\alpha<\arg z<\beta\}$.

Lemma 2. *If the hypothesis of Theorem 3 holds, then the angular boundary values of the function*

$$f(z)=\begin{cases} -e^{-i\sigma z}\Xi(z), z\in\mathbb{C}(0;\pi/2),\\ -e^{-i\sigma z}\left(\Xi(z)-f_2(z)\right), z\in\mathbb{C}(-\pi/2;0),\end{cases}$$

coincide with function f_0 almost everywhere on $i\mathbb{R}$.

Proof. By Lemma 1 we have $\Xi(-\bar z)=0$, $z\in\mathbb{C}_+$. This is why

$$\begin{aligned}\Xi(z) =&\Xi(z)-\Xi(-\bar z)=\frac{1}{2\pi}\int_{0}^{+\infty} f_1(iv)\left(\frac{1}{iv-z}-\frac{1}{iv+\bar z}\right)dv\\ &+\frac{1}{2\pi}\int_{-\infty}^{0} f_3(iv)\left(\frac{1}{iv-z}-\frac{1}{iv+\bar z}\right)dv+\frac{1}{2\pi i}\int_{0}^{+\infty} f_2(u)\left(\frac{1}{u-z}-\frac{1}{u+\bar z}\right)du\\ =&\frac{1}{2\pi}\int_{0}^{+\infty} f_1(iv)\frac{2x}{(iv-z)(iv+\bar z)}dv+\frac{1}{2\pi}\int_{-\infty}^{0} f_3(iv)\frac{2x}{(iv-z)(iv+\bar z)}dv\\ &+\frac{1}{2\pi i}\int_{0}^{+\infty} f_2(u)\frac{2x}{(u-z)(u+\bar z)}du.\end{aligned}$$

The latter integral vanishes [1] on the imaginary axis except possibly the point $z=0$. It is easy to see that

$$\frac{1}{2\pi}\int_{0}^{+\infty} f_1(iv)\frac{2x}{(iv-z)(iv+\bar z)}dv = -\frac{1}{\pi}\int_{0}^{+\infty} f_1(iv)\frac{x}{(v-y)^2+x^2}dv$$

is the Poisson integral and thus it possesses angular boundary values in $\mathbb{C}_+$ almost everywhere on $\partial\mathbb{C}_+$ and these values are equal to $-f_1(iv)$ for $v>0$ and 0 for $v<0$. In the same way one can show that angular boundary values in $\mathbb{C}_+$ on $\partial\mathbb{C}_+$ of the function

$$\frac{1}{2\pi}\int_{-\infty}^{0} f_3(iv)\frac{2x}{(iv-z)(iv+\bar z)}dv$$

are equal to $-f_3(iv)$ for $v<0$ and vanish for $v>0$ almost everywhere. $\square$

Lemma 3. *If the hypothesis of Theorem 3 holds, then the function f defined in Lemma 2 satisfies condition (1).*

Proof. It follows from Lemma 1 that for $\mathrm{Re} z>0$

$$0=\frac{1}{2\pi}\int_0^{+\infty}\frac{f_1(iv)}{iv+z}dv+\frac{1}{2\pi}\int_{-\infty}^{0}\frac{f_3(iv)}{iv+z}dv+\frac{1}{2\pi i}\int_0^{+\infty}\frac{f_2(u)}{u+z}du,$$

$$0=\frac{1}{2\pi}\int_0^{+\infty}\frac{f_1(iv)}{iv+\overline{z}}dv+\frac{1}{2\pi}\int_{-\infty}^{0}\frac{f_3(iv)}{iv+\overline{z}}dv+\frac{1}{2\pi i}\int_0^{+\infty}\frac{f_2(u)}{u+\overline{z}}du.$$

By the above identities we obtain that for $z=x+iy\in\mathbb{C}_+$, $y\neq 0$,

$$\begin{aligned}0=&\frac{1}{2\pi}\int_0^{+\infty}f_1(iv)\left(-\frac{1}{iv+\overline{z}}+\frac{x}{iy}\left(\frac{1}{iv+\overline{z}}-\frac{1}{iv+z}\right)\right)dv\\&+\frac{1}{2\pi}\int_{-\infty}^{0}f_3(iv)\left(-\frac{1}{iv+\overline{z}}+\frac{x}{iy}\left(\frac{1}{iv+\overline{z}}-\frac{1}{iv+z}\right)\right)dv\\&+\frac{1}{2\pi i}\int_0^{+\infty}f_2(u)\left(-\frac{1}{u+\overline{z}}+\frac{x}{iy}\left(\frac{1}{u+\overline{z}}-\frac{1}{u+z}\right)\right)du.\end{aligned}$$

Hence, denoting

$$\chi(w;z):=\frac{2}{\pi i}\frac{wx}{(w+\bar{z})(w-z)(w+z)}=\frac{1}{2\pi i}\left(\frac{1}{w-z}-\frac{1}{w+\bar{z}}+\frac{x}{iy}\left(\frac{1}{w+\bar{z}}-\frac{1}{w+z}\right)\right),$$

we obtain

$$\Xi(z)=i\int_0^{+\infty}f_1(iv)\chi(iv;z)dv+i\int_{-\infty}^{0}f_3(iv)\chi(iv;z)dv+\int_0^{+\infty}f_2(u)\chi(u;z)du.$$

First we consider the case when $0<\varphi<\pi/2$. Then

$$\begin{aligned}\int_0^{+\infty}\left|f(re^{i\varphi})\right|e^{-\sigma r|\sin\varphi|}dr&=\int_0^{+\infty}\left|\Xi(re^{i\varphi})\right|dr\leqslant\int_0^{+\infty}\left|\int_0^{+\infty}f_1(iv)\chi(iv;re^{i\varphi})dv\right|dr\\&+\int_0^{+\infty}\left|\int_{-\infty}^{0}f_3(iv)\chi(iv;re^{i\varphi})dv\right|dr+\int_0^{+\infty}\left|\int_0^{+\infty}f_2(u)\chi(u;re^{i\varphi})du\right|dr\\&\leqslant\int_0^{+\infty}\int_0^{+\infty}\left|f_1(iv)\chi(iv;re^{i\varphi})\right|drdv+\int_{-\infty}^{0}\int_0^{+\infty}\left|f_3(iv)\chi(iv;re^{i\varphi})\right|drdv\\&+\int_0^{+\infty}\int_0^{+\infty}\left|f_2(u)\chi(u;re^{i\varphi})\right|drdu.\end{aligned}$$

It is clear that

$$\frac{\pi}{2}\int\limits_0^{+\infty}\left|\chi(iv;re^{i\varphi})\right|dr=\int\limits_0^{+\infty}\frac{|v|\,r\cos\varphi dr}{(v^2-2vr\sin\varphi+r^2)\sqrt{v^2+2vr\sin\varphi+r^2}}dr.$$

If $v>0$, then

$$\int\limits_0^{+\infty}\frac{|v|\,r\cos\varphi dr}{(v^2-2vr\sin\varphi+r^2)\sqrt{v^2+2vr\sin\varphi+r^2}}=\int\limits_0^{+\infty}\frac{s\cos\varphi ds}{(1-2s\sin\varphi+s^2)\sqrt{1+2s\sin\varphi+s^2}}.$$

Since $\varphi\in(0;\pi/2)$, employing inequality $s<\sqrt{s^2+1}$, we have

$$\int\limits_0^{+\infty}\frac{s\cos\varphi ds}{(1-2s\sin\varphi+s^2)\sqrt{1+2s\sin\varphi+s^2}}\le\int\limits_0^{+\infty}\frac{\cos\varphi ds}{1-2s\sin\varphi+s^2}=\frac{\pi}{2}+\arctan\tan\varphi\le\pi.$$

If $v<0$, then

$$\int\limits_0^{+\infty}\frac{|v|\,r\cos\varphi dr}{(v^2-2vr\sin\varphi+r^2)\sqrt{v^2+2vr\sin\varphi+r^2}}=\int\limits_0^{+\infty}\frac{s\cos\varphi ds}{(1+2s\sin\varphi+s^2)\sqrt{1-2s\sin\varphi+s^2}}$$

$$\le\int\limits_0^{+\infty}\frac{s\cos\varphi ds}{(1+s^2)\sqrt{2s(1-\sin\varphi)}}=\int\limits_0^{+\infty}\frac{s2\sin\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)ds}{(1+s^2)\sqrt{4s\sin^2\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}}$$

$$\le\int\limits_0^{+\infty}\frac{\sqrt{s}\cos\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)ds}{1+s^2}\le\int\limits_0^{+\infty}\frac{\sqrt{s}ds}{1+s^2}=\pi\frac{\sqrt{2}}{2}.$$

Thus, by Fubini theorem for $\varphi\in(0;\pi/2)$ we obtain

$$\int\limits_0^{+\infty}dr\int\limits_0^{+\infty}\left|f_1(iv)\chi(iv;re^{i\varphi})\right|dv\le c<+\infty,\qquad\int\limits_0^{+\infty}dr\int\limits_0^{+\infty}\left|f_3(iv)\chi(iv;re^{i\varphi})\right|dv\le c<+\infty,$$

For $u>0$ we also get

$$\frac{\pi}{2}\int\limits_0^{+\infty}\left|\chi(u;re^{i\varphi})\right|dr=\int\limits_0^{+\infty}\frac{ux}{|(u+\bar{z})(u-z)(u+z)|}dr$$

$$=\int\limits_0^{+\infty}\frac{ur\cos\varphi}{(u^2+2ur\cos\varphi+r^2)\sqrt{u^2-2ur\cos\varphi+r^2}}dr$$

$$=\int\limits_0^{+\infty}\frac{s\cos\varphi}{(s^2+2s\cos\varphi+1)\sqrt{s^2-2s\cos\varphi+1}}dr$$

$$=\int\limits_0^{2}\frac{2}{\sqrt{s^2-2s\cos\varphi+1}}dr+\int\limits_2^{+\infty}\frac{s}{(s^2+1)\sqrt{(s-\cos\varphi)^2+\sin^2\varphi}}dr$$

$$\leqslant 2\int\limits_0^{2}\frac{1}{\sqrt{(s-\cos\varphi)^2+\sin^2\varphi}}dr+\int\limits_2^{+\infty}\frac{1}{s\sqrt{2(s-\cos\varphi)\sin\varphi}}dr$$

$$\leqslant 2\int_0^2 \frac{1}{|\sin\varphi|}dr + \frac{1}{\sqrt{2\sin\varphi}}\int_2^{+\infty}\frac{1}{(s-\cos\varphi)^{3/2}}dr < c_2.$$

Thus, we arrive at

$$\int_0^{+\infty} dr \int_0^{+\infty} \left|f_2(u)\chi(u;re^{i\varphi})\right| du \le c_1 < +\infty, \quad \varphi\in(-\pi/2+\delta;\pi/2-\delta)\,. \tag{3}$$

Consider case $-\pi/2<\varphi<0$. We have

$$\int_0^{+\infty}\left|f(re^{i\varphi})\right|e^{-\sigma r|\sin\varphi|}dr \leqslant \int_0^{+\infty}\left|\Xi(re^{i\varphi})\right|dr + \int_0^{+\infty}\left|f_2(re^{i\varphi})\right|e^{-2\sigma r|\sin\varphi|}dr$$
$$\leqslant \int_0^{+\infty}|f_1(iv)|\int_0^{+\infty}\left|\chi(iv;re^{i\varphi})\right|drdv + \int_{-\infty}^{0}|f_3(iv)|\int_0^{+\infty}\left|\chi(iv;re^{i\varphi})\right|drdv$$
$$+\int_0^{+\infty}|f_2(u)|\int_0^{+\infty}\left|\chi(u;re^{i\varphi})\right|drdu + c.$$

By analogy with the case $0<\varphi<\pi/2$ one can show that if $v>0$, then

$$\int_0^{+\infty}\frac{|v|\,r\cos\varphi dr}{(v^2-2vr\sin\varphi+r^2)\sqrt{v^2+2vr\sin\varphi+r^2}} \le \pi\frac{\sqrt{2}}{2},$$

while if $v>0$, then

$$\int_0^{+\infty}\frac{|v|\,r\cos\varphi dr}{(v^2+2vr\sin\varphi+r^2)\sqrt{v^2-2vr\sin\varphi+r^2}} \le \pi.$$

Taking into consideration also (3), we prove (1) for $\varphi\in(-\pi/2;-\delta)$. □

Proof of Theorem 3 follows from Lemmata 1–3 if one takes into consideration that function f introduced in Lemma 2 is analytic in $\mathbb{C}_+$. Indeed, it is obvious that the function

$$\frac{1}{2\pi}\int_0^{+\infty}\frac{f_1(iv)}{iv-z}dv + \frac{1}{2\pi}\int_{-\infty}^{0}\frac{f_3(iv)}{iv-z}dv$$

is analytic in $\mathbb{C}_+$. Function

$$\frac{1}{2\pi i}\int_0^{+\infty}\frac{f_2(u)}{u-z}du - f_2(z)$$

is analytic in the angles $\mathbb{C}(0;\pi/2)$ and $\mathbb{C}(-\pi/2;0)$. On the positive real semi-axis the boundary values from the upper and the lower angle coincide by Sochocki theorem. Hence, as in the proof of Lemma 8 in [7], we obtain easily that f is analytic in $\mathbb{C}_+$.

BIBLIOGRAPHY

1. A.M. Sedleckiĭ. *An equivalent definition of H^p spaces in the half-plane and some applications* // Matem. Sbornik. 1975. V. 96, No. 1. P. 75-82. [Math. USSR Sb. 1975. V. 25, No. 1. P. 69-76.]
2. P. Koosis. *Introduction to H^p spaces. With an appendix on Wolff's proof of the Corona theorem.* Cambridge University Press, Cambridge, 1980.
3. J.B. Garnett. *Bounded analytic functions.* Academic Press, New York, 1981.
4. B.V. Vinnitskii. *On zeros of functions analytic in a half plane and completeness of systems of exponents* // Ukrain. Matem. Zhurn. 1994. V. 46, No. 5. P. 484-500. [Ukrainian Math. J. 1994. V. 46, No. 5. P. 514-532.]
5. B.V. Vinnitskii. *An extention of Paley-Wiener theorem* // Matematychni Studii. 1995. V. 4. P. 37–44. (in Ukrainian.)
6. B. Vinnitskii, V. Dil'nyi *A generalization of the Beurling-Lax theorem* // Matem. Zametki. 2006. V. 79, No. 3. P. 362-368. [Math. Notes. 2006. V. 79, No. 3-4. P. 335-341.]
7. V. Dilnyi *On cyclic functions in weighted hardy spaces* // J. Math. Phys. Anal. Geom. 2011. V. 7, No. 1. P. 19-33.

Bogdan Vasil'evich Vinnitskii,
Drohobych state pedagogical university named after Ivan Franko,
Stryiskaya str., 3
82100, Drohobych, Ukraine

Vladimir Nikolaevich Dilnyi,
Ivan Franko National University of Lviv,
Universytetska str., 1,
82000, Lviv, Ukraine
E-mail: dilnyi@mail.ru