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CONVERGENCE DOMAIN FOR SERIES OF EXPONENTIAL POLYNOMIALS

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Abstract. In this paper we study the convergence of exponential polynomials series constructed by almost exponential sequences of such polynomials. Particular cases of such series are series of exponential monoms, exponential series, Dirichlet series and power series. We obtain an analogue of Abel theorem for these series implying in particular results on continuation of convergence. An analogue of the Cauchy–Hadamard theorem is obtained as well. We give a formula allowing one to recover the convergence domain for these series by their coefficients. The obtained results include Abel and Cauchy–Hadamard theorems for exponential monoms series, exponential series, Dirichlet series and power series.

Keywords: exponential polynomial, convex domain,exponential series, invariant subspace, convergence domain.

Mathematics Subject Classification: 41A05, 4130

1. Introduction

In the work we study the convergence of the series

$$\sum_{k=1}^{\infty} d_k e_k(z), \tag{1.1}$$

where $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence.

For each convex domain $D \subset \mathbb{C}$ we fix a sequence of convex compacts $K(D) = \{K_p\}_{p=1}^{\infty}$ which strictly exhausts it, i.e., $K_p \subset \operatorname{int} K_{p+1}$, $p = 1, 2, \ldots$, (int indicates the interior of a set) and $D = \cup_{p=1}^{\infty} K_p$. Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers such that $|\lambda_k| \to \infty$ as $k \to \infty$, and e_m is an entire function, $m = 1, 2, \ldots$ We shall say (see [1]) that $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence with the indices $\{\lambda_k\}$ if for each convex domain $D \subset \mathbb{C}$ two conditions hold true:

1) for each $p \geqslant 1$ there exists a constant $a > 0$ and an index s such that

$$\sup_{w \in K_p} |e_k(w)| \leqslant a \exp\left(H_{K_s}(\lambda_k)\right), \quad k = 1, 2, \ldots;$$

2) for each $p \geqslant 1$ there exists a constant $b > 0$ and an index s such that

$$b \exp\left(H_{K_p}(\lambda_k)\right) \leqslant \sup_{w \in K_s} |e_k(w)|, \quad k = 1, 2, \ldots$$

Here $H_M(\lambda)$ denotes the support function of a set M (more precisely, of the set complex conjugate with):

$$H_M(\lambda) = \sup_{w \in M} \operatorname{Re}(\lambda w), \quad \lambda \in \mathbb{C}.$$

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Conditions 1) and 2) mean that sequence $\{e_m\}_{m=1}^{\infty}$ is in some sense similar with the sequence of exponentials $\{\exp(\lambda_m z)\}_{m=1}^{\infty}$. Indeed, by Condition 1) and the definition of the support function we obtain the relations

$$\sup_{w\in K_p} |e_m(w)| \leqslant a\exp(H_{K_s}(\lambda_m)) = a\sup_{w\in K_s}\exp(\mathrm{Re}\,(\lambda_m w)) = a\sup_{w\in K_s}|\exp(\lambda_m w)|, \quad k=1,2,\dots$$

Condition 2) gives a similar lower estimate for the modulus of function $e_m(w)$. It is obvious that the mentioned sequence of exponentials is an almost exponential sequence. As an example of the latter let us consider the family of the functions $\{z^n\exp(\lambda_m z)\}_{m=1,n=0}^{\infty,k_m}$. It was shown in Proposition 2.3 of work [2] that under the condition $k_m/|\lambda_m|\to 0$ this family is an almost exponential sequence. The convergence of series of exponential monomials, i.e., the series w.r.t. the elements of such system, were studied in work [3]. There the analogues of Abel and Cachy-Hadamard theorems for the series of exponential monomials were obtained. Almost exponential sequences of a more general form were considered in work [4]. They consist of linear combinations of exponential monomials whose exponents form so-called "relatively small" groups. Such sequences are used in the representation theory for elements invariant w.r.t. the operator of differentiating subspaces of functions analytic in a convex domain (see [5]) and, in particular, the spaces of solutions to homogeneous convolution equations and their systems. In this connection there appears the issue on studying the convergence of series of exponential polynomials constructed by almost exponential sequence of such polynomials. In the present work we study the convergence domains of the mentioned series. For these series we obtain analogues of Abel and Cauchu-Hadamard theorems.

2. Preliminary results

Let D be a convex domain in $\mathbb{C}$, $K(D)=\{K_p\}_{p=1}^{\infty}$, $\Lambda=\{\lambda_k\}_{k=1}^{\infty}$ and $p=1,2,\dots$ Consider the Banach space of complex sequences

$$Q_p(\Lambda)=\{d=\{d_k\}:\|d_p\|=\sup_{k\geqslant 1}|d_k|\exp H_{K_p}(\lambda_k)<\infty\}.$$

By the symbol $Q(\Lambda,D)$ we denote the projective limit of space Q_p, $p\geqslant 1$. The space $Q(\Lambda,D)$ is the intersection of Q_p, $p\geqslant 1$. The topology in $Q(\Lambda,D)$ is equivalent to that defined by the metric

$$\rho(d,d')=\sum_{p=1}^{\infty}2^{-p}\frac{\|d-d'\|_p}{1+\|d-d'\|_p}.$$

Equipped by this metric, space $Q(\Lambda,D)$ obviously becomes Fréchet space.

Let us show that the sequence of the coefficients of converging series (1.1) belongs to space $Q(D)$ for some special convex domain D.

By the symbol $\mathbb{S}$ we shall denote the circle of unit radius centered at the origin. Let E be a set in $\mathbb{C}$, Θ be a closed subspace of the circle $\mathbb{S}$. Θ-convex envelope of E is the set

$$E(\Theta)=\{z\in\mathbb{C}:\mathrm{Re}\,(z\xi)<H_E(\xi),\xi\in\Theta\}.$$

We note that the interior of E lies in $E(\Theta)$. Indeed, if z is an internal point of E, then the definition of the support function imply the inequalities $\mathrm{Re}\,(z\xi)<H_E(\xi)$ for each $\xi\in\Theta$. It means that $z\in E(\Theta)$. In particular, as $\Theta=\mathbb{S}$, Θ-convex envelope of the set coincides with its usual convex envelope (more precisely, with the interior of this convex envelope) and thus it is a convex domain. The latter holds also in the general situation as the next lemma states.

Lemma 2.1. *Let E be a set in $\mathbb{C}$, Θ be a closed subspace of circle $\mathbb{S}$. Then set $E(\Theta)$ is a convex domain.*

Proof. By the definition, set $E(\Theta)$ is the intersection of half-planes and thus it is convex. The convexity implies the connectivity of $E(\Theta)$. It remains to show that $E(\Theta)$ is an open set. Suppose the opposite. Then there exists a point $z_0\in E(\Theta)$ and a sequence $\{z_k\}$ such that

$z_k \to z_0$ as $k \to \infty$ and $z_k \notin E(\Theta)$ for each $k \geqslant 1$, i.e., $\operatorname{Re}(z_k\xi_k) \geqslant H_E(\xi_k)$ for some $\xi_k \in \Theta$, $k = 1, 2, \ldots$ Passing to a subsequence, we can assume that $\{\xi_k\}$ converges to a point $\xi_0 \in \Theta$. Then the last inequality and the lower semi-continuity of the support function imply

$$\operatorname{Re}(z_0\xi_0) = \lim_{k\to\infty} \operatorname{Re}(z_k\xi_k) = \varliminf_{k\to\infty} \operatorname{Re}(z_k\xi_k) \geqslant \varliminf_{k\to\infty} H_E(\xi_k) \geqslant H_E(\xi_0).$$

It contradicts to the definition of $E(\Theta)$, since $z_0 \in E(\Theta)$, and $\xi_0 \in \Theta$. The proof is complete. □

Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$. By $\Theta(\Lambda)$ we denote the set of all partial limits of sequence $\{\lambda_k/|\lambda_k|\}_{k=1}^{\infty}$ except the point $\lambda_k = 0$ if it is present. It is obvious that $\Theta(\Lambda)$ is a closed subset of circle $\mathbb{S}$. By the symbol $B(x, \delta)$ we shall denote an open circle centered at a point x and a radius δ.

Lemma 2.2. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers, $|\lambda_k| \to \infty$ as $k \to \infty$, $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence with indices $\{\lambda_k\}_{k=1}^{\infty}$. Suppose that the terms of series (1.1) are bounded on each compact set K of an open set $E \subset \mathbb{C}$, i.e., $|d_k e_k(z)| \leqslant A$, $k = 1, 2, \ldots$, $z \in K$. Then the inclusion $d = \{d_k\} \in Q(\Lambda, D)$ holds true, where $D = E(\Theta(\Lambda))$.*

Proof. Suppose $d \notin Q(\Lambda, D)$. Then $d \notin Q_p(\Lambda)$ for some index $p = 1, 2, \ldots$ It means that there exists a subsequence $\{d_{k_l}\}$ such that

$$|d_{k_l}| \exp H_{K_p}(\lambda_{k_l}) \to +\infty, \quad l \to \infty. \tag{2.1}$$

Passing once again to a subsequence, we can assume that $\{\lambda_{k_l}/|\lambda_{k_l}|\}$ converges to a point $x_0 \in \Theta(\Lambda)$. Since K_p is a compact set in the domain $D = E(\Theta(\lambda))$, the definition of the set $E(\Theta(\Lambda))$ and the support function imply that for some $z_0 \in E$ the estimate $\operatorname{Re}(z_0x_0) > H_{K_p}(x_0)$ holds true. Then in view of the continuity of the support function for a compact set there exists $\delta > 0$ such that

$$\operatorname{Re}(z_0x) > H_{K_p}(x), \quad x \in B(x_0, \delta). \tag{2.2}$$

By the assumption E is an open set. This is why it contains some circle $\tilde{D}$ centered at point z_0. Let $K(\tilde{D}) = \{\tilde{K}_m\}_{m=1}^{\infty}$. We choose an index s for which compact set $\tilde{K}_s$ contains z_0. Then inequality

$$H_{\tilde{K}_s}(x) \geqslant \operatorname{Re}(z_0x), \quad x \in \mathbb{C}, \tag{2.3}$$

holds.

Since $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence with the indices $\{\lambda_k\}_{k=1}^{\infty}$, there exist a constant $b > 0$ and an index n such that

$$b \exp(H_{\tilde{K}_s}(\lambda_k)) \leqslant \sup_{w\in\tilde{K}_n} |e_k(w)|, \quad k = 1, 2, \ldots \tag{2.4}$$

We choose an index l_0 such that $\lambda_{k_l}/|\lambda_{k_l}| \in B(x_0, \delta)$, $l \geqslant l_0$. Then it follows from inequalities (2.2)-(2.4) and the positive homogeneity of the support function that for each $l \geqslant l_0$

$$\sup_{w\in\tilde{K}_n} |e_{k_l}(w)| \geqslant b \exp(H_{K_p}(\lambda_{k_l})).$$

Thus, by (2.1) we have

$$|d_{k_l}| \sup_{w\in\tilde{K}_n} |e_{k_l}(w)| \to +\infty, \quad l \to \infty.$$

On the other hand, in accordance with the assumption the inequality

$$|d_{k_l}| \sup_{w\in\tilde{K}_n} |e_{k_l}(w)| \leqslant A, \quad l = 1, 2, \ldots$$

holds true. The proof is complete. □

Lemma 2.2 implies of the results of work [1] (Lemma 1).

Corollary. *Let D be a convex domain in $\mathbb{C}$, $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ is a sequence of complex numbers, $|\lambda_k| \to \infty$, $k \to \infty$, $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence with indices $\{\lambda_k\}_{k=1}^{\infty}$. Suppose that series (1.1) converges uniformly on each compact subset of domain D. Then the inclusion $d = \{d_k\} \in Q(\Lambda, D)$ is valid.*

Proof. It is sufficient to observe that $D \subset D(\Theta(\Lambda))$, and thus the inclusion $Q(\Lambda, D(\Theta(\Lambda))) \subset Q(\Lambda, D)$ holds true. $\square$

It was proven in [1] that under the condition $\sigma(\Lambda) = \lim\limits_{k\to\infty} \ln k/|\lambda_k| = 0$ one has the inverse statement for this corollary and even a stronger statement.

Lemma 2.3. *Let D be a convex domain in $\mathbb{C}$, $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers, $|\lambda_k| \to \infty$, $k \to \infty$, $\{e_k\}_{k=1}^{\infty}$ is an almost exponential sequence with the indices $\{\lambda_k\}_{k=1}^{\infty}$ such that $\sigma(\Lambda) = 0$ and $d = \{d_k\} \in Q(\Lambda, D)$. Then for each $p \geqslant 1$ there exist an index s and a constant $C_p > 0$ independent of $d = \{d_k\}$ for which the inequality*

$$\sum_{k=1}^{\infty} |d_k| \sup_{z\in K_p} |e_k(z)| \leqslant C_p \|d\|_s \tag{2.5}$$

holds true. In particular, series (1.1) converges absolutely and uniformly on each compact subset of domain D.

3. Analogue of Abel theorem

The following result is an analogue of Abel theorem for series (1.1).

Theorem 3.1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers, $|\lambda_k| \to \infty$, $k \to \infty$, such that $\sigma(\Lambda) = 0$, $\{e_k\}_{k=1}^{\infty}$ be an almost exponential sequence with indices $\{\lambda_k\}_{k=1}^{\infty}$. Suppose that the terms of series (1.1) are bounded on each compact set K in an open set $E \subset \mathbb{C}$. Then for each $p = 1, 2, \ldots$ there exist an index s and a number $C_p > 0$ (independent of sequences d) such that (2.5) holds true, where the norms $\|d_p\|$ are constructed by the sequence $K(D) = \{K_p\}_{p=1}^{\infty}$ and $D = E(\Theta(\Lambda))$. In particular, series (1.1) converges absolutely and uniformly on each compact subset of domain D.*

Proof. By Lemma 2.2 the inclusion $d = \{d_k\} \in Q(\Lambda, D)$ holds true. By Lemma 2.3 for each $p = 1, 2, \ldots$ there exist an index s and a number $C_p > 0$ (independent of sequence d) such that (2.5) holds true. The proof is complete. $\square$

Remarks. 1. It follows from Theorem 3.1 that under the condition $\sigma(\Lambda) = 0$ the interior of the set of the uniform convergence of series (1.1) is convex and even Θ is a convex domain (i.e., it is the intersection of half-planes $\{z : \mathrm{Re}\,(z\xi) < h(\xi), \xi \in \Theta\}$).

2. If we omit the condition $\sigma(\Lambda) = 0$ from Theorem 3.1, its statement becomes wrong. It was proven in Lemma 4 of work [1] that the convergence of the series

$$\sum_{k=1}^{\infty} |d_k| \sup_{z\in K_p} |e_k(z)|$$

for each sequence $d = \{d_k\} \in Q(\Lambda, D)$, where D is a bounded domain, implies the identity $\sigma(\lambda) = 0$.

4. Analogue of Cauchy-Hadamard theorem

We provide a result being an analogue of Cauchy-Hadamard theorem for power series. Before formulating it, we introduce additional notation. Let $\xi \in \Theta(\Lambda)$. For the sequence of coefficients $d = \{d_k\}$ of series (1.1) we let

$$h(d, \xi) = \inf \varliminf_{j\to\infty} \frac{\ln(1/|d_{k(j)}|)}{|\lambda_{k(j)}|},$$

where the infimum is taken over all subsequences $\{\lambda_{k(j)}\}$ of sequence $\{\lambda_k\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to ξ as $j \to \infty$. Thus, we have obtained the function $h(d, \xi)$, $\xi \in \Theta(\Lambda)$. It is lower semi-continuous. Indeed, let $\xi, \xi_p \in \Theta(\Lambda)$, $p \geqslant 1$, $\xi_p \to \xi$ and a sequence $\{\xi_p\}$ is so that

$$\overline{\lim_{\zeta \to \xi}} h(d, \zeta) = \lim_{p \to \infty} h(d, \xi_p) = a.$$

By the definition of function $h(d, \zeta)$, for each $p \geqslant 1$ we can find a point $\lambda_{k(p)}$ satisfying the conditions $|\lambda_{k(p)}/|\lambda_{k(p)}| - \xi_p| < 1/p$ and $\ln(1/|d_{k(p)}|)/|\lambda_{k(p)}| < a + 1/p$. Then sequence $\lambda_{k(p)}/|\lambda_{k(p)}|$ converges to ξ and

$$\overline{\lim_{p \to \infty}} \frac{\ln(1/|d_{k(p)}|)}{|\lambda_{k(p)}|} \leqslant a.$$

It means that

$$\overline{\lim_{\zeta \to \xi}} h(d, \zeta) \geqslant h(d, \xi),$$

i.e., $h(d, \zeta)$ is lower semi-continuous. Then as in Lemma 2.1 one can show that the set

$$D(d, \Lambda) = \{z : \mathrm{Re}\,(z\xi) < h(d, \xi), \xi \in \Theta(\Lambda)\}$$

is a $\Theta(\Lambda)$-convex domain. By the symbol $\tilde{D}(d, \Lambda)$ we denote the set of points in the plane such that in the vicinity of each point series (1.1) converges uniformly.

Theorem 4.1. *Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ be a sequence of complex numbers, $|\lambda_k| \to \infty$, $k \to \infty$, such that $\sigma(\Lambda) = 0$, $\{e_k\}_{k=1}^{\infty}$ be an almost exponential sequence with indices $\{\lambda_k\}_{k=1}^{\infty}$. Then domains $\tilde{D}(d, \Lambda)$ and $D(d, \Lambda)$ coincide.*

Proof. Let us show that $d = \{d_k\} \in Q(\Lambda, D(d, \Lambda))$. Let K_p be an arbitrary element of set $K(D(d, \Lambda))$. It is sufficient to show that

$$\overline{\lim_{k \to \infty}} |d_k| \exp H_{K_p}(\lambda_k) < +\infty. \tag{4.1}$$

Suppose the opposite. Then for some sequence $\{k(j)\}$ we have

$$\overline{\lim_{k \to \infty}} |d_k| \exp H_{K_p}(\lambda_k) = +\infty,$$

or, equivalently,

$$\lim_{j \to \infty} (\ln |d_{k(j)}| + H_{K_p}(\lambda_{k(j)})) = +\infty.$$

It yields

$$\overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} (\ln |d_{k(j)}| + H_{K_p}(\lambda_{k(j)})) \geqslant 0. \tag{4.2}$$

Passing to a subsequence once again, we can assume that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to a point $\xi \in \Theta(\Lambda)$. Then in view of semi-continuity and the positive homogeneity of the support function and the definition of $h(d, \xi)$ we obtain

$$\begin{aligned}\overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} (\ln |d_{k(j)}| + H_{K_p}(\lambda_{k(j)})) &\leqslant \overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} \ln |d_{k(j)}| + \overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} H_{K_p}(\lambda_{k(j)}) \\ &\leqslant \overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} \ln |d_{k(j)}| + H_{K_p}(\xi) \leqslant -h(d, \xi) + H_{K_p}(\xi).\end{aligned}$$

Since K_p is a compact set in domain $D(d, \Lambda)$, the inequality $H_{K_p}(\xi) < H_{D(d,\Lambda)}(\xi)$ holds true. Moreover, by the definition of domain $D(d, \Lambda)$ and its support function $H_D(d, \Lambda)$ there holds also the inequality $H_{D(d,\Lambda)}(\xi) \leqslant h(d, \xi)$. Thus, in view of above results,

$$\overline{\lim_{j \to \infty}} |\lambda_{k(j)}|^{-1} (\ln |d_{k(j)}| + H_{K_p}(\lambda_{k(j)})) \leqslant -h(d, \xi) + H_{K_p}(\xi) < -h(d, \xi) + H_{D(d,\lambda)}(\xi) \leqslant 0.$$

It contradicts (4.2). Therefore, (4.1) is true, i.e., $d \in Q(\Lambda, D(d, \Lambda))$. Then in accordance with Lemma 2.3, series (1.1) converges absolutely and uniformly on each compact set in domain $D(d, \Lambda)$. It means that the inclusion $D(d, \Lambda) \subset \tilde{D}(d, \Lambda)$ holds true.

Let us show that the inverse inclusion is valid as well. Let $z \in \tilde{D}(d, \Lambda)$. By the definition of domain $\tilde{D}(d, \Lambda)$, in a neighborhood E of point z series (1.1) converges uniformly. This is why the terms of this series are bounded on set E. Then by Lemma 2.2 sequence $d = \{d_k\}$ is an element of space $Q(\Lambda, E(\Theta(\Lambda))$. As it was noticed above, set E lies in domain $E(\Theta(\Lambda))$. Hence, one of the compact sets $\tilde{K}_p$ of sequence $K(E(\Theta(\Lambda))$ contains the point z in its interior. In accordance with the definition of space $Q(\Lambda, E(\Theta(\Lambda))$ for some $C > 0$ the inequality

$$|d_k| \leqslant C \exp(-H_{\tilde{K}_p}(\lambda_k)), \quad k = 1, 2, \dots \tag{4.3}$$

holds true.

We fix an arbitrary point $\xi \in \Theta(\Lambda)$. In accordance with the definition of quantity $h(d, \xi)$ we find a subsequence $\{k(j)\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}|$ converges to point ξ and

$$h(d, \xi) = \lim_{j \to \infty} \frac{\ln(1/|d_{k(j)}|)}{|\lambda_{k(j)}|}.$$

By (4.3), the positive homogeneity and continuity of the support function of a compact set it implies

$$h(d, \xi) \geqslant \lim_{j \to \infty} \frac{\ln(1/C \exp(-H_{\tilde{K}_p}(\lambda_{k(j)})))}{|\lambda_{k(j)}|} = \lim_{j \to \infty} \frac{(ln(1/C) + H_{\tilde{K}_p}(\lambda_{k(j)}))}{|\lambda_{k(j)}|} =$$

$$= \lim_{j \to \infty} \frac{H_{\tilde{K}_p} \lambda_{k(j)})}{|\lambda_{k(j)}|} = \lim_{j \to \infty} H_{\tilde{K}_p} \left(\frac{\lambda_{k(j)}}{|\lambda_{k(j)}|} \right) = H_{\tilde{K}_p}(\xi).$$

Since point z lies inside compact set $\tilde{K}_p$, inequality $\operatorname{Re}(z\xi) < H_{\tilde{K}_p}(\xi)$ is valid. Therefore, by the previous inequality we have $\operatorname{Re}(z\xi) < h(d, \Lambda)$. We remind that ξ is arbitrary point of set $\Theta(\Lambda)$. And in accordance with the definition, domain $D(d, \Lambda)$ comprises z. The proof is complete. $\square$

Remark. The formula determining $h(d, \Lambda)$ involves as particular cases the corresponding formulae for the series of exponential monomials, series of exponentials and Cauchy-Hadamard formula for power series. In a particular case, for the series $\sum d_k \exp(kz)$ we have

$$h(d, 1) = \varliminf_{k \to \infty} \frac{\ln(1/|d_k|)}{k} = \varliminf_{k \to \infty} (-\ln \sqrt[k]{|d_k|}).$$

Making the transformation $w = \exp z$ reducing this series into the power one, we obtain the following formula for the radius of convergence of the latter

$$R = \exp h(d, 1) = \varliminf_{k \to \infty} \frac{1}{\sqrt[k]{|d_k|}}.$$

Thus, we have obtained Cauchy-Hadamard formula for power series.

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